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Volume 22, 1980

ACADEMIC PRESS
A Subsidiary of Harcourt Brace Jovanovich, Publishers
New York London Toronto Sydney San Francisco

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Printed by the St. Catherine Press, Ltd., Bruges, Belgium

Contents of Volume 22

NUMBER 1, FEBRUARY 1980

ANDREW MCLENNAN. Randomized Preference Aggregation: Additivity of Power and Strategy Proofness. 1
JOSEPH GREENBERG. Beneficial Altruism. 12
EHUD KALAI AND ZVI RITZ. Characterization of the Private Alternatives Domains Admitting Arrow Social Welfare Functions. . . 23
SUSAN I. COHEN. Incentives, Iterative Communication, and Organizational Control. 37
JOHN P. LAITNER. Intergenerational Preference Differences and Optimal National Saving. 56
DAVID DONALDSON AND JOHN A. WEYMARK. A Single-Parameter Generalization of the Gini Indices of Inequality. 67
CHARLES BLACKORBY AND DANIEL PRIMONT. Index Numbers and Consistency in Aggregation. 87

NOTES, COMMENTS, AND LETTERS TO THE EDITOR

MORDECAI KURZ. Income Distribution and Distortion of Preferences: The ℓ Commodity Case. 99
ANUP SHAH. Keynesian Multipliers in Temporary Equilibrium with Consumer Credit Rationing. 107
WILLIAM MACDOUGALL. Utility Theory in Unfair Lotteries. . 113
YAKAR KANNAI. The ALEP Definition of Complementarity and Least Concave Utility Functions. 115

ACKNOWLEDGMENT. 118

NUMBER 2, APRIL 1980

Non-Cooperative Approaches to the Theory of Perfect Competition

ANDREU MAS-COLELL. Noncooperative Approaches to the Theory of Perfect Competition: Presentation. 121
ROY RADNER. Collusive Behavior in Noncooperative Epsilon-Equilibria of Oligopolies with Long but Finite Lives. 136
EDWARD J. GREEN. Noncooperative Price Taking in Large Dynamic Markets. 155
JOSEPH M. OSTROY. The No-Surplus Condition as a Characterization of Perfectly Competitive Equilibrium. 183
LOUIS MAKOWSKI. A Characterization of Perfectly Competitive Economies with Production. 208
LOUIS MAKOWSKI. Perfect Competition, the Profit Criterion, and the Organization of Economic Activity. 222

WILLIAM NOVSHEK AND HUGO SONNENSCHEIN. Small Efficient Scale as a Foundation for Walrasian Equilibrium. 243
KEVIN ROBERTS. The Limit Points of Monopolistic Competition. . 256
OLIVER D. HART. Perfect Competition and Optimal Product Differentiation. 279
WILLIAM NOVSHEK. Equilibrium in Simple Spatial (or Differentiated Product) Models. 313
J. JASKOLD GABSZEWICZ AND J.-F. THISSE. Entry (and Exit) in a Differentiated Industry. 327
PRADEEP DUBEY, ANDREU MAS-COLELL, AND MARTIN SHUBIK. Efficiency Properties of Strategic Market Games: An Axiomatic Approach. 339
PRADEEP DUBEY. Nash Equilibria of Market Games: Finiteness and Inefficiency. 364

NUMBER 3, JUNE 1980

TRUMAN F. BEWLEY. The Permanent Income Hypothesis and Long-Run Economic Stability. 377
KIM C. BORDER AND J. S. JORDAN. Expectations Equilibrium with Expectations Conditioned on Past Data. 395
KOTARO SUZUMURA. Liberal Paradox and the Voluntary Exchange of Rights-Exercising. 407
MURAT R. SERTEL AND ALEXANDER VAN DER BELLEN. On the Route-wise Application of Choice. 423
ELSIE M. KNOER. Economies as Distributions: Implications for Aggregation and Stability. 439
JOHN E. ROEMER. Innovation, Rates of Profit, and Uniqueness of von Neumann Prices. 451
AVIA SPIVAK. Efficient Allocations under a General Transaction Technology. 465
MARTIN F. HELLWIG. On the Aggregation of Information in Competitive Markets. 477
NOTES, COMMENTS, AND LETTERS TO THE EDITOR
ROBERT D. COOTER. How the Law Circumvents Starrett's Nonconvexity. 499

AUTHOR INDEX FOR VOLUME 22. 505

JOURNAL OF ECONOMIC THEORY

Volume 22, Number 1, February 1980

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Published bimonthly at 37 Tempelhof, Bruges, Belgium
by Academic Press, Inc., 111 Fifth Avenue, New York, N.Y. 10003
1980: Volumes 22–23. Price per volume: $61.00 U.S.A.;
$73.00 outside U.S.A. All prices include postage and handling.
Information concerning personal subscription rates may be obtained
by writing to the Publisher.
All correspondence and subscription orders should be addressed to the office of the
Publishers at 111 Fifth Avenue, New York, N.Y. 10003.
Send notices of change of address to the office of the Publishers at least
6–8 weeks in advance. Please include both old and new addresses.
Second class postage paid at Jamaica, N.Y.
Air freight and mailing in the U.S.A. by Publications Expediting, Inc.,
200 Meacham Avenue, Elmont, New York 11003.

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Printed in Bruges, Belgium, by the St. Catherine Press, Ltd.

JOURNAL OF ECONOMIC THEORY 22, 1–11 (1980)

Randomized Preference Aggregation: Additivity of Power and Strategy Proofness

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Received September 1, 1978

In their paper "Preference Aggregation with Randomized Social Orderings" [2], Barbera and Sonnenschein generalize Arrow's Possibility Theorem [1] by considering functions which map profiles of individual strict preference orderings into lotteries on the set of strict preference orderings. Such a function g (called a Strict Social Welfare Scheme [SSWS]) is "Paretian" if, for any alternatives x and y, x is certain to be socially preferred to y whenever all individuals prefer x to y.

Generalizing Arrow's condition of Independence of Irrelevant Alternatives, Barbera and Sonnenschein say that g is "binary" if the probability of x being socially preferred to y depends only on the set of individuals who prefer x to y. They show that if g is Paretian and binary then there exists μ_g, a real-valued function on the set of coalitions of individuals, such that if C is the set of individuals who prefer x to y, then $\mu_g(C)$ will be the probability of x being socially preferred to y. One may think of μ_g as an allocation of power. Barbera and Sonnenshein show that μ_g must be subadditive, and they claim (correctly) that there exist Paretian and binary (SSWSs for which the associated allocation of power is not strictly additive. We show that μ_g must be strictly additive, i.e., a probability measure, whenever there are six or more alternatives. Clearly this strengthens the analogy to Arrow's theorem.

We then compare SSWSs to decision schemes, as defined and analyzed by A. Gibbard.[1] A decision scheme is a function which aggregates individual preferences into lotteries on the set of alternatives. SSWSs, by contrast, are designed to analyze social choice from a feasible set that is arbitrary or unknown at the time of voting. To compare the two, the concept of a "decision scheme" is broadened to allow for different choice probabilities from different feasible sets, giving rise to the notion of a "decision superscheme." A SSWS g induces a decision superscheme f as follows: For each feasible set and each profile of preferences, the probability, under f, of choosing x is set equal to

---

[1] The concepts of "decision scheme" and "strategy proofness" were originated independently by Gibbard [3] and Satterthwaite [6]. Gibbard developed them in the stochastic context [4].

0022-0531/80/010001-11$02.00/0
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the probability, under g, of choosing any preference ordering that ranks x above all other alternatives in the feasible set. A decision scheme is "strategy proof" if no individual can increase the expected value of a von Neumann-Morgenstern utility function consistent with his or her ordinal preferences by misrepresenting those preferences. A decision superscheme is strategy proof, if, for any feasible set, no such misrepresentation can be advantageous, and a SSWS is said to be strategy proof if it induces a decision superscheme that is strategy proof. We show that if a SSWS is Paretian and strategy proof, then its decision probabilities can be duplicated by a lottery on the set of individuals, the winner choosing his or her favorite element from the given feasible set.

I. NOTATION AND A RESTATEMENT OF THE BARBERA–SONNENSCHEIN THEOREM

We follow Barbera and Sonnenschein [2, p. 246]:

> Let $A = \{x, y, z, \ldots\}$ be the set of *alternatives*, and $N = \{1, 2, \ldots, n\}$ be the set of individuals. We assume throughout that A and N are finite and that the number of elements in A (denoted $\#A$) is at least three. Let $\mathscr{B} = \{B, B', \bar{B}, \ldots\}$ be the set of *individual strict preference relations* (complete, asymmetric, and transitive binary relations) on A, and $\mathscr{B}^n$, the n-fold cartesian product of $\mathscr{B}$, be the set of social preference *profiles*. A generic profile is denoted by $\mathbf{B} = (B_1, B_2, \ldots, B_n)$.
>
> Let $L(\mathscr{B})$ be the set of lotteries on $\mathscr{B}$. For $x, y \in A$, let $\mathscr{B}(xBy) = \{B \in \mathscr{B} : xBy\}$. For $l \in L(\mathscr{B})$, let $p(l, xBy) = l(\mathscr{B}(xBy))$.... Given $B \in \mathscr{B}^n$, let $\varphi(\mathbf{B}, xBy) = \{i \in N : xB_iy\}$.
>
> A *Strict Social Welfare Scheme* (*SSWS*) is a function $g : \mathscr{B}^n \to L(\mathscr{B})$. A *SSWS* g is *binary* if for any $x, y \in A$, and any two profiles $\mathbf{B}, \mathbf{B}' \in \mathscr{B}^n$,
>
> $$[\varphi(\mathbf{B}, xBy) = \varphi(\mathbf{B}', xBy)] \text{ implies } [p(g(\mathbf{B}), xBy) = p(g(\mathbf{B}'), xBy)].$$
>
> A *SSWS* is Paretian if for any $x, y \in A$, and any $\mathbf{B} \in \mathscr{B}^n$,
>
> $$[\varphi(\mathbf{B}, xBy) = N] \text{ implies } [p(g(\mathbf{B}), xBy) = 1].$$
>
> Finally, the real-valued function f defined on subsets of X is subadditive if $f(S \cup T) \leq f(S) + f(T)$ for all $S, T \subseteq X$.
>
> THEOREM 1. *Given any binary and Paretian SSWS g, there exists a subadditive function $\mu_g : 2^N \to \mathbb{R}$, such that*
>
> (1) $\mu_g(\varphi(\mathbf{B}, xBy)) = p(g(\mathbf{B}), xBy)$,
>
> *for all $\mathbf{B} \in \mathscr{B}^n$ and $x, y \in A$; furthermore;*
>
> (2) $\mu_g(\phi) = 0$;
>
> (3) $\mu_g(C) + \mu_g(N - C) = 1$, *for all $C \subseteq N$; and*
>
> (4) $\mu_g(C) \geq \mu_g(C')$, *whenever $C' \subseteq C$.*

II. WITH SIX OR MORE ALTERNATIVES μ_g IS ADDITIVE

We begin by strengthening Theorem 1:

THEOREM 2. *Suppose* $\#A \geq 6$, *and let* g *be a binary and Paretian* SSWS. *Then the function* μ_g *defined in Theorem* 1 *must be additive, i.e.*,

$$(5) \quad \mu_g(S \cup T) = \mu_g(S) + \mu_g(T) \textit{ for all disjoint } S, T \subseteq N.$$

Proof of Theorem 2. From Theorem 1, μ_g is known to be subadditive, so it suffices to show that

$$(5') \quad \mu_g(S \cup T) \geq \mu_g(S) + \mu_g(T) \text{ for all } S, T \subseteq N \text{ such that } S \cap T = \varnothing.$$

To produce a contradiction we assume that there exist disjoint $S, T \subseteq N$ such that $\mu_g(S \cup T) < \mu_g(S) + \mu_g(T)$. By assumption we may choose $\{x_1, \ldots, x_6\} \subseteq A$, and $\mathbf{B} \in \mathscr{B}^n$ is selected so that

$$\begin{array}{ll} x_1 B_i x_3 B_i x_2 B_i x_5 B_i x_4 B_i x_6 & \text{for } i \in S, \\ x_4 B_i x_5 B_i x_6 B_i x_3 B_i x_2 B_i x_1 & \text{for } i \in T, \text{ and} \\ x_3 B_i x_5 B_i x_1 B_i x_4 B_i x_2 B_i x_6 & \text{for } i \in N - (S \cup T). \end{array}$$

Let $l_{hk} = p(l, x_h B x_k)$ for $l \in L(\mathscr{B})$ and $1 \leq h, k \leq 6$, $h \neq k$, and define $\theta\colon L(\mathscr{B}) \to \mathbb{R}$ by

$$\theta(l) = (l_{12} + l_{32} + l_{34} + l_{51} + l_{56} + l_{16}) + (l_{25} + l_{63} + l_{41}).$$

Theorem 1 and the reductio assumption imply

$$\begin{aligned} \theta(g(\mathbf{B})) &= 6\mu_g(N) + \mu_g(S) + \mu_g(T) + \mu_g(N - (S \cup T)) \\ &= 6 + \mu_g(S) + \mu_g(T) + 1 - \mu_g(S \cup T) > 7. \end{aligned}$$

For $B \in \mathscr{B}$ we let $l^B \in L(\mathscr{B})$ be the lottery that selects B with probability 1, so l^B_{hk} must be either 0 or 1. Since, for any $l \in L(\mathscr{B})$,

$$l_{hk} = \sum_{B \in \mathscr{B},\, x_h B x_k} l(\{B\}) = \sum_{B \in \mathscr{B}} l^B_{hk} l(\{B\}),$$

it could be shown by redistributing terms that $\theta(l)$ is a convex combination of $\{\theta(l^B)\colon B \in \mathscr{B}\}$. Hence $\theta(g(B)) > 7$ implies the existence of a $B \in \mathscr{B}$ such that $\theta(l^B) > 7$, and this means that at least eight of the terms constituting $\theta(l^B)$ must equal 1. This, however, implies that at least one of the following intransitivities must hold:

$$x_1 B x_2 B x_5 B x_4 B x_1\,;$$

$$x_3 B x_2 B x_5 B x_6 B x_3\,;$$

$$x_3 B x_4 B x_1 B x_6 B x_3\,.$$

The following diagrammatic representation of $\theta(l)$ 9 may make this contradiction more understandable:

$$\begin{tikzcd}
x_1 \arrow[r] & x_2 \arrow[d] & x_3 \arrow[l, dashed] \arrow[r] & x_1 \arrow[d, dashed] \\
x_4 \arrow[u] & x_5 \arrow[l, dashed] \arrow[r, dashed] & x_6 \arrow[u, dashed] & x \arrow[l]
\end{tikzcd}$$

("$x_h \dashrightarrow x_k$" means $l_{hk} = 1$).

The proof is complete.[2,3]

Barbera and Sonnenschein also consider functions which map profiles of individual weak (indifference allowed) preference orderings to lotteries on the set of weak preference orderings, and they prove a theorem analogous to Theorem 1 above. In particular, they derive the existence of a subadditive μ_g satisfying (2)-(4). A reproduction of their definitions would be excessively lengthy, so we will merely state that μ_g must be additive in this case as well provided, of course, that there are six or more alternatives. With minor modifications the proof is the same as that given above.

III. EXISTENCE OF A SSWS g FOR GIVEN μ

Jerry Butters has shown that if $\#A \leq 5$, μ is subadditive, and μ satisfies (2)-(4), then there exists a binary and Paretian SSWS g such that $\mu_g = \mu$; thus the assumptions of Theorem 1 place no additional restrictions on μ_g when $\#A \leq 5$.[4]

It is easy to see that if μ is any probability measure on N there exists a Paretian and binary SSWS g such that $\mu_g = \mu$, regardless of $\#A$. We define g_μ as follows: for each profile **B** let $g_\mu(\mathbf{B})$ be the lottery that selects each $B \in \mathscr{B}$ with probability $\mu(\{i \in N: B_i = B\})$. It is left for the reader to check that g_μ is Paretian and binary. One may think of g_μ as a lottery on the set of individuals, the winner becoming the dictator, and we shall say that SSWS h is *randomly dictatorial* if there exists μ, a probability measure on N, such that $h = g_\mu$.

IV. STRATEGIC CONSIDERATIONS

It is natural to imagine the following sequence of events leading to a social decision: Preferences are collected and recorded; a SSWS converts the profile

---

[2] For more on the set of pairwise preference probabilities that can be produced by lotteries on the set of preference orderings, see [5]. The work presented in that paper, by McFadden and Richter, inspired the proof presented here.

[3] Aspects of this presentation are taken from a proof by S. Barbera (private correspondence).

[4] A proof is available upon request. The method is to show that when $\#A \leq 5$ any set of pairwise preference probabilities that satisfy a condition analogous to transitivity can be produced by a lottery on preference orderings.

of preferences into a lottery on $\mathscr{B}$; the lottery is conducted, arriving at a specific social ordering of the alternatives; the feasible set is determined; and the socially most preferred feasible alternative becomes the social choice. If the feasible set is known before this process begins there is no point in having a full-blown SSWS. All that counts is the ultimate choice, so the social ordering of other alternatives is irrelevant, and the appropriate mechanism is a "decision scheme," as defined and analyzed by Gibbard [4]. Attempting to parallel Gibbard's work, we now focus our attention on what it means for a SSWS to be nonmanipulable or cheat proof.

Consider the following example: $N = \{1, 2\}$, with $\mu(\{1\}) = 1/2 = \mu(\{2\})$; $A = \{x, y, z\}$, and all elements of A are feasible; $\mathbf{B} = (B_1, B_2)$ is the profile of true preferences, where xB_1yB_1z and zB_2yB_2x; U_1, individual 1's von Neumann–Morgenstern utility function, is such that

$$U_1(x) - U_1(y) > U_1(y) - U_1(z),$$

so that

$$1/2U_1(x) + 1/2U_1(z) > 1/3U_1(x) + 1/3U_1(y) + 1/3U_1(z);$$

g maps $\mathbf{B}$ to the lottery that selects each element of $\mathscr{B}$ with probability 1/6, and g maps all other profiles to the lottery that assigns probability 1/2 to each individual's preference ordering. Assuming individual 2 reports his or her true preferences, if individual 1 reports B_1 his or her expected utility will be $1/3U_1(x) + 1/3U_1(y) + 1/3U_1(z)$. Clearly individual 1 can do better by reporting B_1', where $xB_1'zB_1'y$, thus attaining an expected utility of $1/2U_1(x) + 1/2U_1(z)$. This example is clearly Paretian and binary. Arrow's theorem may be viewed as saying that Social Welfare Functions which are Paretian and satisfy Independence of Irrelevant Alternatives must be strategy proof,[5] but this example shows that the analogous proposition is false in our more general framework.

V. SSWSs AND DECISION SCHEMES[6]

One might expect that, unless the set of potential feasible sets could be restricted in some way, only randomly dictatorial SSWSs would be able to eliminate strategic manipulation. This is not the case, as will be shown in the Appendix, but it is not far wrong; we will show that a nonmanipulable SSWS must give a probability distribution of outcomes which could also be produced by a randomly dictatorial SSWS.

---

[5] This point of view is discussed in [6].

[6] This section is an extension of the work of Gibbard [4]. Much terminology is taken from this source, with the meanings of several terms generalized to suit the needs of this paper.

Let $L(A)$ be the set of lotteries on A. For $k \in L(A)$ and $x \in A$ let $p(k, x) = k(\{x\})$. For each $Z \in 2^A - \{\phi\}$, let $L(Z)$ be the set of lotteries on Z, interpreted as a subset of $L(A)$; i.e., $L(Z) = \{k \in L(A): p(k, x) = 0$ for all $x \in A - Z\}$. A *decision scheme* is a function d from the set of profiles into the set of lotteries on A, i.e., $d: \mathscr{B}^n \rightarrow L(A)$.

In order to make a proper comparison between SSWSs and decision schemes, the notion of a "decision scheme" must be expanded to allow different choice probabilities for different feasible sets. A *decision superscheme* (DSS) is defined to be a function $f: (2^A - \{\phi\}) \times \mathscr{B}^n \rightarrow L(A)$ such that $f(Z, \mathbf{B}) \in L(Z)$ for all $Z \in 2^A - \{\phi\}$ and $\mathbf{B} \in \mathscr{B}^n$. Thus a DSS assigns a feasible decision scheme to each feasible set, and we shall write $f(Z, \cdot)$ for the decision scheme d satisfying $d(\mathbf{B}) = f(Z, \mathbf{B})$ for all $\mathbf{B} \in \mathscr{B}^n$.

We need a condition on f that relates $f(Z_1, \cdot)$ to $f(Z_2, \cdot)$ for $Z_1 \neq Z_2$. It is natural for our purposes to require that f can be "rationalized" by a SSWS g. For $Z \in 2^A - \{\phi\}$ and $B \in \mathscr{B}$, let $\pi(Z, B)$ be the (unique) $x \in Z$ such that xBy for all $y \in Z$, $y \neq x$, so $\pi(Z, B)$ is the most preferred element of Z under B.

For $l \in L(\mathscr{B})$ and $\mathscr{C} \subseteq \mathscr{B}$, let $p(l, \mathscr{C}) = l(\mathscr{C})$. We shall say that SSWS g *induces* DSS f if

(6) $p(f(Z, \mathbf{B}), x) = p(g(\mathbf{B}), \{B \in \mathscr{B}: \pi(Z, B) = x\})$ for all $Z \in 2^A - \{\phi\}$, all $x \in Z$, and all $\mathbf{B} \in \mathscr{B}^n$.

Thus, g induces f if, for all $Z \in 2^A - \{\phi\}$ and all $\mathbf{B} \in \mathscr{B}^n$, the probability under $f(Z, \mathbf{B})$ of choosing $x \in Z$ is equal to the probability under $g(\mathbf{B})$ of choosing any $B \in \mathscr{B}$ that has x preferred to all other elements of Z. All SSWSs induce some DSS, but the reverse is not true.

A decision scheme d is *strategy proof* if, for all $\mathbf{B} \in \mathscr{B}^n$, all $i \in N$, all $U_i: A \rightarrow \mathbb{R}$ satisfying

$[U_i(x) > U_i(y)]$ if and only if $[xB_iy]$ for all $x, y \in A$, and all $B_i' \in \mathscr{B}$

$$\sum_{x \in A} U_i(x) \cdot p(d(\mathbf{B}), x) \geq \sum_{x \in A} U_i(x) \cdot p(d(\mathbf{B}'), x),$$

where $\mathbf{B}' = (B_1, \ldots, B_i', \ldots, B_n)$. Less formally, a decision scheme is strategy proof if it is never possible for some individual to increase the expected value of his or her (arbitrary) von Neumann–Morgenstern utility function by reporting a false ordinal preference ordering. The DSS f is said to be strategy proof if $f(Z, \cdot)$ is strategy proof for all $Z \in 2^A - \{\phi\}$, and a SSWS g is said to be strategy proof if the DSS induced by g is strategy proof.

THEOREM 3. *If* SSWS g *is strategy proof, then* g *is binary.*

Proof of Theorem 3. Let f be the DSS induced by g; since g is strategy proof, so is f. Suppose g is not binary; there exist $x, y \in A$ and $\mathbf{B}, \mathbf{B}' \in \mathscr{B}^n$ such that $\varphi(\mathbf{B}, xBy) = \varphi(\mathbf{B}', xBy)$, but $p(g(\mathbf{B}), xBy) \neq p(g(\mathbf{B}'), xBy)$. By (6) this implies that $f(\{x, y\}, \mathbf{B}) \neq f(\{x, y\}, \mathbf{B}')$. Let $\mathbf{B}_i = (B_1', \dots, B_i', B_{i+1}, \dots, B_n)$ for $i = 0, \dots, n$. There must be an $i \in N$ such that $f(\{x, y\}, \mathbf{B}_{i-1}) \neq f(\{x, y\}, \mathbf{B}_i)$. Since $[\varphi(\mathbf{B}, xBy) = \varphi(\mathbf{B}', xBy)]$ implies $[xB_iy$ if and only if $xB_i'y]$, it must be the case that, when preferences are reported as B_j' for $j < i$ and as B_j for $j > i$, and the feasible set is $\{x, y\}$, either individual i with true preferences B_i does better by misrepresenting them as B_j', or vice versa. This contradiction of the strategy proofness of f completes the proof.

A decision scheme d is *Pareto optimific ex post* (POEP) if, for all $x, y \in A$ and $\mathbf{B} \in \mathscr{B}^n$,

$$[xB_iy \text{ for all } i \in N] \text{ implies } [p(d(\mathbf{B}), y) = 0].$$

The following theorem says that a decision scheme that is strategy proof and POEP can be reduced to a lottery on the set of individuals, the winner choosing his or her preferred alternative

THEOREM 4[7]. *If d is a decision scheme that is strategy proof and* POEP, *there exists ν, a probability measure on N, such that*

$$p(d(\mathbf{B}), x) = \nu(\{i \in N: \pi(A, B_i) = x\}),$$

for all $\mathbf{B} \in \mathscr{B}^n$ and all $x \in A$.

For decision superschemes, Pareto dominance is only relevant when the dominating alternative is feasible, so we shall say that a DSS f is POEP if

$$[xB_iy \text{ for all } i \in N] \text{ implies } [p(f(Z, \mathbf{B}), y) = 0],$$

for all $Z \in 2^A - \{\phi\}$, all $x, y \in Z$, and all $\mathbf{B} \in \mathscr{B}^n$.

THEOREM 5. *If* SSWS *g induces* DSS *f, then g is Paretian if and only if f is* POEP.

Proof of Theorem 5. Suppose that g is Paretian. Then, for any $Z \in 2^A - \{\phi\}$, any $x, y \in Z$, and any $\mathbf{B} \in \mathscr{B}^n$,

$$\begin{aligned} &[xB_iy \text{ for all } i \in N] \text{ implies } [p(g(\mathbf{B}), xBy) = 1], \\ &\text{which implies } [p(g(\mathbf{B}), \{B \in \mathscr{B}: \pi(Z, B) = y\}) = 0], \\ &\text{which implies } [p(f(Z, \mathbf{B}), y) = 0]. \end{aligned}$$

---

[7] This theorem, a corollary to Gibbard's theorem due to H. Sonnenschein, is reported and proved, using different terminology and phrasing, in [4, p. 677].

This completes the "only if." Now suppose that f is POEP, and take $x, y \in A$, $x \neq y$. For any $\mathbf{B} \in \mathscr{B}^n$,

$$[xB_iy \text{ for all } i \in N] \text{ implies } [p(f(\{x, y\}, \mathbf{B}), y) = 0],$$
$$\textit{which implies } [p(g(\mathbf{B}), \{B \in \mathscr{B}: \pi(\{x, y\}, B) = x\}) = 1],$$
$$\textit{which implies } [p(g(\mathbf{B}), xBy) = 1].$$

This completes the "if" and the proof.

The following theorem states the central point we wish to make about Paretian and strategy-proof SSWSs.

THEOREM 6. *Suppose that the* SSWS g *is Paretian and strategy proof, and let* f *be the* DSS *that is induced by* g. *There exists a randomly dictatorial* SSWS h *that also induces* f.

Proof of Theorem 6. By Theorem 3, g is binary, so by Theorem 1, there exists a subadditive μ_g satisfying (1)-(4). By definition, f is strategy proof, and by Theorem 5 f is POEP, so $f(A, \cdot)$ must be strategy proof and POEP. By Theorem 4, there exists ν, a probability measure on N, such that

$$p(f(A, \mathbf{B}), x) = \nu(\{i \in N: \pi(A, B_i) = x\}) \qquad \text{for all } \mathbf{B} \in \mathscr{B}^n \text{ and all } x \in A.$$

In order to show that $\nu = \mu_g$ we choose any $C \subseteq N$ and any $x, y \in A$. A profile $\mathbf{B} \in \mathscr{B}^n$ is chosen such that $\varphi(\mathbf{B}, xBy) = C$ and $\varphi(\mathbf{B}, xBw) = N$ for all $w \in N - \{x, y\}$. Then

$$\begin{aligned}
\nu(C) &= \nu(\{i \in N: \pi(A, B_i) = x\}) \\
&= p(f(A, \mathbf{B}), x) \\
&= p(g(\mathbf{B}), \{B \in \mathbf{B}: \pi(A, B) = x\}) \\
&= p(g(\mathbf{B}), xBy) \qquad \text{(since } x \text{ Pareto dominates everything else)} \\
&= \mu_g(\varphi(\mathbf{B}, xBy)) = \mu_g(C).
\end{aligned}$$

Since we chose C arbitrarily, $\nu = \mu_g$, and it follows immediately that μ_g is additive.

We want to show that

$$p(f(Z, \mathbf{B}), x) = \nu(\{i \in N: \pi(Z, B_i) = x\})$$

or any $Z \in 2^A - \{\phi\}$, any $x \in Z$, and any $\mathbf{B} \in \mathscr{B}^n$. Take Z and $\mathbf{B}$, and let $\mathbf{B}'$ be ny profile such that, for all $i \in N$, all $x, y \in Z$, and all $w \in A - Z$,

$$[xB'_iy] \quad \text{if and only if} \quad [xB_iy], \quad \text{and} \quad xB'_iw.$$

($\mathbf{B}'$ is any profile where all elements of Z Pareto dominate all elements for $A - Z$, and B'_i orders Z in the same way as B_i for all $i \in N$.)

Define $\mathbf{B}_i = (B'_1, \ldots, B'_i, B_{i+1}, \ldots, B_n)$. If $f(Z, \mathbf{B}_{i-1}) \neq f(Z, \mathbf{B}_i)$ for some $i \in N$, then there must be a $U_i\colon A \to \mathbb{R}$ satisfying (7) such that

$$\sum_{x \in Z} U_i(x) \cdot p(f(Z, \mathbf{B}_{i-1}), x) \neq \sum_{x \in Z} U_i(x) \cdot p(f(Z, \mathbf{B}_i), x).$$

This means that, if other individuals report their preferences as B'_j for $j < i$ and as B_j for $j > i$, individual i with true preferences B_i could do better by misrepresenting them as B'_i, or vice versa. Thus strategy proofness implies $f(Z, \mathbf{B}_{i-1}) = f(Z, \mathbf{B}_i)$ for all $i \in N$. Therefore, for any $x \in Z$,

$$\begin{aligned}
p(f(Z, \mathbf{B}), x) &= p(f(Z, \mathbf{B}_0), x) = \cdots = p(f(Z, \mathbf{B}_n), x) = p(f(Z, \mathbf{B}'), x)\\
&= p(g(\mathbf{B}'), \{B \in \mathscr{B}\colon \pi(Z, B) = x\})\\
&= p(g(\mathbf{B}'), \{B \in \mathscr{B}\colon \pi(A, B) = x\}) \qquad \text{(since } g \text{ is Peretian)}\\
&= p(f(A, \mathbf{B}'), x)\\
&= \nu(\{i \in N\colon \pi(A, B'_i) = x\})\\
&= \nu(\{i \in N\colon \pi(Z, B_i) = x\}) \qquad \text{(by construction of } \mathbf{B}'\text{)}.
\end{aligned}$$

Letting $h = g_\nu$, so that h is randomly dictatorial,

$$\begin{aligned}
p(h(\mathbf{B}), \{B \in \mathscr{B}\colon \pi(Z, B) = x\}) &= \nu(\{i \in N\colon \pi(Z, B_i) = x\})\\
&= p(f(Z, \mathbf{B}), x),
\end{aligned}$$

for all $\mathbf{B} \in \mathscr{B}^n$, $Z \in 2^A - \{\phi\}$, and $x \in Z$, so h induces f, and this completes the proof.

It is shown in the Appendix that there are SSWSs which are strategy proof and Paretian, but not randomly dictatorial, but Theorem 6 says they might as well be: We have shown that the choice probabilities of any Paretian and strategy proof SSWS can be duplicated by a SSWS that is randomly dictatorial.

VI. CONCLUDING REMARKS

We have shown that if g is a binary and Paretian SSWS then μ_n must be strictly additive, unless $\#A < 5$. Applying the concept of strategy proofness to Paretian SSWSs, we have shown that a Paretian and strategy proof SSWS can be replaced by a randomly dictatorial SSWS that will choose any element from any feasible set with equal probability. We have shed light on both the nature of mechanisms that aggregate individual preferences and the relationship between such mechanisms and social choice procedures.

APPENDIX

It will be shown, nonconstructively, that there exist SSWSs that are Paretian and strategy proof, but not randomly dictatorial. We begin by showing that it is possible for two lotteries on $\mathscr{B}$ to produce identical selection probabilities for all feasible sets.

We let $m = \#A$, and we assume that $m \geq 4$. Lotteries in $L(\mathscr{B})$ will be regarded as vectors $l \in \mathbb{R}^{m!}$ indexed by $B \in \mathscr{B}$, so that $l_B = p(l, \{B\})$, with l satisfying

$$\sum_{B \in \mathscr{B}} l_B = 1. \tag{1A}$$

Let $l \in L(\mathscr{B})$ be such that $l_B > 0$ for all $B \in \mathscr{B}$, and consider the condition

$$p(l', \{B \in \mathscr{B}\colon \pi(Z, B) = x\}) = p(l, \{B \in \mathscr{B}\colon \pi(Z, B) = x\})$$
$$\text{for all } Z \in 2^A - \{\phi\} \text{ and all } x \in Z. \tag{2A}$$

Equation (2A) says that l and l' produce identical selection probabilities for all feasible sets, so if we can prove the existence of $l' \in \mathbb{R}^{m!}_+$, $l' \neq l$, satisfying (1A) and (2A) we shall have achieved our objective.

For each $Z \in 2^A - \{\phi\}$, (2A) gives $\#Z$ linear condition on l', one of which can be eliminated by linear dependence with (1A). For each k, $1 \leq k \leq m$, there are $\binom{m}{k}$ subsets of A with k elements. Hence there are, in total, at most

$$\begin{aligned} 1 + \sum_{k=1}^{m} \binom{m}{k}(k-1) &= 1 + \sum_{k=1}^{m} \binom{m}{k} \cdot k - \sum_{k=1}^{m} \binom{m}{k} \\ &= 1 + \sum_{k=1}^{m} m \cdot \binom{m-1}{k-1} - (2^m - 1) \\ &= 2 - 2^m + m \cdot 2^{m-1} \end{aligned}$$

linearly independent restriction on l'. Computation shows that $2 \cdot (1 + (m-2)\,2^{m-2}) < m!$ whenever $m \geqslant 4$. Since l is a strictly positive solution of the system [(1A) and (2A)], the solution set of this system must have a dimension greater than zero, so there must be an $l' \in \mathbb{R}^{m!}_+$, $l' \neq l$, which satisfies (1A) and (2A).

We construct the desired SSWS as follows: Take the set of individuals to be $N = \{I_B\colon B \in \mathscr{B}\}$; let ν be the probability measure on N which has $\nu(\{I_B\}) =$

I_B ; let **B** be the profile where individual I_B has preferences B; define the SSWS g by

$$\begin{aligned} g(\mathbf{B}') &= g_v(\mathbf{B}'), & \mathbf{B}' &\neq \mathbf{B}, \\ &= l', & \mathbf{B}' &= \mathbf{B}. \end{aligned}$$

Since g produces the same decision probabilities as g_v for any feasible set, g must be Paretian and strategy proof, but since $g(\mathbf{B}) \neq g_v(\mathbf{B})$, g cannot be randomly dictatorial.

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JOURNAL OF ECONOMIC THEORY 22, 12–22 (1980)

Beneficial Altruism

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Received February 6, 1979; revised June 21, 1979

We investigate the question of whether or not it is possible that every allocation that does not involve unilateral transfers of some of the always desired goods, without receiving anything in return, can be Pareto dominated by an allocation that involves such transfers.

1. INTRODUCTION

Consider an economy with n individuals each of whom has a utility function over the l commodities, which is strictly increasing in each commodity. (Thus, more is always strictly preferred to less.) To each subset of individuals (or coalition), there is a (net) production set that describes the possible consumption bundles which are attainable by that group if and when it forms. Due to constraints such as capacity, communications, management, or control, (all reflected by the (net) production correspondence) it may be advantageous for the individuals to carry out activities in coalitions that partition the set of agents, since payoffs under such a coalition structure may dominate the feasible payoffs for the grand coalition. Examples of such situations are: local public goods which due to congestion, either in production or in consumption, are best provided by local communities rather than by the whole nation; production which is carried out in more than one "large" firm; the existence of social and sports clubs, etc.

Assume that such is the situation, and let the n individuals form the J "countries" (coalitions) $B_1, \ldots, B_J$. (Since every individual belongs to one and only one country, $J \leq n$.) The purpose of this paper is to investigate whether or not it is possible that regardless of the initial "country structure" (partition), whenever the total amount produced in each country is distributed among its residents, the situation of *every* individual in society may be improved by forming a different "country structure" in which each country either exports commodities without importing any goods in return, or else imports but does not export any goods. Since such transfers imply that some countries allocate part of the always desired commodities produced in them to other countries, without receiving any counterpart (thus, transfers are

0022-0531/80/010012-11$02.00/0

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distinguished from trade), this property could be understood as meaning that "altruism" is beneficial.

Section 3 brings out a somewhat surprising example in which such indeed is the case. Specifically, by considering a society with three individuals and two commodities, we show that it is possible that every allocation, in which members of a coalition consume exactly the amount produced in that coalition, can be dominated by an allocation that involves net transfers from one coalition to another, giving a higher utility level to every individual.

Section 4 is devoted to a discussion of such possibilities. First, it is shown that when either the number of individuals is less than three or there is only a single commodity, it is never the case that every allocation which does not involve transfers can be Pareto dominated by an allocation involving them. On the other hand, imposing a (strict) concavity assumption on the utility function does not rule out such possibilities. Thus, quite restrictive assumptions seem required to rule out such situations.

Finally, we give three alternative interpretations to the example of Section 3, and show that this example can be understood to imply that transfers among coalitions may be necessary to attain Pareto optimality. This very issue was investigated by Drèze and Greenberg [1] and Greenberg and Kats [2], and we show that the example in this paper can serve as an example also for the situations considered in the above two works.

The general framework is given formally in Section 2.

2. THE MODEL

There are n individuals, given by the set $N := \{1, \ldots, n\}$ and l commodities. Each individual $i \in N$ has a consumption set $X^i \subset R_+^l$, over which his utility function, u_i, is defined. The set of all subsets of N (i.e., the set of all possible coalitions) is denoted by $\mathcal{N}$, and its generic element is denoted by S with cardinality $|S|$ (i.e., $|S|$ denotes the number of members in coalition S).

There exists a production correspondence $Y: \mathcal{N} \twoheadrightarrow R_+^l$ where, for a coalition $S \subset N$, $Y(S) \subset R_+^l$ denotes the set of all combinations of total amounts of the l goods available to S if and when it forms. (Note that initial endowments are implicit in the definition of the production correspondence.)

We shall assume that each coalition S is able to survive on the basis of its own technology and resources ((A.1) below), and that each individual strictly prefers more to less. Formally,

(A.1) For all $S \subset N$ and for every individual $i \in S$, there exists $x^i \in X^i$ with $\sum_{i \in S} x^i \in Y(S)$.

(A.2) For all $i \in N$, u_i is strictly increasing in each coordinate.

We also assume that each individual belongs to one and only one coalition. A partition of N, or a coalition structure, is denoted $B = (B_1, \dots, B_J)$, with generic element B_j. The set of all coalition structures is denoted $\mathscr{B}$.

We contrast three situations. In the first, trade among coalitions is allowed, in the second only transfers (as opposed to trades) are allowed, and in the third situation neither trades nor transfers can take place. For a given coalition structure $B = (B_1, \dots, B_J)$ the sets of feasible allocations in each of the above three situations are $K(B)$, $L(B)$, and $M(B)$, respectively, where

$K(B) = \{a = (a^1, \dots, a^n) \mid$ for all $i \in N$ $a^i \in X^i$ and for all j, $j = 1, \dots, J$, there exists $y(B_j) \in Y(B_j)$ such that $\sum_{i \in N} a^i = \sum_{B_j \in B} y(B_j)\}$;

$L(B) = \{a = (a^1, \dots, a^n) \mid$ for all $i \in N$ $a^i \in X^i$ and for all j, $j = 1, \dots, J$, there exists $y(B_j) \in Y(B_j)$ such that either[1] $\sum_{i \in B_j} a^i \geq y(B_j)$, or: $\sum_{i \in B_j} a^i \leq y(B_j)$ and $\sum_{i \in N} a^i = \sum_{B_j \in B} y(B_j)\}$;

$M(B) = \{a = (a^1, \dots, a^n) \mid$ for all $i \in N$ $a^i \in X^i$ and for all j, $j = 1, \dots, J$, there exists $y(B_j) \in Y(B_j)$ such that $\sum_{i \in B_j} a^i = y(B_j)\}$.

Since transfers are a special form of trade, and transfers do not have to be made even when they are allowed, we have that for every $B \in \mathscr{B}$, $K(B) \supset L(B) \supset M(B)$.

The sets of all attainable allocations from any coalition structure are given, respectively, by

$$K = \bigcup_{B \in \mathscr{B}} K(B), \qquad L = \bigcup_{B \in \mathscr{B}} L(B), \qquad \text{and} \qquad M = \bigcup_{B \in \mathscr{B}} M(B).$$

Thus, for example, $a \in K$ means that there is a coalition structure $B \in \mathscr{B}$ and a feasible choice of production levels in each coalition in B which is distributed among individuals, with individual i receiving the bundle $a^i \in X^i$. Similarly, $a \in L$ implies that there is a coalition structure B which admits the production of $\sum_{B_j \in B} y(B_j)$ and this amount is distributed among the individuals in such a way that for each coalition in B the total amount given to its members is either no greater in every commodity or no less in every commodity than the amount produced by that coalition. Were the amounts received by members of each coalition exactly equal to the amount produced there, we would have $a \in M$.

Denote the images of the sets K, L, and M in the utility space by U, V, and W, respectively, i.e.,

$U = \{b \in R^n \mid$ there exists a feasible allocation $a \in K$ such that for all $i \in N$, $b_i = u_i(a^i)\}$;

$V = \{b \in R^n \mid \exists a \in L$ such that $b_i = u_i(a^i)$, $\forall i \in N\}$;

$W = \{b \in R^n \mid \exists a \in M$ such that $b_i = u_i(a^i)$, $\forall i \in N\}$.

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[1] For $x, y \in R^t$, $x \geqq y$ means $x_k \geq y_k$ $k = 1, \dots, t$; $x \geq y$ means $x_k \geqslant y_k$ for all k and $x \neq y$; $x \gg y$ means $x_k > y_k$, $k = 1, \dots, t$.

Thus, for example, $b \in W$ represents the utility levels attainable by distributing the total output produced by some coalition structure B in such a way that the total output received by members of any coalition in B exactly equals the total output produced by that coalition.

Given a set $A \subset R^n$, the set of maximal elements in A is denoted by $\bar{A}$, i.e. $\bar{A} = \{a \in A \mid \nexists \bar{a} \in A \text{ with } \bar{a} \gg a\}$. In particular $\bar{U}$, $\bar{V}$, and $\bar{W}$ denote, respectively, the sets of Pareto optimal points in U, V, and W.

It is well known that, in general, trades are mutually profitable and hence, in most situations $\bar{U} \cap \bar{W} = \varnothing$. That is, for every utility level, b, which is attainable from a feasible allocation that does not involve trades or transfers among coalitions, there exists another utility level attainable from a feasible allocation which involves trades, under which every individual is at least as well-off as under b, and at least one individual is strictly better-off.

It is equally well known that it is always the case that $\bar{U} \cap \bar{V} \neq \varnothing$. Indeed, $\bar{b} \in \bar{U} \cap \bar{V}$, where $\bar{b}_i = u_i(\bar{a}^i)$ with

$$u_1(\bar{a}^1) = \max_{a \in K} u_1\left(\sum_{i \in N} a^i\right) \quad \text{and} \quad \forall i \neq 1, \quad \bar{a}^i = 0.$$

That is, the total output produced, which is entirely transferred to individual 1, is the one that maximizes Mr. 1's utility.

Recall that for every coalition structure B, $K(B) \supset L(B) \supset M(B)$ implying $K \supset L \supset M$ which in turn yields $U \supset V \supset W$. We know that in general $\bar{U} \cap \bar{W} = \varnothing$ and that it is always true that $\bar{U} \cap \bar{V} \neq \varnothing$. The only interesting question left to be answered is whether or not $\bar{V} \cap \bar{W} = \varnothing$.

Note that if this intersection is empty, then a unilateral transfer of some of the always desired commodities from some coalitions to others (together with a change in the coalition structure) can make *every* individual better-off and this holds uniformly for every coalition structure. Put differently, whenever individuals partition themselves into coalitions and distribute the amount produced in each coalition within that coalition, there is another coalition structure and an allocation that involves transfers under which every individual is made better-off. Thus, $\bar{U} \cap \bar{W} = \varnothing$ can be interpreted to mean that altruism is beneficial! The following example shows that this indeed may well be the case.

3. THE EXAMPLE

There are three individuals ($N = \{1, 2, 3\}$) and two commodities ($l = 2$) denoted by x and y. The consumption set of each individual $i \in N$ is $X^i = R_+^2$ (the nonnegative orthant of the plane) and his utility function is given by

$$u_i(x, y) = (100y + 2)^{x+2}.$$

Claim 4. *Whenever* $n = 2$, $\overline{V} \cap \overline{W} \neq \varnothing$.

Proof. Define, for $i = 1, 2$, $\alpha_i = \text{Max}_{z \in Y(\{i\})} u_i(z)$. Clearly, $\alpha = (\alpha_1, \alpha_2) \in W$. Thus, if $\alpha \in \overline{V}$, $\alpha \in \overline{V} \cap \overline{W}$. If, on the other hand, $\alpha \notin \overline{V}$, then there exists $b \in \overline{V}$, $b_i = u_i = u_i(a^i)$ for some $a \in L$ with $b \geqslant \alpha$. By the definition of α, $a \notin L(\{1\}, \{2\})$. Since $n = 2$, this implies that $a \in L(\{N\})$. But, it is always true that $L(\{N\}) = M(\{N\})$ and hence $b \in W$. Q.E.D.

Claims 3 and 4 imply that the example of Section 3 is the "simplest" possible. This example can easily be modified to show that

Claim 5. $\overline{W} \cap \overline{V}$ *may be empty even when all individuals have identical utility functions which are strictly concave and separable* (i.e., $u(x, y) = \Phi(x) + \Psi(y)$).

Proof. Since Pareto optimality is an ordinal rather than a cardinal notion, any monotonic increasing transformation of the utility function of the example of Section 3 will preserve the property that $\overline{W} \cap \overline{V} = \varnothing$. In particular, one can represent the preference relation in that example by the utility function: $\log \log u(x, y) = \log(x + 2) + \log \log(100y + 2)$. Q.E.D.

It therefore seems that no "reasonable" restrictions on the preferences can rule out the possibility that every allocation which does not involve transfers can be Pareto dominated by an allocation that involves them. Are there restrictions on the production correspondence $Y(\cdot)$ which would imply $\overline{W} \cap \overline{V} \neq \varnothing$? Clearly, this intersection is not empty whenever the production correspondence is superadditive relative to N (i.e., $Y(N) \supset \sum_{B_j \in B} Y(B_j)$ for all $B \in \mathscr{B}$), since then the grand coalition would always form and no transfers or trades among coalitions can take place (as there is only a single coalition in the coalition structure). In fact, when $Y(\cdot)$ is superadditive relative to N we have $K(\{N\}) = L(\{N\}) = M(\{N\}) = K = L = M$ implying that $U = V = W$. Unfortunately, superadditivity does not seem to be a realistic assumption. In fact, if only for crowding phenomena, one would expect subadditivity to be the rule rather than the exception. (Note also that the standard "explanation" given for the superadditivity assumption, namely, that the grand coalition can form and decide to work in subsets, does not hold in the case where $\overline{W} \cap \overline{V} = \varnothing$, since no coalition would agree to transfer for free some of its output to other coalitions. Thus, $Y(S)$ *must* be interpreted as representing what coalition S can produce when all of its members work *together*.)

An assumption weaker than superadditivity is that of monotonicity in production,[2] namely, for $S \subset T \subset N$, $Y(S) \subset Y(T)$; i.e., additional people do not harm. The example of Section 3 can easily be modified to exhibit monotonicity by defining

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[2] I thank the referee for proposing this notion as well as suggesting that I try and prove Claim 6, below.

$$Y(\{i\}) = \{(1, 0)\},$$
$$Y(S) = \{(1, 0), (0, 1)\}, \qquad \text{for} \quad |S| \geqslant 2.$$

Moreover,

CLAIM 6. *It is possible that* $\overline{W} \cap \overline{V} = \varnothing$ *even when the production correspondence is monotonic and convex valued* (*i.e.*, $Y(S)$ *is convex for all* $S \subset N$).

Proof. Replace the utility functions in the example of Section 3 by

$$u_i(x, y) = (1.1y + 1.1)^{10+x}$$

and the production correspondence by

$$Y(\{i\}) = \{(1, 0)\},$$
$$Y(S) = \{(t, 1 - t) \mid 0 \leqslant t \leqslant 1\}, \qquad \text{for} \quad |S| \geqslant 2.$$

Let $u \in \overline{W}$, $u = u(a)$, $a \in M(B)$. Distinguish among the following three cases:

(i) $B = (\{i\}, \{j\}, \{k\})$. Then, $a^i = (1, 0)$ $i = 1, 2, 3$. Define

$$\bar{a}^1 = (1, 0.2), \bar{a}^2 = \bar{a}^3 = (0, 0.4).$$

Clearly, $\bar{a} \in L(\{1\}, \{2, 3\})$ and $u_1(\bar{a}^1) > u_1(a^1)$. Moreover, for $i = 2, 3$,

$$\frac{u_i(\bar{a}^1)}{u_i(a^i)} = \frac{(1.1 \times 1.4)^{10}}{(1.1)^{11}} = \frac{(1.4)^{10}}{1.1} > 1.$$

Thus, $u_i(\bar{a}^i) > u_i(a^i)$ $\forall\, i \in N$, implying $u \notin \overline{V}$.

(ii) $B = (\{i, j\}, \{k\})$, or, w.l.o.g., $B = (\{1, 2\}, \{3\})$. Then

$$a^1 = (0, y_1), \qquad a^2 = (0, y_2), \qquad a^3 = (1, 0), y_1 + y_2 = 1.$$

W.l.o.g., $y_1 \geqslant \frac{1}{2}$. Define

$$\bar{a}^1 = (1, y_1 - 0.02), \qquad \bar{a}^2 = (0, y_2 + 0.01), \qquad \bar{a}^3 = (0, 0.01).$$

Clearly, $\bar{a} \in L(\{1\}, \{2, 3\})$, and $u_2(\bar{a}^2) > u_2(a^2)$. Now, since $y_1 \geqslant \frac{1}{2}$,

$$\frac{u_1(\bar{a}^1)}{u_1(a^1)} = (1.1y_1 + 1.1)\left(1 - \frac{0.02}{y_1 + 1}\right)^{11} > 1.5\left(1 - \frac{0.04}{3}\right)^{11} > 1.$$

Moreover,

$$\frac{u_3(\bar{a}^3)}{u_3(a^3)} = \frac{(1.01)^{10}}{1.1} < 1.$$

Thus, again, $u_i(\bar{a}^i) > u_i(a^i) \forall i \in N$, implying, $u \notin \bar{V}$.

(iii) $B = \{N\}$. Then,

$$a^i = (\alpha_i, \beta_i) \quad \text{with} \quad \sum_{i \in N} \alpha_i + \sum_{i \in N} \beta_i = 1.$$

W.l.o.g., $\alpha_1 < 1$. Define

$$\bar{a}^1 = (1, \beta_1), \qquad \bar{a}^2 = (0, \alpha_2 + \beta_2), \qquad \bar{a}^3 = (0, \alpha_3 + \beta_3).$$

Clearly, $\bar{a} \in L(\{1\}, \{2, 3\})$, and $u_1(\bar{a}^1) > u_1(a^1)$. Define

$$F(\alpha, \beta) = (1.1\alpha + 1.1\beta + 1.1)^{10} - (1.1\beta + 1.1)^{10+\alpha}.$$

Since $F(0, \beta) = 0$, by showing that $F(\cdot, \beta)$ is an increasing function of α whenever $0 \leq \alpha, \beta \leq 1$, $F(\alpha, \beta) \geq 0$ in this range, implying that $u_i(\bar{a}^i) \geq u_i(a^i)$ for $i = 2,3$. Indeed,

$$\begin{aligned}\frac{\partial F}{\partial \alpha} &= 11(1.1\alpha + 1.1\beta + 1.1)^9 \\ &\quad - (1.1\beta + 1.1)^{10+\alpha} \ln(1.1\beta + 1.1) \\ &= (1.1\beta + 1.1)^9 \left[11 \left(1 + \frac{\alpha}{\beta + 1}\right)^9 \right. \\ &\quad \left. - (1.1\beta + 1.1)^{1+\alpha} \ln(1.1\beta + 1.1)\right] \\ &\geq 11 - (2.2)^2 \ln 2.2 > 0.\end{aligned}$$

Hence, $u \notin \bar{V}$. Q.E.D.

In view of Claim 6 and the fact that the production correspondence in the example of Section 3 is a point (rather than a subset) in R_+^l, one feels that unless extremely strong assumptions are imposed on $Y(\cdot)$ (e.g., that it be superadditive), altruism may be beneficial.

5. ALTERNATIVE INTERPRETATIONS TO THE EXAMPLE

(i) *Technological constraints.*[3] Instead of having a production correspondence $Y: \mathcal{N} \twoheadrightarrow R_+^l$, consider the case in which each individual

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[3] This interpretation was offered to me by Jacques H. Drèze.

$i \in \{1, 2, 3\}$ has one unit of input x (labor–leisure) and none of y. Thus, the "initial endowment" of each individual i is $(1, 0)$. Assume also that there is a technology set which is "common knowledge" under which every two individuals, working "full time" (i.e., each contributing his one unit of labor), can produce one unit of y. Thus, no coalition whose cardinality differs from 2 can be productive (a property which should be understood as a technological constraint). Then, using the arguments of Section 3 it can be easily shown that whenever the productive coalition (of cardinality 2) does not transfer (for free) some amount of y to the third individual, using such transfers can make each individual strictly better-off. If, moreover, we let labor be individually specific, so that i's labor cannot be consumed by individual j, $j \neq i$, then the implication of the example in Section 3 becomes that *transfers* (as opposed to trade) *are necessary for Pareto optimality*.

(ii) *Transferable output, nontransferable input.* Consider an uninhabited island with only two possible sites, East and West, with space for one and two individuals, respectively. Three individuals, each endowed with one unit of labor (x), are supposed to settle on the island. It is only in the West that the unique output (y) can be produced, according to the production function $y = \frac{1}{2}x$. Assume also that while it is costless to transfer y from West to East, it is infinitely costly for individuals to commute between the two sites (thus, the individual who resides in the East cannot work in the West). Finally, let the utility function of each individual be given by $u(x, y) = (100y + 2)^{x+2}$. Then, the example in Section 3 implies that unless some of the output produced in the West is transferred (for free!) to the East, each of the three individuals can be made better-off!

This interpretation provides the dual analog to the example in [2], where input is transferable while it is infinitely costly to transfer output across sites.

(iii) *Hedonic aspects.* Whether ruling out transfers might be inconsistent with Pareto optimality was one of the issues in [1]. In that paper, Drèze and I offered a model in which there is a single good (y) and where the utility of each individual depends on the amount of y he consumes as well as on the coalition to which he belongs. (This last argument was referred to in [1] as the hedonic aspect.) Thus, the level of utility attained by individual i who consumes y_i units of y and belongs to coalition S is given by $u_i(y_i, S)$. By means of an example we showed that it was possible that unless transfers of y (across coalitions) were made, Pareto optimality could not be attained. The example of Section 3 serves as another demonstration of such situations. Indeed, let the utility function of individual i be given by

$$u_i(y_i, S) = (100y_i + 2)^{\beta(S)+2},$$

where

$$\beta(S) = 1 \quad \text{if } |S| = 1$$
$$= 0 \quad \text{otherwise,}$$

and let the production correspondence of the unique output be given by

$$Y(S) = 1 \quad \text{if } |S| = 2$$
$$= 0 \quad \text{otherwise.}$$

From the arguments given in Section 3, it follows that unless the coalition with the two individuals transfers some of the always desired good, y, to the coalition that consists of the other individual (without receiving anything in return), Pareto optimality will not be reached.

ACKNOWLEDGMENTS

I am indebted to Jacques H. Drèze, Jean Jaskold Gabszewicz, and Benyamin Shitovitz for beneficial discussions.

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JOURNAL OF ECONOMIC THEORY 22, 23–36 (1980)

Characterization of the Private Alternatives Domains Admitting Arrow Social Welfare Functions*

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Received February 14, 1979

An n-person social choice problem is considered in which the alternatives are n dimensional vectors with the ith component of such a vector being the part of the alternative effecting individual i alone. Assuming that individuals are selfish (i is indifferent between any two alternatives that have the same ith component) we characterize all the families of permissible individual preferences that admit nondictatorial Arrow-type social welfare functions. We also show that the existence of such a function for a given family of preferences is independent of n provided that it is greater than one.

1. INTRODUCTION

A theory of aggregations of individual preferences into group preferences shold be in the core of areas involving group decision making. As such, it is surprising that social choice theory is not used more extensively in economics, game theory, team theory, etc. Clearly incorporation of some of the many models and important results of social choice theory into these other areas may have interesting implications. We feel that one of the reasons for this phenomenon is the lack of structure of the alternatives' space in most of the models discussed in social choice, while, for example in economic aggregation problems, aggregating preferences over allocations of private

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* This research was partly supported by Grants SOC76-20953 and SOC 79-07542 from the National Science Foundation.

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alternatives may differ significantly from aggregating preferences over allocations of public alternatives. One would expect that this structure would be taken into consideration. We hope that the model and results presented in this paper are a step in this direction.

Arrow [1, Chap. II] discusses the "difference between the ordering (done by an individual) of the social states according to direct consumption of the individual and the ordering when the individual adds his general standards of equity." He refers to the former ordering as reflecting his "tastes" and the latter as reflecting his "values." When one deals with a social choice theory whose states are economic consumption bundles, ordering according to tastes, or what is sometimes referred to as a selfishness axiom, becomes a dominant factor in the individual's preferences. This distinction is discussed by Samuelson, who makes this assumption [6, assumption 6, p. 224] for econimic environments. Motivated by this observation Arrow goes on to prove his "Possibility (Impossibility) Theorem for Individualistic Assumptions" (Theorem 3 in his book). The difference between this theorem and the well-known "General Possibility Theorem" is in the fact that individuals care only about their component of the social state and that they are indifferent to changes effecting individuals other than themselves. The social choice model studied in this paper is designed to deal with the aggregation of preferences of selfish individuals.

We assume that each individual i in an n-person society has a set of conceivable preferences, Ω_i, over a set of his own private alternatives A_i. Society's goal is to aggregate these preferences into a social welfare function (SWF) which should rank $A_1 \times A_2 \times \dots \times A_n$—the set of n-person allocations in this society.

We assume that the individuals in this society are symmetric in that all the A_i's and Ω_i's are identical. We provide an answer to the question of which domains of preferences (Ω_i's) admit an Arrow-type nondictatorial SWF. This is done in two stages. We first show that the answer to the question of existence of such a SWF for a given A and Ω is independent of the number of individuals in the society. Then we characterize the A's and Ω's that admit a two-person nondictatorial SWF.

This work parallels that of Maskin [5] and Kalai and Muller [4]. They studied the same questions for the case of public alternatives. The results that we obtain are different than theirs and one model is not a special case of the other (because of the symmetry assumption made by them). In particular we show, as an example of our characterization, that for a large class of single-peaked preferences over private alternatives every SWF is dictatorial. Thus "good" restricted domains of preferences over public alternatives may be "bad" for private alternatives. On the other hand the case of only one conceivable preference ordering ($|\Omega| = 1$) admits nondictatorial SWF's for private alternatives but for public alternatives it must be dictatorial (every individual must be a dictator by the unanimity assumption).

The model presented here has two drawbacks. One is that we assume that individuals have strong preference (indifference is not allowed). This is done for technical reasons, since we cannot solve the more general case. Second, we assume that all A_i's and all Ω_i's are the same. This again is done for technical reasons yet it is not very restrictive. Ω_i is the set of all conceivable preferences of individual i. It is an input into the model and has to be determined by means of some other theory or empirical observations. For example, in economic environments Ω_i may consist of the monotonic convex preferences. It is likely that if these restrictions hold for one individual then they should hold for the others as well. The choice of the A_i's is not critical since we will require our SWF to satisfy a condition of independence of irrelevant alternatives. Thus if the A_i's are too large and contain some nonfeasible alternatives, that will not affect the choice among the feasible ones.

Let A denote a set of alternatives with at least two elements. For an integer $n \geq 2$ let A^n represent the set of all n-tuples of alternatives from A. An element of A^n, $X = (x_1, \ldots, x_n)$, is called an *n-person allocation* of alternatives. Let Σ denote the set of all transitive antisymmetric total (i.e., if $p \in \Sigma$ and $x, y \in A$ then xpy or ypx or $x = y$) binary relations on A. An element of Σ is called a *preference relation*. For $n \geq 2$ let $\Delta^{(n)}$ represent the set of all transitive antisymmetric total binary relations on A^n. Let Ω be a nonempty subset of Σ; the elements of Ω represent the admissible preference relations in the society. For $n \geq 2$, Ω^n represents the set of all n-tuples of preferences from Ω and $P = (p_1, \ldots, p_n)$, an element of Ω^n, is called an *n-person profile*.

An *n-person social welfare function* (SWF) on Ω is a function $f^n \colon \Omega^n \to \Delta^{(n)}$ which satisfies the following two conditions.

1. *Unanimity*. For every $P \in \Omega^n$ and $X, Y \in A^n$, if for $i = 1, \ldots, n$, whenever $x_i \neq y_i$ then $x_i p_i y_i$, then $xf^n(P)\,Y$.

2. *Independence of irrelevant alternatives* (IIA). For $X, Y \in A^n$, and $P, Q \in \Omega^n$ if [$x_i p_i y_i$ iff $x_i q_i y_i$ for $i = 1, \ldots, n$] then [$Xf^n(P)\,Y$ iff $Xf^n(Q)\,Y$].

f^n is *dictatorial* if there exists j, $1 \leq j \leq n$, such that for every $P \in \Omega^n$ and $X, Y \in A^n$, $Xf^n(P)\,Y$ whenever $x_j p_j y_j$. f^n is *nondictatorial* if f^n is not dictatorial.

An n-person *Arrow Social Welfare Function* (ASWF) is a nondictatorial SWF.

Remark. Campbell and Fisburn [3] investigated social welfare functions which are very symmetric in a social choice model that allowed them to distinguish between private and public alternatives. However, their formulation, and the problems they study are very different from ours.

2. INDEPENDENCE OF n

THEOREM 1. For $n \geq 2$

(a) *if there exists a two-person Arrow SWF on Ω then there exists an n-person Arrow SWF on Ω, and*

(b) *if $|A| > 2$ and there exists an n-person Arrow SWF on Ω then there exists a two-person Arrow SWF on Ω.*

COROLLARY 1. *If $|A| > 2$ then for every $n > 2$, Ω admits an n-person Arrow SWF if and only if it admits a two-person Arrow SWF.*

Proof. Part (a). We assume that there exists f^m, an m-person ($m \geq 2$) Arrow SWF on Ω. We construct an $m + 1$ ASWF on Ω, f^{m+1}, as follows. For $X, Y \in A^{m+1}$ and $P \in \Omega^{m+1}$, $Xf^{m+1}(P)\,Y$ if and only if either $(x_1, \dots, x_m) = (y_1, \dots, y_m)$ and $x_{m+1}\, p_{m+1}\, y_{m+1}$ or $(x_1, \dots, x_m) \neq (y_1, \dots, y_m)$ and $(x_1, \dots, x_m) f^m((p_1, \dots, p_m))(y_1, \dots, y_m)$.

It is straightforward to check that f^{m+1} is a well-defined ASWF. Thus, by induction, if there is a two-person ASWF on Ω then there is an n-person one for every $n \geq 2$.

Part (b). Now we assume that $|A| > 2$ and for some $n > 2$ there exists an n-person ASWF on Ω. We have to show that there exists a two-person ASWF on Ω. We first prove it for the following special case.

Ω is said to *contain an inseparable pair* of alternatives if there exists $s, t \in A$ such that

(i). for some $p \in \Omega$, spt and

(ii). for no $p \in \Omega$ and $x \in A$ it is $spxpt$.

(s, t) is called an *inseparable pair* of alternatives.

LEMMA 1. *If Ω contains an inseparable pair of alternatives (s, t), then there exists a two-person ASWF on Ω.*

Proof. For every $X, Y \in A^2$ $(X \neq Y)$ and $P \in \Omega^2$ we define $f^2(P)$ as follows. If $x_1 = y_1$ or $x_2 = y_2$ then f^2 is defined by the unanimity rule according to the preferences of the nonindifferent voter. When $x_1 \neq y_1$ and $x_2 \neq y_2$ we distinguish the special case $\{x_1, y_1\} = \{s, t\}$ and sp_1t. In this case $f^2(P)$ on the pair X, Y is defined to agree with p_2. In all other cases we define $f^2(P)$ on the pair X, Y to coincide with the preferences of p_1. It is easy to check that f^2 is an antisymmetric total relation. It satisfies IIA, nondictatorship, and unanimity by the way it was defined and it remains to be shown that it satisfies transitivity.

Let X, Y, Z be three distinct elements of A^2 and $P \in \Omega^2$ satisfying $Xf^2(P)\, Yf^2(P)\, Z$. We will show $Xf^2(P)\, Z$ by showing it for all the possible cases.

(i) tp_1s (in case there is such a p_1 in Ω).

1. $x_1 \neq y_1 \neq z_1$. Then $Xf^2(P)\, Yf^2(P)\, Z$ implies $x_1 p_1 y_1 p_1 z_1$. By transitivity it is $x_1 p_1 z_1$, which implies, by definition, $Xf^2(P)\, Z$.

2. $x_1 = y_1 \neq z_1$. Then $Yf^2(P)\, Z$ implies $y_1 p_1 z_1$, which is the same as $x_1 p_1 z_1$, and again by definition it implies $Xf^2(P)\, Z$.

3. $x_1 \neq y_1 = z_1$. Then $Xf^2(P)\, Y$ implies $x_1 p_1 y_1$, which is the same as $x_1 p_1 z_1$, and by definition it is $Xf^2(P)\, Z$.

4. $x_1 = y_1 = z_1$. Then $Xf^2(P)\, Yf^2(P)\, Z$ implies $x_2 p_2 y_2 p_2 z_2$. By transitivity it is $x_2 p_2 z_2$, and by definition it is $Xf^2(P)\, Z$. (Note that the case $x_1 = z_1 \neq y_1$) is impossible.)

(ii) sp_1t.

1. At least two of the elements x_1, y_1, z_1 do not belong to $\{s, t\}$. Then we are in the same situation as in case (i) and therefore it is $Xf^2(P)\, Z$.

2. $x_1, y_1 \in \{s, t\}$, $z_1 \notin \{s, t\}$. Then $Yf^2(P)\, Z$ implies $y_1 p_1 z_1$, which implies, by the inseparability condition, both sp_1z_1 and tp_1z. Therefore it is $x_1 p_1 z_1$ and by definition it is $Xf^2(P)\, Z$.

3. $x_1 \notin \{s, t\}$, $y_1, z_1 \in \{s, t\}$. Then $Xf^2(P)\, Y$ implies $x_1 y_1 p_1$, which implies both $x_1 p_1 s$ and $x_1 p_1 t$ and therefore it is $x_1 p_1 z_1$. Then by definition it is $Xf^2(P)\, Z$.

4. $x_1, z_1 \in \{s, t\}$, $y_1 \notin \{s, t\}$. Because of the inseparability condition, this case is impossible.

5. $x_1, y_1, z_1 \in \{s, t\}$. If $x_2 \neq y_2 \neq z_2$ then $Xf^2(P)\, Yf^2(P)\, Z$ implies $x_2 p_2 y_2 p_2 z_2$. By transitivity it is $x_2 p_2 z_2$, and by definition this implies $Xf^2(P)\, Z$. If $x_2 \neq y_2 = z_2$ then $Xf^2(P)\, Y$ implies $x_2 p_2 y_2$, which is the same as $x_2 p_2 z_2$, and again by definition it is $Xf^2(P)\, Z$. If $x_2 = y_2 \neq z_2$ then $Yf^2(P)\, Z$ implies $y_2 p_2 z_2$, or $x_2 p_2 z_2$ and again it is $Xf^2(P)\, Z$. The case $x_2 = y_2 = z_2$ is impossible, since then $Xf^2(P)\, Y$ implies $x_1 = s$, $y_1 = t$, while $Yf^2(P)\, Z$ implies $y_1 = s$, $z_1 = t$, which is a contradiction. Q.E.D.

Continuing with the proof of Theorem 1 we now assume that Ω does not contain any inseparable pair. We also need to introduce some additional definitions and prove the following lemmas. For an integer $n \geq 2$, $z \in A$ and j, $1 \leq j \leq n$, let $A_j^n(z)$ denote the subset of A^n, $A_j^n(z) = \{X \in A^n \mid x_j = z\}$. f^n, a SWF on Ω, is called a *z-dictatorial by j* function if there exists a member k ($1 \leq k \leq n, k \neq j$) such that for every X, $Y \in A_j^n(z)$ and $P \in \Omega^n$, if $x_k p_k y_k$ then it is $Xf^n(P)\, Y$. k is called the *z-dictator by j* and is denoted by $d_j(z)$. Note that if $d_j(z)$ exists then it must be unique. Let $D_j = \{x \in A \mid$ there exists a $d_j(x)\}$.

LEMMA 2. *Let f^n, for $n \geq 4$, be a SWF on Ω. If for every j $(1 \leq j \leq n)$, $D_j \neq \emptyset$, then there is a member k $(1 \leq k \leq n)$, such that for every $j \neq k$ and every $z \in D_j$, $k = d_j(z)$.*

Proof. Suppose that for some $x \in D_1$, $2 = d_1(x)$ and for some $y \in D_3$, $4 = d_3(y)$. Let $z, w \in A$ and $p \in \Omega$ be such that zpw. Let $P = (p, \ldots, p) \in \Omega^n$ $T = (x, z, y, w, \ldots, w) \in A^n$, and $S = (x, w, y, z, \ldots, z) \in A^n$. Since $T, S \in A_1^n(x)$ and $2 = d_1(x)$ therefore it is $Tf^n(P)\,S$, but since $T, S \in A_3^n(y)$ and $4 = d_3(y)$ it is also $Sf^n(P)\,T$, which is a contradiction. Actually this proves that for every $j \notin \{1, 2\}$ and every $y \in D_j$, $d_j(y) \in \{1, 2\}$. Moreover, by applying the above method, it is easy to show that for every $j, k \notin \{1, 2\}$, $y \in D_j$, and $z \in D_k$ it is $d_j(y) = d_k(z)$. Hence, if for every $j \notin \{1, 2\}$ and $y \in D_j$ it is $2 = d_j(y)$, then by applying again the above technique, $2 = d_1(x)$ for every $x \in D_1$. On the other hand, if for every $j \notin \{1, 2\}$ and $y \in D_j$ it is $1 = d_j(y)$, then it also has to be $1 = d_2(x)$ for every $x \in D_2$, which completes the proof of the lemma. Q.E.D.

In the following lemmas we extend the result of Lemma 2 to the case of $n = 3$. Note that although the following lemmas are phrased for a certain order of the members 1, 2, and 3, it is obviously true for any permutation of these members.

LEMMA 3. *Let f^3 be a SWF on Ω. If $D_j \neq \emptyset$, $j = 1, 2, 3$, and if there exist distinct $x, y \in D_1$ such that $2 = d_1(x) = d_1(y)$ and distinct $z, w \in D_2$ such that $3 = d_2(z) = d_2(w)$ then $2 = d_3(t)$ for every $t \in D_3$.*

Proof. Suppose it is not the case and there exists $t \in D_3$ such that $1 = d_3(t)$. Without loss of generality let $P \in \Omega^3$ be such that xp_1y, zp_2w and suppose that for some $s \in A$, tp_3s. Since $(x, z, s), (x, w, t) \in A_1^3(x)$, $2 = d_1(x)$, and zp_2w, it is $(x, z, s)\,f^3(P)(x, w, t)$: since $(x, w, t), (y, z, t) \in A_3^3(t)$, $1 = f_3(t)$, and xp_1y, it is $(x, w, t)\,f^3(P)(y, z, t)$; since $(y, z, t), (x, z, s) \in A_2^3(z)$, $3 = d_2(z)$, and tp_3s, it is $(y, z, t)\,f^3(P)(x, z, s)$; hence, since f^3 is a SWF, by transitivity it is $(x, z, s)\,f^3(P)(x, z, s)$, which is a contradiction. Now suppose Ω is such that for every $p \in \Omega$ and every $s \in A$ it is spt; then, again without loss of generality, suppose $P \in \Omega^3$ is such that xp_1y, zp_2w, and sp_3t for some $s \in A$. Then $(x, w, t), (y, z, t) \in A_3^3(t)$, $1 = d_3(t)$, and xp_1y imply $(x, w, t)f^3(P)(y, z, t)$; $(y, z, t), (y, w, s) \in A_1^3(y)$, $2 = d_1(y)$, and zp_2w imply $(y, z, t)\,f^3(P)(y, w, s)$; $(y, w, s), (x, w, t) \in A_2^3(w)$, $3 = d_2(w)$, and sp_3t imply $(y, w, s)\,f^3(P)(x, w, t)$ and since by transitivity it is $(x, w, t)\,f^3(P)(x, w, t)$ it is again a contradiction, which completes the proof. Q.E.D.

LEMMA 4. *Let f^3 be a SWF on Ω. If $D_j \neq \emptyset$ for $j = 1, 2, 3$ and there exist distinct $r, t \in D_3$ such that $2 = d_3(r) = d_3(t)$ and $x \in D_1$ such that $2 = d_1(x)$ then $2 = d_1(y)$ for every $y \in D_1$.*

Proof. Suppose it is not the case and there exists $y \in D_1$ such that $3 = d_1(y)$. Suppose that for some $p \in \Omega$, xpy, then we can let $a, b, c \in A$, and $P \in \Omega^3$ be such that without loss of generality xp_1y, ap_2bp_2c, and rp_3t. Then $(y, a, t), (x, b, t) \in A_3^3(t)$, $2 = d_3(t)$, and ap_2b imply $(y, a, t) f^3(P)(x, b, t)$; $(x, b, t), (x, c, r) \in A_1^3(x)$, $2 = d_1(x)$, and bp_2c imply $(x, b, t) f^3(P)(x, c, r)$. By unanimity it is $(x, c, r) f^3(P)(y, c, r)$ and $(y, c, r), (y, a, t) \in A_1^3(y)$, $3 = d_1(y)$, and rp_3t imply $(y, c, r) f^3(P)(y, a, t)$. Then by transitivity it is $(y, a, t) f^3(P)(y, a, t)$, which is a contradiction. If Ω is such that for every $p \in \Omega$ it is ypx, then let $P \in \Omega^3$ be such that yp_1x, ap_2bp_2c, and rp_3t. Then by unanimity it is $(y, a, t) f^3(P)(x, a, t)$; $(x, a, t), (x, b, r) \in A_1^3(x)$, $2 = d_1(x)$, and ap_2b imply $(x, a, t) f^3(P)(x, b, r)$; $(x, b, r), (y, c, r) \in A_3^3(r)$, $2 = d_3(r)$ and bp_2c imply $(x, b, r) f^3(P)(y, c, r)$; $(y, c, r), (y, a, t) \in A_1^3(y)$, $3 = d_1(y)$, and rp_3t imply $(y, c, r) f^3(P)(y, a, t)$ and again by transitivity it is $(y, a, t) f^3(P)(y, a, t)$, which is a contradiction, and this completes the proof.

Q.E.D.

The following lemma, by using Lemmas 3 and 4, extends the result of Lemma 2 to the case of $n = 3$.

LEMMA 5. *Let f^3 be a SWF on Ω. If for $j = 1, 2, 3$, $D_j = A$ then there is a member k, $1 \leq k \leq 3$, such that for every $j \neq k$ and every $x \in D_j$, $k = d_j(x)$.*

Proof. Since $|A| \geq 3$ and since $D_j = A$ then obviously there is a member i ($i \neq 1$) such that for at least two distinct alternatives, say x, y, $i = d_1(x) = d_1(y)$. Let us assume without loss of generality that this i is 2. By the same reasoning there is a member k ($k \neq 3$) for which there are at least two alternatives, say z, w, such that $k = d_3(z) = d_3(w)$. Hence there are two possible cases.

Case 1. $k = 2$. Then by Lemma 4, for every $z \in D_1$ and every $t \in D_3$, $2 = d_1(z) = d_3(t)$ and since $A = D_1 = D_3$, then for every $x \in A$ it is $2 = d_1(x)$ and $2 = d_3(x)$.

Case 2. $k = 1$. Then by Lemma 3, for every $x \in D_2$ it is $1 = d_2(x)$, which implies, by Lemma 4, that for every $x \in D_3$ it is $1 = d_3(x)$, which completes the proof.

Q.E.D.

All of the above lemmas lead to the following result.

LEMMA 6. *If f^n, for $n \geq 3$, is a SWF on Ω such that for every member j ($1 \leq j \leq n$), $D_j = A$, then f^n is dictatorial on Ω.*

Proof. Suppose $D_j = A$ for $j = 1, 2, \ldots, n$. Assume without loss of generality that 1 is the k of Lemmas 2 and 5. Suppose $X, Y \in A^n$ and $P \in \Omega^n$ with $x_1 p_1 y_1$. Since we assume that Ω does not contain an inseparable pair we can assume by IIA that for some $z \in A$, $x_1 p_1 z p_1 y_1$. Let $Z := (z, x_2, \ldots, x_{n-1}, y_n)$; then we have $X f^n(P) Z$ by $d_2(x_2) = 1$ and $Z f^n(P) T$ by $d_n(y_n) = 1$. Thus $X f^n(P) Y$ by transitivity. Q.E.D.

We can now complete the proof of Theorem 1 by showing that for $n \geq 3$ if there is an n-person ASWF on Ω, f^n, then there exists an $n - 1$ person ASWF on Ω, f^{n-1}. Since f^n is assumed to be nondictatorial, Lemma 6 implies that there is a person i and an alternative x for which $d_i(x)$ does not exist. Assume without loss of generality that $i = n$. Let $p \in \Omega$ be any fixed preference for n and for $X, Y \in A^{n-1}$ define $X f^{n-1}(P) Y$ if and only if $(x_1, x_2, \ldots, x_{n-1}, x) f^n((p_1, p_2, \ldots, p_{n-1}, p))(y_1, y_2, \ldots, y_{n-1}, x)$. It is easy to check that f^{n-1} defined this way is an ASWF on Ω^{n-1}. Q.E.D.

3. THE CHARACTERIZATION OF THE NONDICTATORIAL DOMAINS

In this section we characterize the domains of preferences Ω that admit two-person Arrow SWFs. By Theorem 1 this is also a characterization of the domains that admit n-person (for every $n \geq 2$) nondictatorial SWFs, provided that the number of alternatives is greater than 2 ($|A| > 2$). Throughout this section we let Ω be an arbitrary fixed nonempty subset of Σ.

A set $R \subset A^2 \times A^2$ is said to be *closed under decisiveness implications if the following two conditions hold for every* $X, Y, Z \in A^2$.

DI1. *If for some* $p \in \Omega$, $x_1 p y_1 p z_1$, $(X, Y) \in R$, *and* $(Y, Z) \in R$, *then* $(X, Z) \in R$.

DI2. If for some $p \in \Omega$, $x_2 p y_2 p z_2$ and $(Z, X) \in R$ then $(Z, Y) \in R$ or $(Y, X) \in R$.

We let $F = \{(X, Y) \in A^2 \times A^2 \mid \text{for some } p \in \Omega, x_1 p y_1\}$. Thus these are the pairs (X, Y) for which it is feasible for voter 1 to prefer X to Y. We let $C = \{(x, y) \in F \mid \text{for some } p \in \Omega, y_2 p x_2\}$. Thus these are the pairs (X, Y) which are feasible for voter 1, but also voter 2 can oppose voter 1 and prefer Y to X.

We say that Ω *has a decomposition* if there is an R which is closed under decisveness implications. We say that Ω has *a nontrivial decomposition* if it decomposes with an R such that

$$(F - C) \subsetneq R \subsetneq F.$$

The intuitive explanation for the above definition is as follows. Suppose there is a two-person ASWF f on Ω. We let R consist of the pairs (X, Y) which are feasible to voter 1 ($x_1 p y_1$, for some $p \in \Omega$) and for which he is also decisive (if he prefers X to Y then so will f). Clearly $F - C \subset R \subset F$. The nondictatorship of voter 1 implies that $R \neq F$ and the nondictatorship of voter 2 implies that $F - C \neq R$. The closure under decisiveness implications of R follows from the transitivity of f and turns out to be equivalent to it. Hence we obtain

THEOREM 2. *Ω admits a two-person (n person if $|A| > 2$) Arrow SWF if and only if Ω has a nontrivial decomposition.*

Proof. Suppose Ω admits a two-person ASWF f. Let $R = \{(X, Y) \in F \mid$ for every $P \in \Omega^2$ if $x_1 p_1 y_1$ then $Xf(P)Y\}$. Clearly $(F - C) \subset R \subset F$. Since 2 is not a dictator there is a pair (X, Y) and a $P \in \Omega^2$ with $x_2 p_2 y_2$ and $Yf(P)X$. Therefore $y_1 p_1 x_1$ (since f satisfies unanimity), and by IIA and unanimity $(Y, X) \in R$ yet $(Y, X) \in C$. Hence $(F - C) \subsetneq R$. Since 1 is not a dictator there is a pair (X, Y) and a $P \in \Omega^2$ with $x_1 p_1 y_1$ but $Yf(P)X$. Hence $(X, Y) \in F$ and $(X, Y) \notin R$. So $R \subsetneq F$. To show that R is closed under decisiveness implications we assume that $X, Y, Z \in A^2$. Suppose that for some $p \in \Omega$, $x_1 p y_1 p z_1$, $(X, Y) \in R$, and $(Y, Z) \in R$. Clearly $(X, Z) \in F$. Let P be any profile with $x_1 p_1 z_1$. By IIA we can assume without loss of generality that $x_1 p_1 y_1 p_1 z_1$ without effecting the (X, Z) outcome. Since $(X, Y) \in R$ and $(Y, Z) \in R$ we have $Xf(P)Yf(P)Z$ and by transitivity $Xf(P)Z$. Thus $(X, Z) \in R$ and R is closed under the first type of decisiveness implications. Now we assume that for some $p \in \Omega$, $x_2 p y_2 p z_2$ and $(Z, X) \in R$. Assume that $(Z, Y) \notin R$ and $(Y, X) \notin R$. Since $(Z, X) \in R$ there is a $p_1 \in \Omega$ with $z_1 p_1 x_1$. Choose p_2 to satisfy $x_2 p_2 y_2 p_2 z_2$ and let $P = (p_1, p_2)$. Then $Zf(P)X$ since $(Z, X) \in \Omega$. If it is not the case that $z_1 p_1 y_1$ then $Yf(P)Zf(P)X$ and $(Y, X) \in R$, a contradiction. So we assume that $z_1 p_1 y_1$. If $Zf(P)Y$ then $(Z, Y) \in R$, a contradiction. Hence $Yf(P)Z$. By transitivity $Yf(P)X$; hence $(Y, X) \in R$, which is a contradiction. So we have that $(Z, Y) \in R$ or $(Y, X) \in R$, and R is closed under decisiveness implications of the second type.

We now assume that R presents a nontrivial decomposition for Ω and demonstrate the existence of an Arrow SWF f. For every $(X, Y) \in A^2 \times A^2$ and every $P \in \Omega^2$ define $Xf(P)Y$ if

(a) $x_i p_i y_i$ for $i = 1, 2$

(b) $x_1 p_1 y_1$ and $(X, Y) \in R$, or

(c) $x_2 p_2 y_2$ and $(Y, X) \notin R$.

We first show that if $X, Y \in A^2$, $X \neq Y$, then $Xf(P)Y$ or $Yf(P)X$ but not both. If neither $Xf(P)Y$ nor $Yf(P)X$ then we have either (i) $x_1 p_1 y_1$ and

$y_2 p_2 x_2$ or (ii) $y_1 p_1 x_1$ and $x_2 p_2 y_2$. If (i) holds yet not $Xf(P)Y$ then $(X, Y) \notin R$ so $Yf(P)X$ by part (c) of the definition of $f(P)$, a contradiction. If (ii) holds yet not $Yf(P)X$ then $(Y, X) \notin R$ but then we should have $Xf(P)Y$, a contradiction. If both $Xf(P)Y$ and $Yf(P)X$ then again there is no unanimity and one of the situations (i) or (ii) described above holds. Situation (i) implies that $(X, Y) \in R$ and $(X, Y) \notin R$, a contradiction. Situation (ii) implies that $(Y, X) \notin R$ and $(Y, X) \in R$, a contradiction again.

It is clear that f satisfies IIA and unanimity. It remains to be shown that f is transitive and nondictatorial.

To show that voter 1 is not a dictator choose $(X, Y) \in F - R$. Since $F - C \subset R$, then $(X, Y) \in C$. Thus for some $p_2 \in \Omega$, $y_2 p_2 x_2$. Since $(X, Y) \in F$, for some $p_1 \in \Omega$, $x_1 p_1 y_1$. Let $P = (p_1, p_2)$; then $Yf(P)X$, so 1 is not a dictator. To see that 2 is not a dictator choose $(X, Y) \in R - (F - C)$. For some $p_1 \in \Omega$, $x_1 p_1 y_1$ and for some $p_2 \in \Omega$, $y_2 p_2 x_2$. Let $P = (p_1, p_2)$; then $Xf(P)Y$, so 2 is not a dictator.

To show transitivity we assume to the contrary that for some $X, Y, Z \in A^2$ and $P \in \Omega^2$, $Xf(P)\,Yf(P)\,Zf(P)\,X$.

Case 1. $z_1 p_1 x_1$ and $z_2 p_2 x_2$

Subcase (a) $y_1 p_1 x_1$. Since $Xf(P)Y$ then $x_2 p_2 y_2$ and $(Y, X) \notin R$. Since $Yf(P)Z$ then $y_1 p_1 z_1$ and $(Y, Z) \in R$. Since $(Z, (x_1, z_2)) \in R$, then by decisiveness implications of the first type, DI1, $(Y, (x_1, z_2)) \in R$. By DI2 (with X being (x_1, z_2), Y being X and Z being Y) either $(Y, X) \in R$ or $(X, (x_1, z_2)) \in R$, a contradiction.

Subcase (b) not $y_1 p_1 x_1$. Therefore $y_2 p_2 z_2$, which implies $x_1 p_1 y_1$. Hence $(X, Y) \in R$ and $(Z, Y) \notin R$. Since $((z_1, x_2), Y) \in R$. Now by DI2 we get that either $((z_1, x_2), Z) \in R$ or $(Z, Y) \in R$, a contradiction.

Case 2. Not $z_2 p_2 x_2$

Therefore $z_1 p_1 x_1$ and $(Z, X) \in R$. If not $y_2 p_2 z_2$ then $y_1 p_1 z_1$ and $(Y, Z) \in R$, which implies by DI1 that $(Y, X) \in R$, which contradicts $Xf(P)Y$. Therefore $y_2 p_2 z_2$. If not $x_2 p_2 y_2$ then $x_1 p_1 y_1$ and $(X, Y) \in R$, which implies by DI1 that $(Z, Y) \in R$, which contadicts $Yf(P)Z$ since $z_1 p_1 y_1$. So we are left with the situation that $x_2 p_2 y_2 p_2 z_2$ and $z_1 p_1 x_1$. If not $y_1 p_1 z_1$ and not $x_1 p_1 y_1$ then $(Z, Y) \notin R$ and $(Y, X) \notin R$. But then DI2 contradicts the fact that $(Z, X) \in R$. So we must have $x_2 p_2 y_2 p_2 z_2$ and either $y_1 p_1 z_1 p_1 x_1$ or $z_1 p_1 x_1 p_1 y_1$. If $y_1 p_1 z_1 p_1 x_1$ then $(Y, X) \notin R$. But by DI2 $((z_1, y_2), X) \in R$ and by DI1 (with X being Y, Y being (z_1, y_2), and Z being X) $(Y, X) \in R$, a contradiction. So we must have $x_2 p_2 y_2 p_2 z_2$ and $z_1 p_1 x_1 p_1 y_1$. But now DI2 implies that $(Z, (x_1, y_2)) \in R$ and DI1 (with X being Z, Y being (x_1, y_2), and Z being (y_1, y_2)) implies that $(Z, Y) \in R$. This contradicts $Yf(P)Z$, which shows that case 2 is impossible.

Case 3. Not $z_1 p_1 x_1$

Therefore $z_2 p_2 x_2$ and $(X, Z) \notin R$. If not $x_1 p_1 y_1$ then $x_2 p_2 y_2$ and $(Y, X) \notin R$. By condition DI2 (with X being Z, Y being X, and Z being Y), $(Y, Z) \notin R$. Therefore $Zf(P)\, Y$, a contradiction. So $x_1 p_1 y_1$. If not $y_1 p_1 z_1$ then $y_2 p_2 z_2$ and $(Z, Y) \notin R$. By condition DI2 (with X being Y, Y being Z, and Z being X) we have that $(X, Y) \notin R$, which contradicts $Xf(P)\, Y$. So we must have $x_1 p_1 y_1 p_1 z_1$. If not $y_2 p_2 z_2$ and not $x_2 p_2 y_2$ then $(Y, Z) \in R$ and $(X, Y) \in R$, which contradict by DI1 the fact that $(X, Z) \notin R$. So we have either $y_2 p_2 z_2 p_2 x_2$ or $z_2 p_2 x_2 p_2 y_2$. If $y_2 p_2 z_2 p_2 x_2$ then $(X, Y) \in R$. By DI1 $(X, (z_1, y_2)) \in R$, by DI2 (with X being (z_1, y_2) Y being (z_1, z_2), and Z being X), $(X, Z) \in R$, a contradiction. So we must have $x_1 p_1 y_1 p_1 z_1$ and $z_2 p_2 x_2 p_2 y_2$. Since $Yf(P)\, Z$, $(Y, Z) \in R$. By DI1 $((x_1, y_2), Z) \in R$, and by DI2 (with X being Z, Y being (x_1, x_2), and Z being (x_1, y_2)), $(X, Z) \in R$, a contradiction.

Q.E.D.

We summarize the question of existence in

THEOREM 3. *For* $n \geq 2$

(a) *If* $|A| > 2$ *then there exists an n-person Arrow SWF if and only if* Ω *has a nontrivial decomposition.*

(b) *If* $|A| = 2$, *say* $A = \{x, y\}$, *then*

(i) *if* $\Omega = \Sigma$ *then there exists an n-person Arrow SWF for* Ω, *and*

(ii) *if* $\Omega \neq \Sigma$ *then there exists an n-person ASWF for* Ω *if and only if* $n \geq 3$.

Proof. Part (a) is an immediate consequence of Theorem 1 and 2. For part (b)(i) for any pair of allocations $X, Y \in A^2$ with $X \neq Y$ we define $f(P)$ to be determined by the unanimity rule if it is applicable. Otherwise we define $f(P)$ to coincide with p_1 if $xp_1 y$ and to coincide with p_2 if $yp_1 x$. It is straightforward to check that this defines a two-person Arrow SWF on A. The extension to n-person follows by Theorem 1. To show part (ii) assume w.l.o.g. that $\Omega = \{p\}$ with xpy. In this case the only possible profile is $p = (p, p, \ldots, p)$. For $n = 2$ if f is a SWF then $(x, y) f(P) (y, x)$ or $(y, x) f(P) (x, y)$. In either case either voter 1 is a dictator or voter 2 is a dictator. For $n = 3$ we define $\succ = f(P)$ by $(x, x, x) \succ (x, x, y) \succ (x, y, x) \succ (y, x, x) \succ (y, y, x) \succ (y, x, y) \succ (x, y, y) \succ (y, y, y)$. It is easy to check that this is a well-defined three-person ASWF. By Theorem 1 part (a), (ii) holds true for every $n \geq 3$.

Q.E.D.

4. APPLICATIONS

To show the usefulness of Theorems 1, 2, and 3, we discuss some examples.

EXAMPLE 1. *Arrow impossibility theorem for private alternatives.*

If $|A| > 2$ and $\Omega = \Sigma$ then Ω is dictatorial.

Proof. Suppose Ω is nondictatorial: then Ω has a nontrivial decomposition $F - C \subsetneqq R \subsetneqq F$, which means that there is $(X, Y) \in C$ such that $(X, Y) \in R$. Since $\Omega = \Sigma$ then $F = \{(X, Y) \in A^2 \times A^2 \mid x_1 \neq y_1\}$ and $C = \{(X, Y) \in F \mid x_2 \neq y_2\}$. Therefore by DI1 and DI2, $(X, Y) \in R$ implies that for every $(X, Z) \in C$ also $(X, Z) \in R$. But again by DI1 and DI2 this implies that for every $(W, Z) \in C$ it is also $(W, Z) \in R$, or that $R = F$, a contradiction.

Q.E.D.

EXAMPLE 2. *Ω contains an inseparable pair of alternatives (s, t).*

We define the set R which corresponds to f^2, the Arrow SWF defined in Lemma 3, as follows. $R = (F - C) \cup \{(X, Y) \in F \mid$ such that $x_1 = s$ and $y_1 = t\}$, the proof that Ω has a nontrivial decomposition is straightforward.

EXAMPLE 3. *Single-peaked preferences.*

The set of single-peaked preferences is one of the most celebrated examples of domains of public alternatives, admitting Arrow SWFs independently of the size of the set of alternatives (see Black [2] and the other standard texts). Therefore the following result is somewhat of a surprise.

Let $q \in \Sigma$, and define the *set of single-peaked preferences relative to the linear order q* by

$$\Omega_q = \{p \in \Sigma : \text{for every three distinct alternatives } x, y, z \text{ if } xqyqz \text{ then it is not the case that } xpy \text{ and } zpy\}.$$

That Ω_q admits an ASWF when A, the set of alternatives, is finite is immediate because of Example 2. Even when A is infinite but has two alternatives $x, y \in A$ such that for every $z \in A$ it is the case of $xqyqz$ (or for every z it is $zqyqx$), is immediate because of Example 2. However, in general, the single-peak condition does not imply the existence of a SWF for private alternatives.

THEOREM 4. *If A is a (infinite) set of alternatives and the linear order q is such that there is no maximal alternative x (for any other alternative z, xqz) and there is no minimal alternative y (for any other alternative w, wqy) then Ω_q admits only dictatorial SWFs.*

Proof. Suppose Ω_q admits an Arrow SWF; then by Theorem 2 it has a nontrivial decomposition with an R such that $F - C \subsetneqq R \subsetneqq F$. Suppose $((a, b), (c, d)) \in R \cap C$; then there are four possible cases. That (aqc and bqd) or (aqc and dqb) or (cqa and dqb) or that (cqa and dqb). Since the proofs for the different cases are almost identical, we assume without loss of generality that it is the case of (aqc and bqd). In all of the following steps we use extensively the assumption of no minimal or maximal alternatives in A. Note that $F = \{(X, Y) \in A^2 \times A^2 \mid x_1 \neq y_1\}$ and $C = \{(X, Y) \in F \mid x_2 = y_2\}$; therefore if for some $p \in \Omega_q x_1 p y_1 p z_1$ then by DI1

(1a) $(X, Y) \in R$ implies $((X, (z_1, y_2)) \in R$, and

(1b) $(Y, Z) \in R$ implies $((x_1, y_2), Z) \in R$;

and if for some $p \in \Omega_q$, $x_2 p y_2 p z_2$ then by DI2

(2a) $(Z, X) \in R$ implies $(Z, (x_1, y_2)) \in R$, and

(2b) $(Z, X) \in R$ implies $((z_1, y_2), X) \in R$.

Step 1. Let $y \in A$ such cqy; then there exist $p, p' \in \Omega_q$ such that $apcpy$ and $cp'ap'y$. Therefore by (1a), $((a, b), (c, d)) \in R$ implies $((a, b), (y, d)) \in R$, which in turn, by (1b), implies $((c, b), (y, d)) \in R$.

Step 2. Let $y, z \in A$ such that cqy and zqy ($z \neq c$). Then there exists $p \in \Omega_q$ such that $zpcpy$ and by (1b), $((c, b), (y, d)) \in R$ implies $((z, b), (y, d)) \in R$.

Step 3. Let $z, y \in A$ such that cqy and zqy (z may also be c). Since for every $v \in A$ such that vqz or yqv, there exists $p \in \Omega_q$ such that $zpypv$, then by (1a), $((z, b), (y, d)) \in R$ implies $((z, b), (v, d)) \in R$.

Step 4. Let $z, s \in A$ such that $zqsqc$. There exist $v \in A$ and $p \in \Omega_q$ such that vqz and $zpvps$; then, by (1a), $((z, b), (v, d)) \in R$ implies $((z, b), (s, d)) \in R$.
Steps 1, 2, 3, and 4 toghether imply that for every $z, s \in A$ such that $((z, b), (s, d)) \in C$, then $((z, b), (s, d)) \in R$.

Step 5. Let $u \in A$ such that bqu; then there exists $p \in \Omega_q$ such that $dp\,upb$ and, by (2a), $((z, b), (s, d)) \in R$ implies $((z, b), (s, u)) \in R$.

Step 6. Let $u, w \in A$ such that bqu and bqw; then there exists $p \in \Omega_q$ such that $upwpb$ and by (2b), $((z, b), (s, u)) \in R$ implies $((z, w), (s, u)) \in R$.

Step 7. Let $x, u \in A$ such that xqb and bqu (or u is b); then there exist $w \in A$ and $p \in \Omega_q$ such that $xquqw$ and $upxpw$, and by (2b), $((z, w), (s, u)) \in R$ implies $((z, x), (s, u)) \in R$.

Step 8. Let $r, t \in A$; then there exist $x, u \in A$ such that xqr, xqt, and bqu. By Step 7, $((z, x), (s, u)) \in R$. Considering alternative x as the alternative b, Step 6 implies that $((z, r), (s, t)) \in R$.

This completes the proof that for every $r, s, t, z \in A$ such that $((z, r), (s, t)) \in C$ it is also $((z, r), (s, t)) \in R$, hence $R = F$, a contradiction.

Q.E.D.

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JOURNAL OF ECONOMIC THEORY **22**, 37–55 (1980)

Incentives, Iterative Communication, and Organizational Control

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Received March 22, 1978; revised April 10, 1979

I. INTRODUCTION

This paper defines a class of incentive mechanisms for decentralized organizations when communication between the members of the organization must, a priori, be iterative in nature. Payoff functions for the members of the organization are defined and these functions along with the set of possible member strategies are viewed as defining an N-person game. We define a solution concept for this game that has the following properties: the members of the organization (nontrivially) maximize their payoffs at a "solution" *and* the overall organizational goals are achieved by any "solution."

Acting in accordance with the behavioral rules of the organization, i.e., "telling the truth," is shown to be among the "solutions" to the game. Thus, the members of the organization are shown to have an incentive to follow these rules, since doing so is an individually optimal strategy that is, in some sense, easy to calculate.

Arrow [1] has defined the problem of organizational control as consisting of two parts: (1) the choice of operating (behavioral) rules, i.e., communication and decision making rules and (2) the choice of enforcement rules, i.e., rules that induce the members of the organization to follow the operating rules.[1] There is a vast literature in economics concerned with the definition and/or analysis of operating rules for decentralized organizations, theoretical planning procedures being the most obvious [2, 14, 18].[2] These various

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* This paper contains results from my Ph.D. thesis submitted to Northwestern University. I want to thank my advisor Theodore Groves for his encouragement and help. This paper has benefited greatly from comments by Matthew Sobel, David Nachman, and an anonymous referee. All errors, of course, remain my own.

[1] Obviously, the enforcement rules defined for a particular organization must be tied to individual members' preferences, either directly or indirectly.

[2] The mathematical programming literature contains many different algorithmic procedures whose economic interpretations are as decentralized planning procedures: e.g., the Dantzig–Wolfe decomposition algorithm [4]. In addition many economic planning procedures are based on mathematical programming algorithms: for example, Malinvaud's

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literatures are concerned with normative analyses: Do procedures exist that have certain desirable properties?[3] The procedures defined in the literature have several important characteristics in common: they inevitably involve iterative communication and they generally involve decentralized decision making, due to the size of the problems considered. None of these procedures deal with the provision of incentives (the definition of enforcement rules) to ensure that the operating rules are followed; they thus ignore the second part of the organizational design question.

An important question is thus raised: Are there enforcement rules that make compliance with the behavioral rules in the best interests of the members of the organization? If the answer to this question is no, then it is not necessarily true that these procedures will lead to a solution of the organization's overall problem.

Until recently, very little work has been done on the specification of the enforcement rules, referred to in other contexts as the incentive problem. The existing work has examined the properties of classes of incentive mechanisms [8, 13], investigated incentives in a team [9], and solved the "free rider" problem in a general equilibrium model with both public and private goods [12]. In all of this work, communication is noniterative in nature.[4]

The model and analyses developed in this paper are in the same spirit as the model and analyses of Groves [10, 11] and Groves and Loeb [13]. These latter models will be referred to as "Groves schemes." Implementation of a Groves scheme requires each member of the organization to have sufficient computational, storage (memory), and communcation capabilities both to determine the graph of the function it must transmit to the central decision maker and to make this transmission (without error) as a single message. The informational capabilities of most economic decision making units are severely limited in reality. The purpose of this paper is the exploration of the economic implications of these limitations on the production and communication of information.

As a first step in this exploration, this paper examines the incentive properties of a particular (rather general) scheme in which communication is iterative. This scheme implicitly reflects some limitations on computational capacity, memory, and communication capabilities of the members of the organization. It is only necessary that the members be able to calculate the

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"Procedure Implying Mathematical Programming at the Center" [18] is an inner-linearization algorithm for a concave programming problem. For a discussion of some of the economic procedures, see [14]; of the mathematical programming algorithms, see [6].

[3] "Nice" properties may be of the following type: efficient use of information, monotonicity, feasibility, etc. (see [14]).

[4] Groves and Loeb [13] use an iterative procedure to reconstruct functions (at least locally) by a Taylor series expansion. Dreze and de la Vallee Poussin [5] have dealt with incentives in a tâtonnement-type procedure.

value of the function of interest at a particular point in its domain, and communicate an associated message. The iterative nature of the communication process implicitly captures the fact that the members of the organization can do no more than that without incurring real economic or time costs that may change the nature of the function itself. This would change the nature of the organization's problem, since information production and communication efforts would have to be considered explicitly.

The results obtained in this paper can be applied to establish the incentive properties of some decentralized planning procedures. Malinvaud's [18] "Procedure Implying Mathematical Programming at the Center" is a case in point. The firms in the economy compute and transmit, to a central authority, well-chosen points in their production possibilities sets in response to well-chosen price vectors sent by the central authority. It is easily shown that this procedure is a special case of the procedure developed in this paper (see Cohen [3]). While it is straightforwqrd to contract Malinvaud's procedure into a Groves scheme, this action would require the firms to have a communicable description of their entire production possibilities sets. In practice, however, knowledge of production possibilities is very often limited to the ability to determine outputs or an input–output vector for a given vector of prices. Hence, the use of an iterative scheme becomes feasible as well as prudent. Over an extended period of time, assuming fixed technology, it is possible that some firms may have more or less complete knowlledge of some portions of their production possibilities sets. However, for a Groves scheme to be implemented, every firm must have complete, communicable knowledge of all of its production possibilities set.

That the consequences of these informational limitations are not trivial is demonstrated by a comparison of the properties of a Groves scheme and the iterative scheme developed here. The existence of dominant strategy equilibria can be shown for a Groves scheme. In addition, "telling the truth" is one such equilibrium. Even though the iterative scheme developed in this paper is under certain circumstances informationally equivalent to a Groves scheme, one must settle for weaker results.

In Section II, a model of a decentralized organization is defined. Strictly for ease of exposition, the analysis in this paper is couched in terms of a multidivisional firm that seeks to maximize overall firm profit.

Given the iterative operating rules, the organizational design problem is defined in Section III as one of finding enforcement rules that lead to compliance with the operating rules. We then define (in Section IV) the problem for any member of the organization as the choice of messages to be sent to the central decision maker (the Center) at each iteration of the communication process, i.e., the choice of a strategy. The operating and enforcement rules and the set of possible member strategies are viewed as defining an N-person (non-zero sum) game in normal form.

It is assumed that this game is played noncooperatively. There are two possible ways of viewing the game. It could be viewed as a game in which payoff can be received only when the communication process has been run to its end and terminal decisions have been taken and/or implemented. This formulation does not allow us to deal with truncated communication and suboptimal decision making. Alternatively, one could view the game as one in which decisions can be taken and payoff recieved at any iteration.

As a first attempt at an analysis of incentives when communication is iterative, we view the game as being of the first type. A natural solution concept for such an N-person noncooperative game is that of the Nash equilibrium. From a normative, or design, point of view, just knowing that a Nash equilibrium exists is not enough; in addition, one would like to know whether such an equilibrium will be attained. Thus, we might ask for the existence of a dominant strategy equilibrium that is, in some sense, easy to calculate. As was noted above, the iterative nature of communication precludes demonstrating the existence of such equilibria for the present game. In Section V, a "solution" to the game is defined as a Nash equilibrium that weakly dominates in the class of Nash equilibria; this definition avoids some of the possible ambiguities of multiple Nash points.

In Section VI, a general class of operating rules is defined, and we demonstrate the existence of enforcement rules with certain desirable properties. It is shown that any "solution" for the game, defined by these operating and enforcement rules, leads to a decision solving the organization's overall problem. In addition, a dominant strategy equilibrium is shown to exist for a restricted set of player strategies. Moreover, the equivalence class of "truthful" strategies is shown to be a subset of the class of restricted dominant strategy equilibria, as well as a subset of the set of "solutions."

Finally, in Section VII, we discuss the applicability of our operating rules to existing decentralized planning procedures and mathematical programming algorithms.

II. A MODEL OF A MULTIDIVISIONAL FIRM WITH N-DIVISIONS

Consider a multidivisional firm with N divisions and a corporate center (the Center). Each division has a profit contribution function $\Pi_i(\cdot)$, where $\Pi_i(x)$ is the profit contribution by division i given the decisions x.[5] The Center's direct contribution to profit is denoted by the function $\Pi_0(\cdot)$.[6]

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[5] It is not specified, as yet, who makes the decisions.

[6] It has been pointed out that inclusion of the Center's contribution function does not preclude the application of a Groves (noniterative) mechanism, and that is indeed the case. The purpose of the present work is to investigate the incentive properties of mechanisms in which communication must be iterative. The $\Pi_0(\cdot)$ function is included because, especially in the context of the multidivisional firm, one can envision situations in which the Center contributes to overall profit directly.

The L-dimensional vector x may represent an input vector of resources or x may be the level of a "public good" used by all the divisions and the Center. The total profit of the firm is the sum of each division's, and the Center's, profit contribution; i.e., total profit $= \sum_{i=1}^{N} \Pi_i(\cdot) + \Pi_0(\cdot)$. The Center's problem is to choose the vector x, subject to constraints, to maximize the firm's profit. Formally, the Center's problem is assumed to be to solve

$$\text{(A)} \qquad \operatorname*{Max}_{x} \sum_{i=1}^{N} \Pi_i(x) + \Pi_0(x) \qquad \text{s.t.} \quad x \in X$$

where $\Pi_i : \mathbb{R}^L \to \mathbb{R}$ is concave for all $i = 0, 1, \ldots, N$ and X is a con[illegible] compact set. More realistic models of decentralized organizations would permit nonconvexities in X and departures from concavity in the $\{\Pi_i(\cdot)\}$ but (A) is a reasonable starting point to develop a theory of control with iterative communication.

Suppose that each division of the firm knows[7] its own profit contribution function $\Pi_i(\cdot)$, but not the functions of the other divisions. Also, suppose that these functions are not known by the Center either; however, it does know that they are concave. Only the Center knows that function $\Pi_0(\cdot)$ and the set X.

Since the Center does not know the divisions' $\Pi_i(\cdot)$ functions, it must seek some information from the divisions about them. Once the Center has this information, it must make a decision, i.e., choose a "good" x vector.

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III. THE CONTROL MECHANISM

A *control mechanism* is defined by a complete specification of both operating and enforcement rules. These rules are known explicitly by the Center. For the iterative procedures considered here, the operating rules consist of a message set for the divisions, a message rule for the Center for each possible iteration, and a decision rule for the Center.[8] Thus, an *iterative control mechanism* for problem (A) is defined by a quadruple: The first component is the message set for the divisions; that is, a set from which a division at each iteration may select a message to be sent to the Center. The second component is a sequence of rules for messages sent by the Center, specifying at every iteration the message to be sent to the divisions given the messages received from the divisions at all previous iterations. The third component is the Center's decision rule. Given the messages received from the divisions at all iterations, this rule specifies the Center's decision. The final component of the quadruple is an N-tuple of evaluation measures, one for each division. Each division's evaluation, as specified by these measures, will depend on its realized contribution Π_i, which we assume can be observed ex post by the Center, and on the messages of all the divisions. This last component of the quadruple implicitly specifies the enforcement rules for the control mechanism; i.e., maximization of a division's evaluation measure enforces the use of the prescribed operating rules.

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[7] Where "know" means that for each x, division i can calculate $\Pi_i(x)$.

[8] For a discussion of control mechanisms for noniterative procedures, see Groves [10]

Formally, an *iterative control mechanism* C is defined by

$$C = (\mathcal{M}, \{y^t(\cdot)\}_{t=1}^{\infty}, x(\cdot), \langle E_i(\cdot)\rangle_{i=1}^{N}), \tag{1}$$

where

(a) $\mathcal{M} \subset \{\text{finite-dimensional Euclidean space}\}$ is the *message set of the divisions*. $m_{it} \in \mathcal{M}$ is the message of division i at iteration t.

$$m_t = (m_{1t}, \dots, m_{Nt}),$$
$$m^t = (m_1, \dots, m_t),$$
$$m_i^t = (m_{i1}, \dots, m_{it}).$$

(b) $y^t: \mathcal{M}^{N(t-1)} \to Y$ is the *message rule of the Center* for $t \geq 2$. y^1 is a given, fixed element of Y.

(c) $x: \mathcal{M}^{\infty} \to X$ is the *Center's decision rule.*[9]

(d) $E_i: \mathbb{R} \times \mathcal{M}^{\infty} \to \mathbb{R}$ is *division i's evaluation measure.*

Under certain iterative procedures, the Center may be able to take a "good" decision after a finite number of iterations. Since x is defined only for an infinite string of messages, it would appear that the control mechanism cannot handle situations in which the Center has collected sufficient information for optimal decision making after a finite number of iterations of the communication process. That is, if (1a)–(1c) are viewed as partially defining a (decentralized) nonlinear programming algorithm, what happens if the algorithm converges to a solution of (A) in a finite number of iterations? For the procedures considered here it will suffice to assume that the Center acts as though the final message of each division has been repeated an infinite number of times. Formally,

$$x: \mathcal{M}^t \times (m_t \times m_t \times \cdots) \to X. \tag{2}$$

A stopping criterion is associated with any iterative procedure: if this criterion is satisfied at iteration t, the Center's decision is $x(m^t, (m_t \times m_t \times \cdots))$.

[9] $\mathcal{M}^{\infty}$ is the Cartesian product of $\mathcal{M}$ with itself an infinite number of times.

It would then remain to be shown for a particular procedure that, if the stopping criterion is satisfied at iteration t, the decision $x(m^t, (m_t \times m_t \times \cdots))$ is optimal.

IV. The Division's Problem

Once the control mechanism C has been specified, the problem for each division is to decide how to choose the messages $m_{it} \in \mathcal{M}$ to be sent to the Center at each iteration. Each division wants to send messages to the Center that induce the Center to take a decision that maximizes the division's evaluation. Thus it chooses, for each iteration t, a *response rule*, or a function u_{it}, that determines at iteration t how it should respond to the messages $(y^1, \ldots, y^t)$ from the Center (and implicitly to the past messages of the other divisions, since each y^t, for $t \geqslant 2$, depends on the messages of all divisions for all previous iterations).

$$u_{it}: \bigtimes_{\tau=1}^{t} Y \to \mathcal{M}. \tag{3}$$

A *strategy* u_i is an infinite sequence of response rules, one for each possible iteration of the communication process:

$$u_i = \{u_{it}\}_{t=1}^{\infty}. \tag{4}$$

Each message of any division can be expressed in terms of the messages (and thus the response rules) of all the divisions at previous iterations, the response rule of the division at the current iteration, the messages of the Center at previous iterations, and the starting point y^1. Since associated with each control mechanism is a unique starting point y^1 and a set of message rules $y^t(\cdot)$ for $t \geqslant 2$, a control mechanism C and the N-tuple of divisional strategies $u = (u_1, \ldots, u_N)$ uniquely determine the infinite sequence of messages that the Center will receive. The available strategies for a division to choose among are members of the set

$$U = \{\text{the space of infinite sequences defined by (3) and (4)}\}. \tag{5}$$

Given the joint strategy $u = (u_1, \ldots, u_N)$, the infinite string of messages can be defined recursively as follows (where the dependence on the control mechanism C is suppressed):

$$m_{it}: U^N \to \mathcal{M}, \tag{6}$$

where

$$\begin{aligned} m_{i1}(u) &= u_{i1}(y^1), \\ m_{i2}(u) &= u_{i2}(y^1, y^2(m^1(u))), \\ &\vdots \\ m_{it}(u) &= u_{it}(y^1, y^2(m^1(u)), \ldots, y^t(m^{t-1}(u))), \end{aligned}$$

and

$$m^\alpha : U^N \to \mathcal{M}^\alpha,$$

where

$$m^\alpha(u) = (m_{11}(u),\ldots, m_{N1}(u), m_{12}(u),\ldots, m_{N2}(u),\ldots).$$

The payoff for division i, as a function of the vector of strategies u, is division i's evaluation when the messages are $m^\alpha(u)$:[10]

$$W_i[u] = E_i(\Pi_i(x(m^\alpha(u))), m^\alpha(u)). \tag{7}$$

V. DEFINITION OF OPTIMALITY

The choice of divisional strategies can be thought of in game-theoretic terms. The payoff functions $W_i[\cdot]$ and the set of divisional strategies U define an N-person (non-zero sum) game in normal form, which we assume is played noncooperatively.

A vector of strategies u^* will be called a "*solution*" to the game defined by the payoff functions $W_i[\cdot]$ and the set of strategies U if

> u^* is a Nash equilibrium and $W_i[u^*] \succcurlyeq W[u]$ for any other Nash equilibrium u, for all i.[11,12] (8)

An iterative control mechanism will be called *optimal* if a "solution" u^* exists and if any "solution" leads the Center to take a decision that solves problem (A). Formally, the iterative control mechanism

$$C^* = (\mathcal{M}^*, \{y^{t*}(\cdot)\}_{t=1}^{\infty}, x^*(\cdot), \langle E_i^*(\cdot)\rangle_{i=1}^{N}) \tag{9}$$

is an optimal control mechanism if the following two properties are satisfied:

Decisiveness: There exists a u^* satisfying (8).

Efficiency: If u^* satisfies (8) then $x^*(m^\alpha(u^*))$ solves problem (A).

At first glance problem (A) might seem limited in its applicability, since the only decisions are made by the Center. However, it is easily extended to cover models in which both the Center and the divisions must take decisions, when it is assumed that the Center takes its decision first and that the divisions take their decisions on the basis of the Center's decision.[13] (For a complete demonstration of this fact, see Cohen [3].)

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[10] Since each u generates a sequence of messages $m^\alpha(u)$, $x(\cdot)$ can be calculated for every u, since $x: \mathcal{M}^\infty \to X$.

[11] $\hat{u}$ is a Nash equilibrium iff $W_i[\hat{u}] \succcurlyeq W_i[\hat{u}/u_i]$ for all $u_i \in U$, for all i.

[12] Since, in general, intermixtures $(\hat{u}_1, u_2^*,\ldots, \hat{u}_i,\ldots, u_j^*,\ldots, \hat{u}_N)$ of Nash equilibria $\hat{u}$ and u^* are *not* Nash equilibria, to avoid ambiguity the divisions should be able to choose among multiple Nash points. The solution concept for this game is thus strengthened by requiring the existence of a best Nash equilibrium.

VI. DEFINITION OF A CLASS OF OPTIMAL CONTROL MECHANISMS—$\mathscr{C}^*$

In this section a special class of control mechanisms is examined. The specification of this class requires a few preliminary definitions:

$$\mathscr{H} = \{H_i \mid H_i : \mathbb{R}^L \to \mathbb{R}\},$$
$$\mathscr{H}_0' = \{H_i \mid H_i \in \mathscr{H} \text{ and } H_i \text{ is concave}\}.$$

A strategy u_i was defined as an infinite sequence of functions u_{it}, where $u_{it} : \times_{\tau=1}^{t} Y \to \mathscr{M}$. A particular type of strategy is one in which $u_{it} : Y \to \mathscr{M}$ for all t and u_{it} is identical for all t, i.e., $u_{it} = u_{is}$ for all $t, s \in [1, \infty)$. The strategy u_i is thus a stationary strategy, in the language of dynamic programming.

$$\tilde{U} = \{u_i \mid u_i \in U, u_{it} : Y \to \mathscr{M} \text{ and } u_{it} = u_{is} \text{ for all } t, s \in [1, \infty)\}.$$

(The tth element of the infinite sequence u_i, when $u_{it} = u_{is}$ for all $t, s \in [1, \infty)$, will be referred to as u_i. Therefore $u_i(y^t)$ should be interpreted as $u_{it}(y^t)$.)

The class of control mechanisms considered can first be characterized by their operating rules. In the previous work on incentive mechanisms, most notably that of Groves [10] and Groves and Ledyard [12], the following question is never addressed: How does the Center actually calculate the vector x once it has received the divisions' reported profit contribution functions. It is implicitly assumed that the Center can find a solution to the resultant mathematical programming problem. Under such an implicit assumption, it is immediate that the Center can find a solution to its overall problem if the divisions report their true profit contribution functions. With iterative communication, the calculation of the vector x is an inherent part of the control problem. Restriction 1 merely ensures the existence of divisional strategies that will lead the Center to calculate the solution to problem (A).

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[13] If the divisions and the Center make their respective decisions simultaneously, the Center's problem will not necessarily be solved by these decisions. Suppose that the decisions were taken simultaneously. It would not necessarily be true that each division's decision and the Center's decision would be *jointly* feasible for that division. If the divisions take their decisions first there will not necessarily be *one* decision for the Center that is jointly feasible with the local decisions of all the divisions. When the Center takes its decision first, local decisions of the divisions and the Center's decision will be jointly feasible for all divisions. Sequencing of decision making allows sufficient information to be exchanged so that the Center's problem can be solved. Groves [9] deals with the problem of simultaneous decision making by the Center and the divisions as a problem in decision making under uncertainty.

Restriction 1. Given the iterative control mechanism C, there exists a non-empty-valued mapping

$$\Psi : \mathscr{H} \rightarrow \tilde{U}$$

such that if $H_i \in \mathscr{H}_0 \subset \mathscr{H}$ for all i, then for any $u_i \in \Psi[H_i]$,

$$x(m^{\infty}(u)) \text{ maximizes } \sum_{i=1}^{N} H_i(x) + H_0(x) \text{ subject to } x \in X.$$

A *truthful strategy for i* will be any $u_i \in \Psi[H_i]$.

In order to prove the dominant strategy property of "telling the truth" (i.e., reporting true profit contribution functions) the work on noniterative procedures requires the Center to be able to calculate the value of reported profit at the Center's decision x. This calculation is straightforward since the Center knows the *entire* reported profit contribution function of each division. When communication is iterative, the Center can only approximate this function. Restriction 2 ensures that when the decision $x(\cdot)$ is taken, the Center has an exact approximation of each division's reported profit contribution at $x(\cdot)$. The rules defined below in Restriction 2 are approximation rules for the reported profit contribution functions of the divisions.

Restriction 2. Given the iterative control mechanism C, there exist rules $P_i : X \times \mathscr{M}^{\infty} \rightarrow \mathbb{R}$ such that

If $u_i \in \Psi[H_i]$, where $H_i \in \mathscr{H}_0$, then for all
$u \backslash u_i \in U^{(N-1)}$, $P_i(x(m^{\infty}(u)), m_i^{\infty}(u)) = H_i(x(m^{\infty}(u)))$,
where $m_{it}(u) = u_i(y^t(m^{t-1}(u)))$ for all t, and all i.[14]

It cannot be *assumed* that a division will play a particular type of strategy; it must be shown that playing this type of strategy is in the best interests of the division. Evaluation measures are used to provide the divisions with an incentive to play a particular type of strategy. What can be required is that the messages sent to the Center by the divisions obey some consistency requirements.

Given an iterative control mechanism C satisfying
Restriction 1, the infinite sequence of messages m_i^{∞} (10)

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[14] $(u \backslash u_i) = (u_1, \ldots, u_{i-1}, u_{i+1}, \ldots, u_N)$.

is said to be *consistent with* $H_i(\cdot)$ *given* $\{y^t\}_{t=1}^{\infty}$ if there exists a $u_i \in U$ such that

(a) $u_i \in \Psi[H_i]$ and

(b) $m_{it} = u_i(y^t)$ for all t.

Thus, the sequence of messages m_i^{∞} is *consistent with a concave function given* $\{y^t\}_{t=1}^{\infty}$ if there exist a concave function H_i and a strategy u_i satisfying (10).[15,16]

Restriction 3. Given the iterative control mechanism C satisfying Restrictions 1 and 2, $E_i(\Pi_i, m^{\infty}) = F_i(R_i(\Pi_i, m^{\infty}))$, where

$$R_i(\Pi_i, m^{\infty}) = \begin{cases} \Pi_i + \sum\limits_{j \neq i} P_j(x(m^{\infty}), m_j^{\infty}) + \Pi_0(x(m^{\infty})) \\ \qquad \text{if } m_i^{\infty} \text{ is consistent with a concave function} \\ \qquad \text{given } \{y^t\}_{t=1}^{\infty}\text{; } \textit{and } \Pi_i \geqslant P_i(x(m^{\infty}), m_i^{\infty}) \\ -\infty \qquad \text{otherwise} \end{cases}$$

and $F_i : \mathbb{R} \to \mathbb{R}$ predetermined and strictly increasing.[17]

Since $E_i(\cdot)$ must be defined for all m^{∞}, whether or not the rules of the game have been followed, the $P_j(\cdot)$ functions must be defined for *all* m_j^{∞}. The inclusion of the condition that actual profit contributions be at least as large as reported profit contribution is not as innocuous as it may seem at first glance. This condition creates a discontinuity in the evaluation measure, since any deviation (even a small one) downward from reported profits will cause the division to have an evaluation of $-\infty$. This definition also requires ex post observation by the Center of divisional contribution. Thus, given Restriction 2, these evaluations are a type of "profit sharing." Why not just define $E_i(\cdot)$ to be some fixed fraction of actual profit, i.e., define $E_i(\Pi_i, m^{\infty}) = \alpha_i(\sum \Pi_i(x(m^{\infty})))$, $0 < \alpha_i < 1$? Evaluations of the type defined by Restriction 3 are not dependent upon the local decisions of the other divisions. Thus the optimal strategy for division *i* does not depend on the rationality of the other divisions. Profit sharing does not have the independence property. Since this model is easily generalizable to situations that include divisional decision making, profit sharing is not a viable alternative.

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[15] In general, given $\{y^t\}_{t=1}^{\infty}$, m_i^{∞} may be consistent with many different H_i, some in $\mathscr{H}_0$ and some not. It is thus not possible in general to determine if *i* is playing according to some $H_i \in \mathscr{H}_0$ by observing only m_i^{∞} and $\{y^t\}_{t=1}^{\infty}$.

[16] If m_i^{∞} is consistent with a function $H_i \in \mathscr{H}_0$ for *all* sequences $\{y^t\}_{t=1}^{\infty}$, where $y^t \in Y$, then division *i must* be playing according to the strategy $u_i \in \Psi[H_i]$, where $m_{it}(y^t) = u_i(y^t)$.

[17] It may not be decidable in general whether or not a particular infinite string of messages m_i^{∞} is consistent with some $H_i(\cdot)$ given $\{y^t\}_{t=1}^{\infty}$. Thus, Restriction 3 implicitly excludes mechanisms for which this is not decidable.

The class of iterative control mechanisms defined by Restrictions 1-3 will be denoted by $\mathscr{C}^*$:

$$\mathscr{C}^* = \{C : C \text{ is an iterative control mechanism and } C \text{ satisfies Restrictions 1-3}\}.$$

VII. PROPERTIES OF THE CLASS $\mathscr{C}^*$

An optimal control mechanism has been defined as a control mechanism for which:

(a) there exists a "best" Nash equilibrium that dominates (in terms of divisional payoff) all other Nash equilibria for every division; and

(b) any "best" Nash equilibrium leads to an optimal decision by the Center.

The class of mechanisms $\mathscr{C}^*$ can be shown to be optimal. It is not possible to show the existence of a dominant strategy equilibrium. However, if all players are restricted, a priori, to playing strategies associated with concave functions H_i, i.e., $u_i \in \Psi[\mathscr{H}_0] \quad \tilde{U}^*$, not only does a best Nash equilibrium exist, but this best Nash equilibrium is also a dominant strategy equilibrium for the restricted class.

We define the equivalence class of truthful strategies for division i as the set of strategies $\mathscr{U}_i^*$, where

$$\mathscr{U}_i^* = \Psi[\Pi_i + c_i] \qquad \text{for all constants } c_i. \tag{11}$$

A restricted equivalence class of truthful strategies for division i is defined by restricting $\Pi_i(\cdot)$ to the domain X. If $\bar{\Pi}_i$ is the restricted form of Π_i, then

$$\bar{\mathscr{U}}_i^* = \Psi[\bar{\Pi}_i + c_i] \qquad \text{for all constants } c_i.$$

The following lemma is immediate from Restrictions 1 and 3 and (11), and is given without proof. Unless otherwise stated, the proofs of all subsequent lemmas and theorems appear in the Appendix.

LEMMA 1. *Given any iterative control mechanism C satisfying Restrictions* 1 *and* 3, *if*

$$u_i^1, u_i^2 \in \mathscr{U}_i^*$$

then

$$W_i[u/u_i^1] = W_i[u/u_i^2] \qquad \textit{for all } u\backslash u_i \in U^{(N-1)}, \qquad \textit{for all } i.^{18}$$

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[18] $u\ \tilde{u}_i = (u_1, \dots, u_{i-1}, \tilde{u}_i, u_{i+1}, \dots, u_N)$.

THEOREM 1. *Given a control mechanism* $C \in \mathscr{C}^*$, *for all* $u \backslash u_i \in \tilde{U}^{*(N-1)}$

$$W_i[u/u_i^*] \succcurlyeq W_i[u] \qquad \text{for all} \quad u_i \in \tilde{U}^* \text{ iff } u_i^* \in \bar{\mathscr{U}}_i^*$$

Thus, a dominant strategy equilibrium exists when the divisions are restricted to playing strategies in $\tilde{U}^*$. Does there exist a strategy in $\tilde{U}^*$ that is best against a particular $(N-1)$-tuple of strategies that is in $U^{(N-1)}$ but not in $\tilde{U}^{*(N-1)}$? If division i wishes to maximize its payoff, it must send messages m_i^α to the Center that are consistent with a concave function given $\{y^t\}_{t=1}^{\alpha}$; this ensures the existence of a best strategy in $\tilde{U}^*$ for division i. The following is straightforward, and is given without proof:

LEMMA 2. *Given a control mechanism* $C \in \mathscr{C}^*$, *for each* $u \backslash u_i \in U^{(N-1)}$ *there exists a particular* $\hat{u}_i \in \tilde{U}^*$ *such that*

$$W_i[u/\hat{u}_i] \succcurlyeq W_i[u] \qquad \forall u_i \in U. \tag{12}$$

The proof of the (if) part of Theorem 1, i.e., the proof that for all $u \backslash u_i \in \tilde{U}^{*(N-1)}$, $W_i[u/u_i^*] \succcurlyeq W_i[u]$ for all $u_i \in \tilde{U}^*$, relies on the following two conditions:

(a) $x(m^\alpha(u/u_i^*))$ maximizes

$$\Pi_i(x) + \sum_{j \neq i} H_j(x) + \Pi_0(x) \text{ over } X; \tag{13}$$

(b) $P_j(x(m^\alpha(u/u_i^*)), m_j^\alpha(u/u_i^*)) = H_j(x(m^\alpha(u/u_i^*)))\ \forall j.$

If $u \backslash u_i \notin \tilde{U}^{*(N-1)}$ the conditions in (13) do not *necessarily* hold. Therefore it is not possible to prove that for any $u \backslash u_i \in U^{(N-1)}$ condition (12) holds.

Theorem 1 establishes the fact that the "truth" is a Nash equilibrium, and in addition that the "truth" is the best concave Nash equilibrium. Lemma 1 establishes that there exists, for each particular $(N-1)$-tuple of divisional strategies, a best concave strategy ? (This is the result of the divisions sending messages consistent with a concave strategy.)

Is there reason to believe that the divisions will play the particular Nash equilibrium of "truthful" strategies? The answer is in the affiffirmative for the following reasons: first, the divisions already know their respective "truthful" strategies, and playing one of those strategies requires no additional calculation on their part. Second, to play any other Nash equilibrium vector of strategies, each division must know what the other divisions are doing, in order to calculate the best strategy in the particular situation. Thus, the N-tuple of "truthful" strategies is the only Nash equilibrium that is "independent" of the other divisions' strategies.

LEMMA 3. *For any control mechanism* $C \in \mathscr{C}^*$ *any* N*-tuple of strategies* u^*, *where* $u_i^* \in \mathscr{U}_i^*\ \forall i$, *is a Nash equilibrium.*

LEMMA 4. *For any control mechanism $C \in \mathscr{C}^*$ any Nash equilibrium u^*, where $u_i^* \in \mathscr{U}_i^* \; \forall i$, is a "solution," i.e.,*

$$W_i[u^*] \geq W_i[u] \qquad \forall \textit{ Nash equilibria } u. \ \forall i.$$

LEMMA 5. *For any control mechanism $C \in \mathscr{C}^*$, if $\tilde{u}$ is a "solution," then $x(m^{\alpha}(\tilde{u}))$ solves problem* (A).

The major result of this section is stated in the following theorem:

THEOREM 2. *Any control mechanism C in the class $\mathscr{C}^*$ is* optimal.

In the previous work that has been done on incentive mechanisms no specific assumptions have been made about the Center's ex post knowledge of actual divisional profit contribution. The divisions are assumed to send entire profit contribution functions, and it can be shown that the N-tuple of "truthful" strategies is a dominant strategy equilibrium. As was pointed out above, it is assumed in the previous work that the Center possesses some method of solving the mathematical programming problem it receives from the divisions. The important point is that the Center knows each division's *entire* strategy, and thus can calculate a global solution to the reported problem. When communication is iterative in nature, the Center knows only a local approximation of the strategies near the decision x, and is not guaranteed a global solution. Thus, if some of the divisions play strategies that do not ensure convergence to a solution (i.e., nonconcave strategies), the truth may not be the best strategy for the remaining divisions. The lack of complete information about strategies ex post leads to a weaker result than can be obtained about the "complete information," or noniterative, mechanisms.

Finally, we must give an interpretation of the evaluation measures $E_i(\cdot)$. One alternative would be to view the functions $R_i(\cdot)$ as providing an index number and the transformation $F_i(\cdot)$ as translating that number into the argument of the division manager's utility function. For example, the manager of division i may receive satisfaction from paid vacation days, while the manager of division j wants to maximize the probability of receiving a key to the executive lavatory.

Alternatively, the evaluation measures and thus the payoff functions, may represent real resource flows. In that case, one must worry about the question of a "balanced budget" at any equlilibrium.

LEMMA 6. *If $F_i(\cdot) = \alpha_i$ for all i, where $0 < \alpha_i < 1$ and $\sum_{i=1}^{N} \alpha_i = 1$, then, at any Nash equilibrium u,*

$$\sum_{i=0}^{N} \Pi_i(x(m^{\alpha}(u))) \geq \sum_{i=1}^{N} W_i[u].$$

VIII. Applications

One may rightfully ask the following question: Is the set $\mathscr{C}^*$ of optimal control mechanisms nonempty? If so, how exclusionary are Restrictions 1 and 2 on the class of admissible operating rules?

The answer to the first question is yes; it is straightforward to verify that the Dantzig–Wolfe decomposition algorithm [4], as well as Malinvaud's "Procedure Implying Mathematical Programming at the Center" [18] satisfy Restrictions 1 and 2 (see Cohen [3]). It can also be shown that two simple nonlinear algorithms satisfy the restrictions: an outer-linearization algorithm and an inner-linearization (price decomposition) algorithm (see Cohen [3]). (Both the Dantzig–Wolfe algorithm and Malinvaud's procedure are special cases of the latter.)

Restriction 1 requires only that the operating rules actually define a convergent algorithm for solving a nonlinear concave programming problem. Restriction 2 limits attention to procedures that calculate the value of the objective function at the solution. The inner- and outer-linearization algorithms cited above satisfy Restriction 2, thus leading one to postulate that a wide range of algorithms satisfy the restriction.

Appendix

For ease of exposition, we make the following definitions:

$$x(m^{\infty}(u)) \quad x(u),$$

$$A_i(x) \quad F_i\Big(\Pi_i(x) + \sum_{j \neq i} P_j(x, m_j^{\infty}) + \Pi_0(x)\Big),$$

$$B_i(x) \quad F_i\Big(\Pi_i(x) + \sum_{j \neq i} H_j(x) + \Pi_0(x)\Big),$$

$$C_i(x) \quad F_i\Big(\Pi_i(x) + \sum_{j \neq i} \Pi_j(x) + \Pi_0(x)\Big).$$

Then by definition

$$W_i[u] = A_i(x(u)).$$

(Bars over $A_i(\cdot)$, $B_i(\cdot)$, and $C_i(\cdot)$ denote a bar over $\Pi_i(\cdot)$).

Proof of Theorem 1. (If) Let $u_i^* \in \Psi[\Pi_i]$. From Lemma 1, $W_i[u/u_i'] = W_i[u/u_i^*]$ for all $u_i' \in \mathscr{U}_i^*$. We therefore need only prove the theorem for u_i^*. From Restriction 2,

$$A_i(x(u/u_i^*)) = B_i(x(u/u_i^*)).$$

From Restriction 1 and $F_i(\cdot)$ strictly increasing,

$$\bar{B}_i(x(u/u_i^*)) \succcurlyeq \bar{B}_i(x) \qquad \forall x \in X.$$

Since $\Pi_i(x) \quad \bar{\Pi}_i(x)\ \forall x \in X$,

$$B_i(x(u/u_i^*)) \quad \bar{B}_i(x(u/u_i^*)) \succcurlyeq \bar{B}_i(x) = B_i(x) \qquad \forall x \in X.$$

In particular,

$$B_i(x(u/u_i^*)) \succcurlyeq B_i(x(u)) \qquad \forall u_i \in U.$$

Since $u_j \in \tilde{U}^*$ for all $j \neq i$, Restriction 2 gives

$$B_i(x(u)) - A_i(x(u)) \qquad \forall u_i \in U.$$

Therefore for all $u \backslash u_i \subset \tilde{U}^{*(N-1)}$

$$W_i[u/u_i^*] \succcurlyeq W_i[u] \qquad \forall u_i \in U.$$

(Only if) Let $\tilde{u}_i \in \tilde{U}^*$ be such that for all $u \backslash u_i \in \tilde{U}^{*(N-1)}$

$$W_i[u/\tilde{u}_i] \quad W_i[u/u_i^*] \succcurlyeq W_i[u] \qquad \forall u_i \in U$$

(where $u_i^* \subset \Psi[\Pi_i]$).

We must show that $\tilde{u}_i \subset \bar{\Psi}_i^*$. From the (if) part of this theorem, its proof, and Restriction 2 we get

$$W_i[u/\tilde{u}_i] = B_i(x(u/\tilde{u}_i)) \succcurlyeq B_i(x) \qquad \forall x \in X.$$

Restriction 1 yields $x(m^\infty(u/\tilde{u}_i))$ as the maximizer of

$$\tilde{\Pi}_i(x) + \sum_{j \neq i} \Pi_j(x) + \Pi_0(x) \qquad \text{s.t.} \quad x \in X,$$

For the assertion of the theorem to be true for all $u \backslash u_i$ associated with concave functions, it must be true for all $u \backslash u_i$ associated with strictly concave, differentiable functions. If $D_{\Pi_i} \quad \{x \mid \Pi_i(x) \text{ is differentiable}\}$ and $D_{\tilde{\Pi}_i}$ and D_{Π_0} are defined similarly, then concavity of Π_i, $\tilde{\Pi}_i$, and Π_0 implies that

(a) $D_{\Pi_i}^C$, $D_{\tilde{\Pi}_i}^C$, and $D_{\Pi_0}^C$ (where C denotes complement) are all sets of measure zero; and

(b) D_{Π_i}, $D_{\tilde{\Pi}_i}$, and D_{Π_0} are each contained in the interior domain of their respective functions of definition. (See Rockafellar [21, p. 246, Theorem 25.5].)

Therefore,

$$(D_{\Pi_i} \cap D_{\tilde{\Pi}_i} \cap D_{\Pi_0})^C = (D_{\Pi_i}^C \cup D_{\tilde{\Pi}_i}^C \cup D_{\Pi_0}^C)$$

is a set of measure zero. For each $\tilde{x} \in D_{\Pi_i} \cap D_{\tilde{\Pi}_i} \cap D_{\Pi_0} \cap X$ define the strategies $\tilde{u} \backslash u_i$ such that $\tilde{u}_j \in \Psi[\tilde{\Pi}_j]$ and

$$\sum_{j \neq i} \tilde{\Pi}_j(\cdot) \text{ is strictly concave and differentiable;}$$

$$\nabla \sum_{j \neq i} \tilde{\Pi}_j(\tilde{x}) = -\nabla \Pi_i(\tilde{x}) - \nabla \Pi_0(\tilde{x}). \tag{14}$$

(To show the existence of such strategies we need only show the existence of the $\tilde{\Pi}_j(\cdot)$, which is obvious.) A necessary and sufficient condition for a unique interior maximum of a strictly concave, differentiable function $f(\cdot)$ at x^* is that $\nabla f(x^*) = \mathbf{0}$. Since

$$\nabla \Pi_i(\tilde{x}) + \nabla \sum_{j \neq i} \tilde{\Pi}_j(\tilde{x}) + \nabla \Pi_0(\tilde{x}) = \mathbf{0}$$

$\tilde{x}$ is the unique maximizer of $\Pi_i(x) + \sum_{j \neq i} \tilde{\Pi}_j(x) + \Pi_0(x)$ over X. In addition, $\tilde{x} \in \{\text{interior domain of } \tilde{\Pi}_i(\cdot)\}$ and $\tilde{x}$ maximizes $\sum_{i=1}^{N} \tilde{\Pi}_i(x) + \Pi_0(x)$ over X. Therefore,

$$\nabla \sum_{i=1}^{N} \tilde{\Pi}_i(\tilde{x}) + \nabla \Pi_0(\tilde{x}) = \mathbf{0}.$$

This gives the result that

$$\nabla \tilde{\Pi}_i(\tilde{x}) = \nabla \Pi_i(\tilde{x}).$$

Since functions $\tilde{\Pi}_j(\cdot)$, $j \neq i$, satisfying (14) can be defined for each $x \in D_{\Pi_i} \cap D_{\tilde{\Pi}_i} \cap D_{\Pi_0} \cap X$,

$$\nabla \tilde{\Pi}_i(\tilde{x}) = \nabla \Pi_i(\tilde{x}) \qquad \text{a.e. in } X.$$

Since $\Pi_i(\cdot)$ and $\tilde{\Pi}_i(\cdot)$ are concave and defined over all of X, they are necessarily continuous over X (see Rockafellar [21, p. 83, Corollary 10.11]). Therefore,

$$\tilde{\Pi}_i(x) = \Pi_i(x) + c_i \qquad \text{for all } x \in X, \text{ where } c_i \text{ is a constant.}$$

This gives the result that

$$\tilde{u}_i \in \bar{\mathscr{W}}_i^*. \quad \blacksquare$$

Proof of Lemma 3. From the (if) part of the proof of Theorem 1 and Restriction 2, for all $u_i \in U$

$$W_i[u^*] \geqslant C_i(x(u^*/u_i)) = A_i(x(u^*/u_i)) = W_i[u^*/u_i]. \quad \blacksquare$$

Proof of Lemma 4. Let $\tilde{u}$ be any Nash equilibrium. Then for any i

(1) $m_i^\infty(\tilde{u})$ is consistent with a concave function given $\{y^t\}_{t=1}^\infty$; and

(2) $\Pi_i \geqslant P_i(x(\tilde{u}), m_i^\infty(\tilde{u}))$.

Thus, for all i, if $u_i^* \in \Psi[\Pi_i]$ for all i,

$$W_i[u^*] = C_i(x(u^*)) \geqslant C_i(x(\tilde{u})) \geqslant A_i(x(\tilde{u})) = W_i[\tilde{u}]. \quad \blacksquare$$

Proof of Lemma 5. Since any u^*, such that $u_i^* \in \mathscr{U}_i^* \; \forall i$, is a solution,

$$W_i[\tilde{u}] = W_i[u^*] \qquad \forall i.$$

Assume $x(\tilde{u})$ does not solve problem (A), i.e., that

$$\sum_{i=0}^{N} \Pi_i(x(u^*)) > \sum_{i=0}^{N} \Pi_i(x(\tilde{u})),$$

where $u_i^* \in \Psi[\Pi_i]$.

From the proof of Theorem 1, $F_i(\cdot)$ strictly increasing, Restriction 1, and $\tilde{u}$ a Nash equilibrium (and thus $\Pi_i \geqslant P_i(x(\tilde{u}), m_i^\infty(\tilde{u}))$ for all i) we get

$$W_i[\tilde{u}] = W_i[u^*] = C_i(x(u^*)) > C_i(x(\tilde{u})) \geqslant A_i(x(\tilde{u})) = W_i[\tilde{u}],$$

which is a contradiction.

Therefore, $x(\tilde{u})$ must solve problem (A). $\blacksquare$

Proof of Lemma 6. Let u be a Nash equilibrium. Then

$$\Pi_i(x(u)) \geqslant P_i(x(u), m_i^\infty(u)) \qquad \forall i.$$

Therefore

$$\begin{aligned}\sum_{i=1}^{N} W_i[u] &= \sum_{i=1}^{N} \alpha_i \left[\Pi_i(x(u)) + \sum_{j \neq i} P_j(x(u), m_j^\infty(u)) + \Pi_0(x(u))\right] \\ &\leqslant \sum_{i=1}^{N} \alpha_i \left[\sum_{j=0}^{N} \Pi_j(x(u))\right] \\ &= \sum_{i=0}^{N} \Pi_i(x(u)). \quad \blacksquare\end{aligned}$$

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JOURNAL OF ECONOMIC THEORY 22, 56–66 (1980)

Intergenerational Preference Differences and Optimal National Saving*

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Received June 19, 1978; revised January 25, 1979

This paper investigates the problem facing national planners who realize that their successors may employ different social welfare criteria for determining economic policies than they do. In such a situation, we show that the present planners may want to leave future generations a smaller capital stock than they would if they thought the stock would be managed in a way consistent with their own preferences. On the other hand, we also identify cases in which the opposite is true. We obtain the results by studying an optimal aggregate growth model the utility function of which changes randomly from one generation to the next. We analyze the model using a system of functional equations in place of the conventional Bellman equation from dynamic programming theory.

Optimal growth models (see, for example, [3, 12]) posit the existence of a societal utility function registering a nation's tastes over possible time paths of average per capita consumption. At a given time such a function presumably represents an amalgamation of the preference orderings of existing citizens, the citizens being able to weigh the benefits of alternative levels of present consumption against those of corresponding amounts of consumption for future cohorts.

Customarily optimal growth models adopt two key postulates: (1) each generation's societal utility function is additively separable between time periods and stationary in the Koopmans' [6, postulate 4] sense, and (2) all cohorts have the same function. These assumptions ensure that as one generation adjusts its aggregate saving to maximize national welfare, it is coincidentally behaving exactly as previous generations would have wanted it to. That fact enables modelers to use Bellman's "principle of optimality" to define the indirect utility function of each generation with a single recursive functional equation, the so-called Bellman equation. The equation often provides useful characterizations of society's best behavior.

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* I am indebted to my colleague Alan Deardorff and to two anonymous referees for many helpful comments on earlier drafts of this paper.

0022-0531/80/010056-11$02.00/0
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The purpose of this paper is to drop the second assumption listed in the preceding paragraph, asking instead the following question: If succeeding generations might have different preferences, how should the present one behave. Because a society's membership changes over time, logically national preferences could too. But if they do, one cohort cannot assume future societies will consume and save exactly the amounts it would like. So, the Bellman equation will no longer be valid. We show that if national preferences change randomly over time, however, we can develop a system of two simultaneous functional equations replacing Bellman's single one. Employed together, the new equations can determine equilibrium behavior for all generations. Under restrictive, but familiar, additional hypotheses we exhibit a solution to the system of equations and show that in various cases intergenerational preference variability unambiguously raises or lowers optimal aggegate saving.

1. THE MODEL

The problem of how much one cohort is to save if it knows its descendants may have different preferences seems very difficult. In order to make any headway, therefore, we make the following assumptions.

First, we assume each generation's societal (von Neumann Morgenstern) preference ordering is a stationary member of the Bergson class:

$$U = \sum_{i=0}^{r} h^i \cdot u(C_i, b) \tag{1}$$

with

$$u(C, b) = b \cdot C^b, \qquad b \in (-\infty, 0) \cup (0, 1).^1 \tag{2}$$

The variable C_0 measures average per capita consumption for the present generation, C_1 is average per capita consumption for the next, and so on. We will often find $\sigma = \sigma(b) = 1/(1 - b)$, negative the inverse of the elasticity of marginal utility with respect to current consumption, more convenient to work with than the constant b. We require that the subjective discount factor h be between 0 and 1.

Second, we assume a linear technology: if Q_t is average per capita output at time t, then

$$Q_{t+1} = \lambda \cdot (Q_t - C_t) \tag{3}$$

---

[1] We omit the logarithmic, or $b = 0$, case. The Appendix and the discussion in Section 3 show that $b = 0$ is the dividing line for case-by-case analyses.

(provided $Q_t - C_t \geq 0$). In words, $Q_t - C_t$ is the capital stock bequeathed per capita at time t, and the positive constant λ is the average productivity of capital divided by 1 plus the rate of population growth.

Suppose, for the moment, that all generations have the same preference-ordering parameters h and b. Then if $v(Q, h, b)$ stands for the maximum possible utility subject to line (3) of a generation with average income Q,

$$v(Q, h, b) = \underset{S \geq 0}{\text{Max}} \{u(Q - S, b) + h \cdot v(\lambda S, h, b)\}. \tag{4}$$

In fact, if $h\lambda^b < 1$,

$$v(Q, h, b) = b \cdot (1/(1 - (h\Lambda)^\sigma))^{1/\sigma} \cdot Q^b, \tag{5}$$

where $\Lambda = \Lambda(b) = \lambda^b$ (see Levhari and Srinivasan [7]). If $s(Q, h, b)$ gives the unique maximizing value of S for the right-hand side of line (4),

$$s(Q, h, b) = (h\Lambda)^\sigma \cdot Q. \tag{6}$$

We will use lines (5) and (6) for comparisons in Section 3. To prevent confusion, from this point forward we will name all alternative indirect utility functions and optimal saving rules with capital letters.

2. DIFFERING PREFERENCE ORDERINGS

We now let every cohort have its own values for h and b. For the sake of simplicity we assume that each generation's pair (h, b) is an independent realization from a fixed distribution (although we could develop functional equations for any case in which pairs are generated by a Markov process). We also assume that each cohort makes its consumption-saving decision without knowing the parameter pairs which will apply to future generations, although each cohort does know the distribution from which all pairs are sampled.

The assumption that the sampling distribution for pairs (h, b) does not change over time gives the new model a recursive structure, which we exploit as follows. Let (h_t, b_t) characterize the preferences of generation t. Let $h_t \cdot R(\lambda S_t, h_t, b_t)$ stand for the discounted expected utility accruing to generation t (vicariously) from the consumption of future cohorts (i.e., from $C_{t+1}, C_{t+2}, \ldots$). Note that the values of $C_{t+1}, C_{t+2}, \ldots$ will depend not only on the average bequest per capita generation t leaves, S_t, but also on $(h_{t+1}, b_{t+1}), (h_{t+2}, b_{t+2}), \ldots$. Because of the stationarity of generation t's preference ordering, however, its vicarious expected utility from $C_{t+2}, C_{t+3}, \ldots$ must be $h_t^2 \cdot R(\lambda \cdot (Q_{t+1} - C_{t+1}), h_t, b_t)$. The function $R(\cdot)$ needs no time subscript because the time horizon is infinite. So, if $S(Q_{t+1}, h_{t+1}, b_{t+1})$ gives the saving of generation $t + 1$ when its taste parameters are h_{t+1} and b_{t+1}, the function $R(\cdot)$ must obey

$$
\begin{aligned}
R(Q_{t+1}, h_t, b_t) = {} & E[u(Q_{t+1} - S(Q_{t+1}, \tilde{h}_{t+1}, \tilde{b}_{t+1}), b_t)] \\
& + h_t \cdot E[R(\lambda S(Q_{t+1}, \tilde{h}_{t+1}, \tilde{b}_{t+1}), h_t, b_t)],
\end{aligned}
\tag{7}
$$

where $\tilde{h}_{t+1}$ and $\tilde{b}_{t+1}$ are random variables. Note that the present cohort uses h_t and b_t in all of its utility calculations, although it cannot force subsequent generations to determine their saving rates using the same constants.[2]

Generation $t + 1$ determines the function $S(\cdot)$ from its own utility maximization problem. Assuming it uses $h_{t+1} \cdot R(\lambda S_{t+1}, h_{t+1}, b_{t+1})$ to compute its vicarious expected utility from $C_{t+2}, C_{t+3}, \ldots$, we have

$$
\begin{aligned}
& u(Q_{t+1} - S(Q_{t+1}, h_{t+1}, b_{t+1}), b_{t+1}) \\
& \qquad + h_{t+1} \cdot R(\lambda S(Q_{t+1}, h_{t+1}, b_{t+1}), h_{t+1}, b_{t+1}) \\
& \qquad = \operatorname*{Max}_{S \geq 0} \{u(Q_{t+1} - S, b_{t+1}) + h_{t+1} \cdot R(\lambda S, h_{t+1}, b_{t+1})\}.
\end{aligned}
\tag{8}
$$

Combining (7) and (8), and streamlining the notation slightly, we have a recursive system of two functional equations,

$$
R(Q, h, b) = E[u(Q - S(Q, \tilde{h}, \tilde{b}), b)] + h \cdot E[R(\lambda S(Q, \tilde{h}, \tilde{b}), h, b)], \tag{9}
$$

$$
\begin{aligned}
& u(Q - S(Q, h, b), b) + h \cdot R(\lambda S(Q, h, b), h, b) \\
& \qquad = \operatorname*{Max}_{S \geq 0} \{u(Q - S, b) + h \cdot R(\lambda S, h, b)\},
\end{aligned}
\tag{10}
$$

in two "variables," $R(\cdot)$ and $S(\cdot)$.

If h and b are the same for every cohort, the right-hand side of Eq. (9) equals the left-hand side of (10). In that case substitution will reduce the system to the single functional equation of Section 1, Eq. (4). If h and b can vary over time, however, the right-hand side of (9) and the left-hand side of (10) are not necessarily equal, so collapsing the system to one equation is not feasible.

Suppose $(R(\cdot), S(\cdot))$ solves Eqs. (9) and (10). Then as shown below, we can think of $S(Q, h, b)$ as giving the desired saving for a generation with taste parameters h and b and per capita national product Q. The indirect utility function for such a generation is

---

[2] The present generation uses h_t and b_t in its calculations because its direct utility function (see line (1)) depends on $C_t, C_{t+1}, \ldots$. If U depended on C_t and the utility of future generations instead, then the present cohort's calculations in line (7) would depend on $(\tilde{h}_{t+1}, \tilde{b}_{t+1})$ and $(\tilde{h}_{t+2}, \tilde{b}_{t+2})$, but not (h_t, b_t).

$$V(Q, h, b) = u(Q - S(Q, h, b), b) + h \cdot R(\lambda \cdot S(Q, h, b), h, b). \tag{11}$$

To rule out anomalies, however, such as $R(Q, h, b) = \infty$ all Q, which satisfies lines (9) and (10) if, for example, $S(Q, h, b) = Q/2$, we require that to constitute a solution for our model, $(R(\cdot), S(\cdot))$ must satisfy the following restriction in addition to the two functional equations above: for any $Q > 0$,

$$R(Q, h, b) = \sum_{i=1}^{\infty} h^i \cdot E[u(\tilde{C}_i, b)], \tag{12}$$

where $\tilde{Q}_0 = Q$, $\tilde{h}_0 = h$, $\tilde{b}_0 = b$, and $\tilde{Q}_i = \lambda \cdot S(\tilde{Q}_{i-1}, \tilde{h}_{i-1}, \tilde{b}_{i-1})$ and $\tilde{C}_i = \tilde{Q}_i - S(\tilde{Q}_i, \tilde{h}_i, \tilde{b}_i)$ all $i \geq 1$. This new restriction comes from line (1) and the definition of $R(\cdot)$.

We could easily modify our functional equations to encompass nonlinear technologies and direct utility functions outside of the Bergson class. The advantage of making the strong assumptions of lines (1)–(3), however, is that, given them, we can often prove the existence of a solution $(R(\cdot), S(\cdot))$ to lines (9), (10), and (12), and, as Section 3 shows, we can compare the resulting saving function $S(\cdot)$ with $s(\cdot)$ from section 1.[3]

To solve the simultaneous equations of lines (9) and (10) let us try a functional form for $R(\cdot)$ similar to the one used in Section 1:

$$R(Q, h, b) = b \cdot r(h, b) \cdot Q^b. \tag{13}$$

Substituting this into Eq. (10), the optimal saving rule is

$$S(Q, h, b) = \left(\frac{(h \cdot \Lambda \cdot r(h, b))^{\sigma}}{1 + (h \cdot \Lambda \cdot r(h, b))^{\sigma}}\right) \cdot Q, \tag{14}$$

where, as always, $\sigma = 1/(1 - b)$ and $\Lambda = \lambda^b$. Inserting (13) and (14) into Eq. (9), we find that

$$r(h, b) = E\left[\left(\frac{1}{1 + (\tilde{h} \cdot \tilde{\Lambda} \cdot r(\tilde{h}, \tilde{b}))^{\tilde{\sigma}}}\right)^{b}\right] + r(h, b) \cdot h \cdot \Lambda(b)$$
$$\cdot E\left[\left(\frac{(\tilde{h} \cdot \tilde{\Lambda} \cdot r(\tilde{h}, \tilde{b}))^{\tilde{\sigma}}}{1 + (\tilde{h} \cdot \tilde{\Lambda} \cdot r(\tilde{h}, \tilde{b}))^{\tilde{\sigma}}}\right)^{b}\right] \tag{15}$$

---

[3] The general procedure for solving the Bellman equation is the "method of successive approximations" (see [1]). That will not work for Eqs. (9) and (10) because the set of finite time horizon indirect utility functions generated by the method will not form a monotone sequence the way it does in Bellman's problems.

(where $\tilde{\Lambda} = \Lambda(\tilde{b})$ and $\tilde{\sigma} = \sigma(\tilde{b})$), so that $r(\cdot, \cdot)$ must satisfy

$$r(h, b) = \frac{E[(1/(1 + (\tilde{h} \cdot \tilde{\Lambda} \cdot r(\tilde{h}, \tilde{b}))^{\tilde{\sigma}}))^{b}]}{1 - h \cdot \Lambda(b) \cdot E[(((\tilde{h} \cdot \tilde{\Lambda} \cdot r(\tilde{h}, \tilde{b}))^{\tilde{\sigma}})/(1 + (\tilde{h} \cdot \tilde{\Lambda} \cdot r(\tilde{h}, \tilde{b}))^{\tilde{\sigma}}))^{b}]}. \tag{16}$$

Equation (16) shows that if a solution function $R(\cdot)$ of the type shown in line (13) exists, $r(\cdot, \cdot)$ must have the form

$$r(h, b) = \frac{\phi(b)}{1 - h \cdot \Lambda(b) \cdot \psi(b)}, \tag{17}$$

where

$$\phi(b) = f^1(\phi(\cdot), \psi(\cdot), b) = E\left[\left(1 + \left(\frac{\tilde{h} \cdot \tilde{\Lambda} \cdot \phi(\tilde{b})}{1 - \tilde{h} \cdot \tilde{\Lambda} \cdot \psi(\tilde{b})}\right)^{\tilde{\sigma}}\right)^{-b}\right], \tag{18}$$

$$\psi(b) = f^2(\phi(\cdot), \psi(\cdot), b) = E\left[\left(\left(\frac{1 - \tilde{h} \cdot \tilde{\Lambda} \cdot \psi(\tilde{b})}{\tilde{h} \cdot \tilde{\Lambda} \cdot \phi(\tilde{b})}\right)^{\tilde{\sigma}} + 1\right)^{b}\right]. \tag{19}$$

If we can find a solution pair $(\phi(\cdot), \psi(\cdot))$ for the latter two equations, then the functions of lines (13) and (14) will solve Eqs. (9) and (10).

The following proposition supplies sufficiency conditions for the existence of a solution to lines (18) and (19).

PROPOSITION. *Let all pairs* (h, b) *be contained in* $H \times B = [h_L, h_U] \times [b_L, b_U]$. *Let* $\sigma^* = \sigma(b_U)$, $\alpha = h_L \cdot \mathrm{Min}_{b \in B}\{\Lambda(b)\}$, *and* $\theta = h_U \cdot \mathrm{Max}_{b \in B}\{\Lambda(b)\}$. *Suppose either*

(i) $B \subset (0, 1)$ *and* $\theta < 1$, *or*

(ii) $B \subset (-\infty, 0)$ *and* $\theta \cdot [(1/\alpha)^{\sigma^*} + 1]^{-b_L} = \theta^* < 1$.

Then Eqs. (18) *and* (19) *have a solution. The solution defines a pair of functions* $(R(\cdot), S(\cdot))$ *solving Eqs.* (9) *and* (10), *and* $R(\cdot)$ *satisfies Eq.* (12).

The Appendix provides a proof.

For those cases in which a solution does exist, the Appendix's fixed-point arguments give few clues about uniqueness. In fact our method of solving Eqs. (9) and (10) does not allow us to rule out solutions with functional forms other than those of lines (13) and (14) anyway. Regardless of whether a solution pair $R(\cdot), S(\cdot))$ is unique, however, we can interpret it as a Nash equilibrium: Suppose $(R^*(\cdot), S^*(\cdot))$ satisfies (9)–(12). If $S^*(\cdot)$ will determine the saving of all future cohorts, lines (1) and (3) and the definition of $R(\cdot)$ show that $R(\cdot)$ must be uniquely determined for the present cohort. But, line (12) shows $R(\cdot) = R^*(\cdot)$.[4] Thus, $S(\cdot) = S^*(\cdot)$ will work on the left-hand side of line (10). So, if all generations except one adopt the saving function $S^*(\cdot)$, the remaining cohort cannot do bettert than to adopt $S^*(\cdot)$ for itself.[5]

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[4] Note that in terms of the notation we are using here, line (12) requires that $R^*(Q, h, b) = \sum_{i=1}^{\infty} h^i \cdot E[u(\tilde{C}_i, b)]$.

3. The Average Propensity to Save

Section 2 developed a specific class of equilibrium solutions for our economy with intergenerational preference differences. Within the class all optimal aggregate saving functions are linear. We will now compare the implied average propensity to save with that obtained from the conventional analysis outlined in Section 1.

Intergenerational variability of h and b lowers optimal saving for the *present generation if for that generation* $b > 0$, *and raises optimal saving if* $b < 0$. To prove this, for $\phi(\cdot)$ and $\psi(\cdot)$ generated in the proof of our proposition, define

$$g(z) = \left(1 - \frac{z\psi(b)}{\phi(b)}\right)^{\sigma} + z^{\sigma}. \tag{20}$$

(Note that $\phi(b) > 0$ and $\psi(b) > 0$ all b in our solutions.) Then Eq. (14) shows that

$$S(Q, h, b) = [(h\Lambda)^{\sigma}/g(\Lambda h)] \cdot Q. \tag{21}$$

Since Section 1 establishes that the optimal saving rule with a constant discount rate h and elasticity σ is $s(Q, h, b) = (h\Lambda)^{\sigma} \cdot Q$, we can compare $S(\cdot)$ and $s(\cdot)$ by comparing $g(h\Lambda)$ and 1.

Differentiating $g(\cdot)$ with respect to z we find that $g(\cdot)$ is strictly convex if $b > 0$ and strictly concave if $b < 0$. In each case $g(\cdot)$ has a single critical point, say, z^*, and

$$g(z^*) = (\phi(b)^{1/b} + \psi(b)^{1/b})^{-\sigma b}. \tag{22}$$

If $\tilde{z}$ is a random variable and $G(\cdot)$ is convex, $G(E[\tilde{z}]) \leq E[G(\tilde{x})]$. Therefore, if $\tilde{x}(\phi, \psi) = [\tilde{h} \cdot \tilde{\Lambda} \cdot \phi(\tilde{b})/(1 - \tilde{h} \cdot \tilde{\Lambda} \cdot \psi(\tilde{b}))]^{\sigma}$, we have $(E[(1/(1 + \tilde{x}))^{b}])^{1/b} \leq E[(1/(1 + \tilde{x}))^{b/b}] = E[1/(1 + \tilde{x})]$ and $(E[(\tilde{x}/(1 + \tilde{x}))^{b}])^{1/b} \leq E[\tilde{x}/(1 + \tilde{x})]$. Thus, lines (18) and (19) show that

$$\phi(b)^{1/b} + \psi(b)^{1/b} \leq E\left[\frac{1}{1 + \tilde{x}(\phi, \psi)}\right] + E\left[\frac{\tilde{x}(\phi, \psi)}{1 + \tilde{x}(\phi, \psi)}\right] = 1. \tag{23}$$

---

[5] Note that for the solutions derived in the Appendix $(R^*(\cdot), S^*(\cdot))$ will be a perfect Nash equilibrium. In fact, $(R^*(\cdot), S^*(\cdot))$ also constitutes what Blackorby *et al.* [2], Hammond [5], and Peleg and Yaari [9] call a "sophisticated equilibrium," which is a slightly stronger concept than a Nash equilibrium (see [9]).

Lines (22) and (23) show that

$$g(z^*) \geqslant (\leqslant) 1 \quad \text{for} \quad b > (<) 0. \tag{24}$$

Convexity and concavity then establish

$$g(z) \geqslant (\leqslant) 1 \quad \text{for} \quad b > (<) 0 \quad \text{all} \quad z > 0. \tag{25}$$

So,

$$S(Q, h, b) \leqslant (\geqslant) s(Q, h, b) \quad \text{if} \quad b > (<) 0. \tag{26}$$

The following is an intuitive explanation of the effects of uncertainty on the average propensity to save. The definitions of $v(\cdot)$ (see Section 1) and $V(\cdot)$ (see line (11)) show that $v(Q) \geqslant V(Q)$ all $Q > 0$: $v(Q)$ gives the indirect utility of the present generation under the ideal (for it) conditions that all future cohorts will behave just as it wants them to.[6] $V(\cdot) \leqslant v(\cdot)$ implies $R(\cdot) \leqslant v(\cdot)$ (see lines (11), (10), and (4)). But $R(\cdot)$ and $v(\cdot)$ have the same functional form with respect to Q. Thus, when $b > 0$, variability of h and b can lower $R(\cdot)$ only by making $\partial R(Q, h, b)/\partial Q \leqslant \partial v(Q, h, b)/\partial Q$, which, in turn, leads to a lower level of saving. On the other hand, if $b < 0$, $v(\cdot)$ and $R(\cdot)$ are negative so $R(\cdot) \leqslant v(\cdot)$ requires $\partial R(Q, h, b)/\partial Q \geqslant \partial v(Q, h, b)/\partial Q$, which means saving will be greater in the variable-parameter case.

4. Conclusion

Existing papers on the subject of intergenerational preference differences deal for the most part with three issues: identifying various behavioral strategies given changing preferences—see Strotz [14] and Pollak [11]; proving the existence of Nash or "sophisticated" equilibrium solutions for such models—see Phelps and Pollak [10], Peleg and Yaari [9], and Blackorby *et al.* [2]; and establishing the Pareto efficiency (or, more likely, the lack of Pareto efficiency) of such equilibria or proving the existence of overall orderings which could induce sophisticated behavior through time and which have "sensible" properties—see Phelps and Pollack [10], Blackorby *et al.* [2], Hammond [5], and Goldman [4]. The present work differs from this literature as follows. Although we study intergenerational preference differences, we assume the differences appear randomly over time. Thus, at any given time, the policymakers in our economy do not have the information-requirement burdens of, for instance, their counterparts in the Peleg-Yaari economy: when calculating equilibrium behavior in the latter situation, policymakers know the exact utility functions of all future generations; in our model, makers only need to know the present generation's utility function he distribution function for (h, b) pairs. Our framework also enables derive specific comparative static results in Section 3: we show that enerational preference variability will cause society's equilibrium ge propensity to save to decline (rise) if the present generation's elas- of marginal utility with respect to consumption—$b - 1$—is between nd 0 (is less than -1). On the other hand, as is the case in other studies above, we do not claim (or expect) that our equilibria are Pareto nt.

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[6] We could establish $v(Q) \geqslant V(Q)$ all $Q > 0$ rigorously using the inequalities of line (25).

APPENDIX

following is a proof of the proposition in Section 2.

of. Let $F(\phi(\cdot), \psi(\cdot)) = (\phi^*(\cdot), \psi^*(\cdot))$, where $\phi^*(b) = f^1(\phi(\cdot), \psi(\cdot), b)$ $^*(b) = f^2(\phi(\cdot), \psi(\cdot), b)$ (see lines (18) and (19)).

p 1. Suppose case (i). Let $\Gamma = \{(\phi(\cdot), \psi(\cdot))$: $\phi(\cdot), \psi(\cdot)$ are continuous; $\psi(b) \in [0, 1]$ all $b \in B\}$. Note that $(\phi(\cdot), \psi(\cdot)) \in \Gamma$ implies $h\Lambda(b)\psi(b) \leq$ 1 all $(h, b) \in H \times B$.
fine $\phi_L(\cdot) = [1 + (1/(1 - \theta))^{\sigma*}]^{-b}$. Let $\gamma = h_L \cdot \mathrm{Min}_{b \in B}\{\Lambda(b)\} \cdot \phi_L(b_U)$.
e $\psi_L(\cdot) = [(1/\gamma)^{\sigma*} + 1]^{-b}$. Then $\phi_L(b), \psi_L(b) > 0$ all $b \in B$. Define $\{(\phi(\cdot), \psi(\cdot)) \in \Gamma$: $\phi(b) \geq \phi_L(b)$, $\psi(b) \geq \psi_L(b)$ all $b \in B\}$. Then lines 19) show $F(\Gamma^*) \subset \Gamma^*$.

p 2. Suppose case (ii). Define $\Gamma = \{(\phi(\cdot), \psi(\cdot))$: $\phi(\cdot), \psi(\cdot)$ are contin- $h\Lambda(b)\psi(b) \leq \theta^*$ all $(h, b) \in H \times B$; and, $\phi(b)$, $\psi(b) \geq 1$ all $b \in B\}$. $\phi(\cdot), \psi(\cdot)) \in \Gamma$ and $(\phi^*(\cdot), \psi^*(\cdot)) = F(\phi(\cdot), \psi(\cdot))$. Then $\phi^*(b), \psi^*(b) \geq 1$ B (see lines (18)-(19)).
$\psi_U(b) = [(1/\alpha)^{\sigma*} + 1]^{-b}$. Then $\psi(b) \leq \psi_U(b)$ all $b \in B$ implies $\psi(b) \leq \theta^* < 1$ all $(h, b) \in H \times B$ by hypothesis (ii).
fine $\gamma^* = [1/(1 - \theta^*)]^{\sigma*}$ and $\beta = -b_L/(1 - b_L)$. Then $\beta \in (0, 1)$.
e $g(x) = 1 + \gamma^* x^{\beta}$. Then $\lim_{x \to \infty} g(x)/x = 0$. So, there exists $x^* \geq 1$ $(x^*) < x^*$. Fix such an x^*.
fine $\phi_U(b) = x^{*^{-b}}$. Define $\Gamma^* = \{(\phi(\cdot), \psi(\cdot)) \in \Gamma$: $\phi(b) \leq \phi_U(b), \psi(b) \leq$ all $b \in B\}$. Let $(\phi(\cdot), \psi(\cdot)) \in \Gamma^*$ and $(\phi^*(\cdot), \psi^*(\cdot)) = F(\phi(\cdot), \psi(\cdot))$. Then 19) shows $\psi^*(b) \leq \psi_U(b)$ all $b \in B$. Line (18) shows $\phi^*(b) \leq E[\{1 +$ $(\tilde{b}))^{\sigma}\}^{-b}] \leq \{1 + \gamma^* x^{*^{\beta}}\}^{-b} = \{g(x^*)\}^{-b} < x^{*^{-b}} = \phi_U(b)$. So, $F(\Gamma^*) \subset \Gamma^*$.

p 3. In case (i) let Γ^* be defined as in Step 1; in case (ii) let Γ^* be d as in Step 2. Let $(\phi(\cdot), \psi(\cdot)) \in \Gamma^*$. Then $\phi(b)$, $\psi(b) > 0$ all $b \in B$ $\Lambda(b)\psi(b) < 1$ all $(h, b) \in H \times B$. So, if we adopt the uniform norm on $\phi(\cdot), \psi(\cdot))$, $(\bar{\phi}(\cdot), \bar{\psi}(\cdot)) \in \Gamma^*$ implies $|(\phi(\cdot), \psi(\cdot)) - (\bar{\phi}(\cdot), \bar{\psi}(\cdot))| =$

$\text{Max}\{\sup_{b\in B} |\phi(b) - \bar{\phi}(b)|, \ \sup_{b\in B} |\psi(b) - \bar{\psi}(b)|\}$—then $F(\cdot)$ is continuous on Γ^*. Γ^* is convex. Thus, Schauder's fixed-point theorem (see Smart [13]) shows that $F(\cdot)$ will have a fixed point on Γ^* if $F(\Gamma^*)$ is contained in a compact subset of Γ^*. Each fixed point for $F(\cdot)$ defines a solutions to Eqs. (18) and (19).

Step 4. Since $(\phi(\cdot), \psi(\cdot)) \in \Gamma^*$ implies $\phi(b)$ and $1 - h\Lambda(b)\psi(b)$ are bounded away from 0 for all $(h, b) \in H \times B$, for each case (i.e., case (i) and case (ii)) there exists a finite constant δ such that $|\partial f^1(\phi(\cdot), \psi(\cdot), b)/\partial b|$, $|\partial f^2(\phi(\cdot), \psi(\cdot), b)/\partial b| < \delta$ all $(\phi(\cdot), \psi(\cdot)) \in \Gamma^*$ and $b \in B$. Then $|b - b^*| < \epsilon/\delta$ implies $|(\phi^*(b) - \phi^*(b^*)|$, $|\psi^*(b) - \psi^*(b^*)| < \epsilon$ if $(\phi(\cdot), \psi(\cdot)) \in \Gamma^*$ and $(\phi^*(\cdot), \psi^*(\cdot)) = F(\phi(\cdot), \psi(\cdot))$. So, $\Gamma^{**} = F(\Gamma^*)$ is equicontinuous. Thus, $\Gamma^{***} =$ Closure (Γ^{**}) is also. So, Ascoli's theorem (see Munkres [8]) shows Γ^{***} is compact. Thus, Step 3 shows $F(\cdot)$ has a fixed point on Γ^*.

Step 5. Let $(\phi(\cdot), \psi(\cdot)) \in \Gamma^*$ be a fixed point for $F(\cdot)$. Use lines (13), (14), and (17) to define a pair $(R(\cdot), S(\cdot))$ from $(\phi(\cdot), \psi(\cdot))$. Then $(R(\cdot), S(\cdot))$ solves Eqs. (9) and (10). In terms of the notation of line (12), Eq. (9) shows $R(Q, h, b) = \sum_{i=1}^{T} h^i \cdot E[u(\tilde{C}_i, b)] + h^{T+1} \cdot E[R(\tilde{Q}_{T+1}, \tilde{h}_{T+1}, \tilde{b}_{T+1})]$ any $T > 1$. Thus, if we can show $h^{T+1} \cdot E[R(\tilde{Q}_{T+1}, \tilde{h}_{T+1}, \tilde{b}_{T+1})] \to 0$ for each $Q > 0$ as $T \to \infty$, we will have established that $R(\cdot)$ satisfies line (12).

Suppose case (i). Then for all $(h, b) \in H \times B$, $r(h, b) \geq 0$ (see line (17)) because $\phi(b) \geq 0$ and $1 > \theta \geq \theta \cdot \psi(b)$. Let $r_U = 1/(1 - \theta)$. Then $0 \leq r(h, b) \leq r_U$ all $(h, b) \in H \times B$. Hence, $0 \leq h^{T+1} \cdot E[R(\tilde{Q}_{T+1}, \tilde{h}_{T+1}, \tilde{b}_{T+1})] \leq h^{T+1} \cdot b \cdot r_U \cdot Q^b \cdot E[(\tilde{s}_0 \cdot \tilde{s}_1 \cdot \cdots \cdot \tilde{s}_T)^b] \cdot \Lambda(b)^{T+1}$, where $\tilde{s}_i = (\tilde{h}_i \cdot \Lambda(\tilde{b}_i) \cdot r(\tilde{h}_i, \tilde{b}_i))^{\tilde{\sigma}_i}/[1 + (\tilde{h}_i \cdot \Lambda(\tilde{b}_i) \cdot r(\tilde{h}_i, \tilde{b}_i))^{\tilde{\sigma}_i}]$, $\tilde{\sigma}_i = 1/(1 - \tilde{b}_i)$ all $i = 0, \ldots, T$. Then $\tilde{s}_i \in [0, 1)$, so $0 \leq h^{T+1} \cdot E[R(\tilde{Q}_{T+1}, \tilde{h}_{T+1}, \tilde{b}_{T+1})] \leq \theta^{T+1} \cdot r_U \cdot Q^b$. But, $\theta^{T+1} \cdot r_U \cdot Q^b \to 0$ as $T \to \infty$ for any $Q > 0$.

Suppose case (ii). Then $r(h, b) \geq 0$ (see line (17)) for all $(h, b) \in H \times B$. Let $r_U = \text{Max}_{b\in B}\, \phi_U(b)/(1 - \theta^*)$. Then $r(h, b) \leq r_U$ all $(h, b) \in H \times B$. Using the definition of $\tilde{s}$, from the preceding paragraph, $0 \geq h^{T+1} \cdot E[R(\tilde{Q}_{T+1}, \tilde{h}_{T+1}, \tilde{b}_{T+1})] \geq h^{T+1} \cdot b \cdot r_U \cdot Q^b \cdot E[(\tilde{s}_0 \cdot \tilde{s}_1 \cdot \cdots \cdot \tilde{s}_T)^b] \cdot \Lambda(b)^{T+1} \geq \theta^{T+1} \cdot b \cdot r_U \cdot Q^b$. But, $|\theta^{T+1} \cdot b \cdot r_U \cdot Q^b| \to 0$ as $T \to \infty$ for each $Q > 0$. Q.E.D.

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JOURNAL OF ECONOMIC THEORY 22, 67–86 (1980)

A Single-Parameter Generalization of the Gini Indices of Inequality*

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Received March 26, 1979; revised July 1979

1. INTRODUCTION

Inequality measures are really families of indices, one index for each population size. The family of Gini indices of relative inequality (the Gini coefficient) is the most commonly used family of inequality indices. Like all relative indices, members of this family depend on income shares only. For each population size, the relative Gini index implicitly defines a class of ordinally-equivalent social-evaluation functions; social-evaluation functions represent ethical orderings of alternative income distributions. This class of Gini social-evaluation functions in turn implies a Gini index of absolute inequality[1] which depends, as do all absolute indices, on income differentials only. In this paper, we present a class of relative inequality indices characterized by a single parameter which contains the family of Gini relative indices as a special case and a corresponding class of absolute inequality measures, also characterized by a single parameter, which contains the family of Gini absolute indices as a special case.

The generalized Gini relative and absolute indices of inequality, introduced by Weymark [18], contain a subclass, the single-series Ginis. In general, families of these indices do not satisfy Dalton's [9] principle of population. This principle requires indices of inequality to be unchanged when a popula-

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* During the preparation of this article, Weymark was a visitor at the University of British Columbia.

[1] For a discussion, see Kolm [12, 13], Blackorby and Donaldson [3, 4, 5], and Blackorby, *et al.* [6].

0022-0531/80/010067-20$02.00/0

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tion is replicated, income by income.[2] We demonstrate that the only members of the single-series classes which satisfy the population principle are the relative and absolute single-parameter Ginis, the *S*-Ginis. Then, we establish the conditions which the complete class of generalized Gini inequality indices must meet in order to satisfy the principle of population.

If incomes are ranked in descending order, a generalized Gini social-evaluation function can be written as a linear function of the list of incomes, where the weights are restricted to be nondecreasing. For an *S*-Gini, the weights for a population of size m are the first m terms in an infinite sequence of numbers while the weights for a population of size n are the first n terms of the same infinite sequence. If incomes are now ranked in ascending order, it is a simple matter to rewrite the formula for any generalized Gini to reflect this change. However, for an m- and n-person *S*-Gini it will no longer be true, in general, that the new weights are drawn from a single infinite sequence. Consequently, we consider a new single-series class of Ginis based on an ascending order of incomes and we characterize the class of these indices which satisfy the principle of population.

The Gini indices have many advantages: they are easy to compute, they satisfy the principle of population, they accomodate nonpositive incomes easily (a common feature of income distribution data), the relative and absolute indices imply and are implied by the same class of ordinally-equivalent social-evaluation functions, a relative Gini is bounded by zero and one for nonnegative incomes, and they can be decomposed in a simple fashion by source of income.[3] The Atkinson [2] and Theil [17] relative indices are not defined for nonpositive incomes and the coefficient of variation may exceed one. Hence, among the currently used relative indices, the Gini family is the most useful family of indices for data with nonpositive incomes. The Kolm Pollak family of absolute indices [4, 13] performs well with nonpositive incomes; however, there is no simple family of relative indices corresponding to its family of social-evaluation functions [4]. Disadvantages of the Ginis include insensitivity (because of their linear structure) to changes in the income distribution, absence of any parameter which can be varied to incorporate alternative distributional judgements, and nonseparability (resulting in difficulties with subindices [6]). Our *S*-Ginis parameterize distributional judgments while preserving all of the Ginis' advantages.

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[2] For a discussion of indices that satisfy the principle of population, see Kurabayashi and Yatsuka [14] and Cowell [8].

[3] If the rank-order of incomes is the same for all sources of income, the absolute Gini for income from all sources is the simple sum of the absolute Ginis for each type of income while the relative Gini for income from all sources is a mean-weighted sum of the relative Ginis for each type of income, where the weights are normalized to sum to unity. See Weymark [18].

2. Gini Indices and Their Social-Evaluation Functions

An index of inequality is called ethical or exact if it implies, and is implied by, a social-evaluation function. Indices such as the Gini, Theil, Atkinson, or Kolm–Pollak are really families of indices, one index for each population size. Strictly speaking, for indices such as the Atkinson, fixing the distributional parameter results in a family of indices while considering all possible parameter values yeidls a collection of such families. To be precise, we shall refer to a list consisting of one index for each population size as a family while collections of such lists are called a class. For each family of indices we consider a corresponding family of social-evaluation functions. Letting $W^n: \mathbb{R}^n \to \mathbb{R}$, $n \in I$ be such a family,[4] the "equally distributed equivalent income," ξ, corresponding to a given income distribution $y \in \mathbb{R}^n$ is given by

$$W^n(\xi \mathbf{1}) = W^n(y) \tag{1}$$

where $\mathbf{1}$ is an n-dimensional vector of ones. Thus, ξ is that per capita income which, if distributed equally, is ethically indifferent to the given distribution. We assume that W^n is increasing along the ray of equality and that, for each y, ξ exists; all continuous increasing W^n satisfy these assumptions. Then (1) can be solved uniquely for ξ and we write

$$\xi = \Xi^n(y). \tag{2}$$

The function Ξ^n is a particular numerical representation of W^n.

The Atkinson–Kolm–Sen (AKS) [2, 11, 16] index of inequality for the social-evaluation function W^n is given by

$$\begin{aligned} I^n(y) &= 1 - \frac{\xi}{\mu^n(y)} \\ &= 1 - \frac{\Xi^n(y)}{\mu^n(y)}, \end{aligned} \tag{3}$$

where

$$\mu^n(y) = \frac{1}{n} \sum_{i=1}^{n} y_i$$

is the mean of the distribution $y \in \mathbb{R}^n$. I^n is a relative index, that is, it is homogeneous of degree zero, if and only if W^n is homothetic. As μ^n is positively linearly homogeneous, I^n is a relative index if and only if Ξ^n is positively linearly homogeneous as well.

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[4] For some indices, the domain of definition for W^n must be the nonnegative or positive orthant. $I = (1, 2, 3, \ldots)$.

Another general index of inequality [4] is given by

$$\begin{aligned} A^n(y) &= \mu^n(y) - \xi \\ &= \mu^n(y) - \Xi^n(y). \end{aligned} \tag{4}$$

It is the (dollar) difference between per capita income and the equally distributed equivalent income. A^n is an absolute index, depending on income differentials only, if and only if

$$A^n(y + \alpha\mathbf{1}) = A^n(y) \tag{5}$$

for all $\alpha \in \mathbb{R}$. A function f^n is translatable if and only if it can be written as an increasing transformation of a unit-translatable function ϕ^n. A function ϕ^n is unit-translatable if and only if

$$\phi^n(x + \alpha\mathbf{1}) = \phi^n(x) + \alpha. \tag{6}$$

From (4)–(6), it is clear that A^n is an absolute index if and only if Ξ^n is unit-translatable. Equivalently, A^n is an absolute index if and only if W^n is translatable.[5]

The Gini index of relative inequality for $y \in \mathbb{R}^n$ is

$$I_G^n(y) = \frac{1}{2n^2\mu^n(y)} \sum_{i=1}^{n} \sum_{j=1}^{n} |y_i - y_j|. \tag{7}$$

This can be rewritten [16, p. 31] as

$$\begin{aligned} I_G^n(y) &= 1 + \frac{1}{n} - \frac{2}{n^2\mu^n(y)} [\tilde{y}_1 + \cdots + i\tilde{y}_i + \cdots + n\tilde{y}_n] \\ &= 1 - \frac{(1/n^2)[\tilde{y}_1 + 3\tilde{y}_2 + \cdots + (2i-1)\,\tilde{y}_i + \cdots + (2n-1)\,\tilde{y}_n]}{\mu^n(y)}, \end{aligned} \tag{8}$$

where $\tilde{y}$ is a permutation of y such that $\tilde{y}_1 \geqslant \tilde{y}_2 \geqslant \cdots \geqslant \tilde{y}_n$. Thus, $\Xi_G^n(y)$ and $W_G^n(y)$ are given by

$$\begin{aligned} \Xi_G^n(y) &= (1/n^2)[\tilde{y}_1 + \cdots + (2i-1)\,\tilde{y}_i + \cdots + (2n-1)\,\tilde{y}_n] \\ &= \frac{\tilde{y}_1 + \cdots + (2i-1)\,\tilde{y}_i + \cdots + (2n-1)\,\tilde{y}_n}{1 + \cdots + (2i-1) + \cdots + (2n-1)} \end{aligned} \tag{9}$$

(since the sum of the first n odd numbers is n^2), and

$$W_G^n(y) = \overset{*}{W}{}^n[\tilde{y}_1 + \cdots + (2i-1)\,\tilde{y}_i + \cdots + (2n-1)\,\tilde{y}_n], \tag{10}$$

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[5] For a discussion, see Blackorby and Donaldson [4] and Blackorby, *et al.* [6].

here $\overset{*}{W}{}^n$ is an arbitrary increasing transformation. The function Ξ_G^n is ositively linearly homogeneous *and* unit translatable. Therefore, the family indices $\{A_G^1, A_G^2, \ldots\}$ given by

$$A_G^n(y) = \mu^n(y) - \frac{1}{n^2}\sum_{i=1}^{n}(2i-1)\tilde{y}_i \tag{11}$$

a family of absolute indices; it is the family of Gini absolute indices. The coefficients in (9) seem arbitrary and can be replaced by general efficients. This procedure yields a class of generalized Gini indices,[6] with ually distributed equivalent income given by

$$\begin{aligned}\Xi_{\mathscr{G}}^n(y) &= \frac{a_1^n\tilde{y}_1 + a_2^n\tilde{y}_2 + \cdots + a_n^n\tilde{y}_n}{a_1^n + a_2^n + \cdots + a_n^n}\\ &= \frac{\sum_{i=1}^n a_i^n\tilde{y}_i}{\sum_{i=1}^n a_i^n}.\end{aligned} \tag{12}$$

e assume $a_i^n > 0$, $i = 1,\ldots, n$ (the strong Pareto requirement). S-concavity $\Xi_{\mathscr{G}}^n$ (and hence $W_{\mathscr{G}}^n$), the requirement that $\Xi_{\mathscr{G}}^n$ agree with the weak renz quasi ordering, requires

$$a_1^n \leqslant a_2^n \leqslant \cdots \leqslant a_n^n \tag{13}$$

r all $n \in I$. Without loss of generality, we may set $a_1^n = 1$, $\forall n \in I$. With ese assumptions, $\Xi_{\mathscr{G}}^n$ and $W_{\mathscr{G}}^n$ are also quasi-concave. The indices $I_{\mathscr{G}}^n$ d $A_{\mathscr{G}}^n$ are defined by (3) and (4).

Let $\{\Xi_G^1, \Xi_G^2, \ldots\}$ be a family of generalized Gini equally distributed-uivalent-income functions. A natural restriction on the members of this mily is the requirement that the coefficients $a_1^1, \ldots, a_n^n$ be independent of n. other words, the family is characterized by a single nondecreasing sequence $_1, a_2, \ldots\}$ with $a_1 = 1$ and $a_i^n = a_i$, $\forall n \in I$. A family defined in this fashion a single-series Gini; the class of single-series Ginis is a subclass of the neralized Ginis. So, for a family of single-series Ginis, the equally distri-ted equivalent income is given by

$$\begin{aligned}\Xi_{\mathscr{S}}^n(y) &= \frac{a_1\tilde{y}_1 + \cdots + a_n\tilde{y}_n}{a_1 + \cdots + a_n}\\ &= \frac{\sum_{i=1}^n a_i\tilde{y}_i}{\sum_{i=1}^n a_i},\end{aligned} \tag{14}$$

$= 1$, $a_1 \leqslant a_2 \leqslant \cdots \leqslant a_i \leqslant \cdots$. The corresponding indices $I_{\mathscr{S}}^n$ and $A_{\mathscr{S}}^n$ e defined by (3) and (4). The family of Gini indices is that member of the ass of single-series Ginis with $a_i = (2i-1)$.

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[6] For a discussion, see Weymark [18].

3. GINI INDICES AND THE PRINCIPLE OF POPULATION

The principle of population for a family of indices requires that the value of the n-person index be equal to the value of the $2n$-person index *if the income distribution in the larger population is a twofold replication of the ncome distribution in the smaller population. This is clearly an important rinciple since we want to be able to use indices of inequality on sample istributions or on grouped data. Formally, we may state*

THE PRINCIPLE OF POPULATION. A family of symmetric functions $\{h^n\}$, ^{n}: $\mathbb{R}^n \to \mathbb{R}$, $n \in I$ satisfies the principle of population if and only if

$$h^{qn}\underbrace{(y,\ldots,y)}_{q} = h^n(y) \tag{15}$$

r all $q, n \in I$.

The family $\{\mu^n\}$ satisfies the principle as does $\{\Xi_G{}^n\}$; hence, both $\{I_G{}^n\}$ nd $\{A_G{}^n\}$ do as well. The families $\{\Xi_{\mathscr{G}}{}^n\}$ and $\{\Xi_{\mathscr{S}}{}^n\}$ do not, in general, tisfy the principle. Consequently, their inequality indices do not satisfy it ther. We wish to identify all families of generalized Ginis and all families of ngle-series Ginis which satisfy this principle; that is, we want to find the alues of $a_i{}^n$, $i, n \in I$, and a_i, $i \in I$, so that the principle is satisfied. ince $\{\mu^n\}$ satisfies the principle of population, the families $\{\Xi_{\mathscr{G}}{}^n\}$, $\{I_{\mathscr{G}}{}^n\}$, and $_{\mathscr{G}}{}^n\}$ (or $\{\Xi_{\mathscr{S}}{}^n\}$, $\{I_{\mathscr{S}}{}^n\}$, and $\{A_{\mathscr{S}}{}^n\}$) must either (a) all satisfy the population rinciple or (b) simultaneously fail to satisfy the population principle. To mplify the exposition, we restrict our attention to social-evaluation funcons; the extension of the analysis to families of inequality measures is raightforward.

We begin with the single-series Ginis. For the family $\{\Xi_{\mathscr{S}}{}^n\}$ we define the inction f: $\{0, I\} \to \mathbb{R}$ by[7]

$$\begin{aligned} f(m) &:= 0, & m &= 0, \\ &:= \sum_{i=1}^{m} a_i, & m &\in I. \end{aligned} \tag{16}$$

hus

$$a_i = f(i) - f(i-1) \tag{17}$$

r all $i \in I$. We show that if $\{\Xi_{\mathscr{S}}{}^n\}$ satisfies the principle of population, f ust satisfy a simple functional equation.

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[7] We adopt Eichhorn's [10] convention, $a := b$ means a is defined by the statement b.

THEOREM 1. *The family* $\{\Xi_{\mathscr{P}}^n\}$, *where*

$$\Xi_{\mathscr{P}}^n(y) = \frac{\sum_{i=1}^n a_i \tilde{y}_i}{\sum_{i=1}^n a_i} = \frac{\sum_{i=1}^n [f(i) - f(i-1)]\tilde{y}_i}{f(n)}, \tag{18}$$

$f(0) = 0$, $f(1) = 1$, *satisfies the principle of population if and only if*

$$f(qn) = f(q)f(n) \tag{19}$$

for all $q, n \in I$.

Proof. See the Appendix.

We now solve the functional equation (19) when $f(n)$ is required to satisfy the restrictions in (13).

THEOREM 2. *The function* f, *with* $f(0) = 0, f(1) = 1$, *and*

$$f(n+1) - f(n) \geq f(n) - f(n-1)^{8} \tag{20}$$

for all $n \in I$, *satisfies the equation*

$$f(qn) = f(q)f(n) \tag{19}$$

for all $q, n \in I$ *if and only if*

$$f(n) = n^{\delta}, \tag{21}$$

where $\delta \geq 1$ *is an arbitrary constant.*

Proof. See the Appendix.

The functional equation (19) is similar to the Cauchy equation $f(xy) = f(x)f(y)$, $x, y > 0$. For $f(1) = 1$ and $f(x)$ nondecreasing, its solution is $f(x) = x^{\delta}$, $\delta \geq 0$ [10, p. 22]. Moser and Lambek [15] also considered the functional equation (19) defined on the positive integers but relax (20) to the requirement that $f(n)$ be nondecreasing. For $f(1) = 1$, they establish, using techniques quite different from ours, that $f(n) = n^{\delta}$, $\delta \geq 0$, the same solution as is found for the domain $\mathbb{R}_+$. It is also of interest to consider the general solution to (19) without the restriction (20); this is presented as Lemma 1 in the Appendix.[9]

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[8] This is equivalent to (13), $a_{n+1} \geq a_n$.

[9] Moser and Lambek [15] do not consider the general solution.

The choice of the single parameter $\delta \geq 1$ in (21) is sufficient to determine a_i for all i and, thus, defines a family of single-series Ginis. This family is therefore defined by

$$\Xi_\delta^n(y) = \frac{\sum_{i=1}^n [i^\delta - (i-1)^\delta]\,\tilde{y}_i}{n^\delta} \tag{22}$$

with I_δ^n and A_δ^n defined by (3) and (4). A family derived in this fashion will be called a single-parameter Gini, an *S*-Gini.

The distributional sensitivity of Ξ_δ^n increases as a δ increases from one to plus infinity. If $\delta = 1$, Ξ_δ^n is the "utilitarian" rule with

$$\Xi_1^n(y) = \frac{1}{n}\sum_{i=1}^n \tilde{y}_i = \mu^n(y). \tag{23}$$

In this case, $I_1^n(y) = A_1^n(y) = 0$. If $\delta = 2$, $\Xi_2^n = \Xi_G^n$ and we have the Gini indices. If $\delta = \infty$,

$$\Xi_\infty^n(y) = \lim_{\delta \to \infty} \Xi_\delta^n(y) = \tilde{y}_n = \min_i\{y_i\}, \tag{24}$$

the maximin rule. The corresponding indices of inequality are

$$I_\infty^n(y) = 1 - \frac{\min_i\{y_i\}}{\mu^n(y)}, \tag{25}$$

and

$$A_\infty^n(y) = \mu^n(y) - \min_i\{y_i\}. \tag{26}$$

All of the useful properties of the Gini are inherited by the *S*-Gini. Again, nonpositive incomes are easily accomodated, for nonnegative incomes the relative index is bounded by zero and one, and inequality can be decomposed by income type if the rank-order of incomes does not vary by source of income. The only difference is the added distributional flexibility introduced by the choice of the parameter δ.

We now drop the assumption that each member of our family of indices has coefficients drawn from the same infinite sequence. For the family $\{\Xi_{\mathscr{G}}^n\}$ we define the function $g\colon I \times \{0, I\} \to \mathbb{R}$ by

$$\begin{aligned} g(n, m) &:= 0, & m &= 0, \quad n \in I, \\ &:= \sum_{i=1}^m a_i^n & m &\in I, \quad n \in I, \quad m \leq n. \end{aligned} \tag{27}$$

Thus,

$$\Xi_{\mathscr{G}}^n(y) = \frac{\sum_{i=1}^n [g(n, i) - g(n, i-1)]\,\tilde{y}_i}{g(n, n)}, \tag{28}$$

with $I_{\mathscr{G}}^n$ and $A_{\mathscr{G}}^n$ expressed analogously using (3) and (4). Since $a_1^n = 1$ for all $n \in I$, we have $g(n, 1) = 1$ for all $n \in I$.

Again, the principle of population requires g to satisfy a functional equation.

THEOREM 3. *The family* $\{\Xi_{\mathscr{G}}^n\}$, *where* $\Xi_{\mathscr{G}}^n$ *is defined in* (28), $g(n, 0) = 0$, *and* $g(n, 1) = 1$ *for all* $n \in I$, *satisfies the principle of population if and only if*

$$g(qn, qm) = g(qn, q)g(n, m) \tag{29}$$

for all $q, n \in I$ *and for all* $m \leqslant n$, $m \in \{0, I\}$.

Proof. See the Appendix.

If g is independent of its first argument, we have (19) and the S-Gini. In general, we obtain

THEOREM 4. *The function* $g\colon I \times \{0, I\} \to \mathbb{R}$, *with* $g(n, 0) = 0$ *and* $g(n, 1) = 1$ *for all* $n \in I$, *and with*

$$g(n, m+1) - g(n, m) \geqslant g(n, m) - g(n, m-1)^{10} \tag{30}$$

for all $m, n \in I$, *satisfies the equation*

$$g(qn, qm) = g(qn, q)g(n, m) \tag{29}$$

for all $q, n \in I$ *if and only if*

$$g(n, m) = \frac{\Psi(m/n)}{\Psi(1/n)}, \tag{31}$$

where Ψ *is defined on the nonnegative rational numbers with* $\Psi(0) = 0$ *and* (a) Ψ *is an arbitrary convex function*[11] *with* $\Psi(x) > 0$ *for* $x > 0$ *or* (b) Ψ *is an arbitrary concave function with* $\Psi(x) < 0$ *for* $x > 0$.

Proof. See the Appendix.

If, in (31), we pick $\Psi(x) = x^\delta$, $\delta \geqslant 1$, then

$$g(n, m) = \frac{(m/n)^\delta}{(1/n)^\delta} = m^\delta, \tag{32}$$

he S-Gini.

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[10] This is the requirement that $a_{m+1}^n \leqslant a_m^n$.

[11] For a definition of a convex function which is applicable to this domain, see the proof of Theorem 2 in the Appendix.

As another example,[12] let $\Psi(x) = e^{\alpha x} - 1$, $\alpha > 0$, so

$$g(n, m) = \frac{e^{\alpha(m/n)} - 1}{e^{\alpha/n} - 1}. \tag{33}$$

In (33) we have constructed a single-parameter class of generalized Ginis which statisfies the principle of population. For this class, $a_m^n = e^{\alpha((m-1)/n)}$ and

$$\Xi_\alpha^n(y) = \frac{\sum_{i=1}^n e^{\alpha((i-1)/n)} \tilde{y}_i}{\sum_{i=1}^n e^{\alpha((i-1)/n)}}. \tag{34}$$

As with Ξ_δ^n, the distributional sensitivity of Ξ_α^n increases as the parameter value increases. If $\alpha \to 0$,

$$\Xi_0^n(y) = \frac{1}{n} \sum_{i=1}^n \tilde{y}_i = \mu^n(y), \tag{35}$$

while if $\alpha = \infty$

$$\Xi_\infty^n(y) = \lim_{\alpha \to \infty} \Xi_\alpha^n(y) = \tilde{y}_n = \min_i \{y_i\}. \tag{36}$$

The limiting cases of $\Xi_\alpha^n(y)$ are the "utilitarian" and maximin rules, the same limits found for Ξ_δ^n. The Gini indices, however, are not members of this new class of functions. This example also illustrates the ease with which one can construct single-parameter classes of inequality measures.

It is apparent from (31) that the principle of population places strong restrictions on the coefficients of a family of generalized Gini indices. Further, since any social-evaluation function can be linearly approximated at a point by a generalized Gini, the result derived here may be of some use in characterizing the restrictions imposed by the principle of population in the general case.

4. ILLFARE-RANKED MEASURES

In constructing the class of generalized Ginis (12), incomes were ranked in nonincreasing order. We say that $\tilde{y}$ is welfare ranked. Alternatively incomes can be illfare ranked; $\hat{y}$ is a permutation of y such that $\hat{y}_1 \leqslant \hat{y}_2 \leqslant \cdots \leqslant \hat{y}_n$. Then, we can define a class of illfare-ranked generalized Gini indices, with equally distributed equivalent income given by

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[12] Charles Blackorby suggested to us that it would be useful to construct an example of a family of generalized Ginis which satisfies the population principle, but which is not also an S-Gini.

$$\begin{aligned}\hat{\Xi}_{\mathscr{G}}^{n}(y) &= \frac{b_1^{n}\hat{y}_1 + b_2^{n}\hat{y}_2 + \cdots + b_n^{n}\hat{y}_n}{b_1^{n} + b_2^{n} + \cdots + b_n^{n}} \\ &= \frac{\sum_{i=1}^{n} b_i^{n}\hat{y}_i}{\sum_{i=1}^{n} b_i^{n}}. \end{aligned} \tag{37}$$

We assume $b_i^n > 0$, $i = 1,\ldots, n$ and require

$$b_1^n \geqslant b_2^n \cdots \geqslant b_n^n \tag{38}$$

for all $n \in I$, so $\hat{\Xi}_{\mathscr{G}}^n$ is S-concave. Without loss of generality, we may set $b_1^n = 1$, $\forall n \in I$.

Now it is clear that the class of social-evaluation functions corresponding to this class of illfare-ranked generalized Ginis is identical to the class of social-evaluation functions corresponding to the class of welfare-ranked generalized Ginis considered earlier. Selecting from both classes those members which satisfy the principle of population would preserve this equality.

We may also construct families of illfare-ranked single-series Ginis. Such a family is characterized by a single nonincreasing sequence of positive numbers $\{b_1, b_2, \ldots\}$ with $b_1 = 1$. For a family of illfare-ranked Ginis, the equally distributed equivalent income is given by

$$\begin{aligned}\hat{\Xi}_{\mathscr{S}}^{n}(y) &= \frac{b_1\hat{y}_1 + \cdots + b_n\hat{y}_n}{b_1 + \cdots + b_n} \\ &= \frac{\sum_{i=1}^{n} b_i\hat{y}_i}{\sum_{i=1}^{n} b_i}, \end{aligned} \tag{39}$$

$b_i > 0$, $1 = b_1 \geqslant b_2 \geqslant \cdots \geqslant b_i \ldots$. The class of social-evaluation functions corresponding to the class of illfare-ranked S-Ginis does not correspond to the class of social-evaluation functions for the S-Ginis. Indeed, their only common member corresponds to "utilitarianism." For example, when incomes are welfare ranked the Gini is an S-Gini with weights $\{1, 3, \ldots, (2i - 1), \ldots\}$. However, if incomes are illfare ranked, for the Gini $(b_1^2, b_2^2) = (1, 1/3)$ while $(b_1^3, b_2^3, b_3^3) = (1, 3/5, 1/5)$. The Gini is an illfare-ranked generalized Gini but not an illfare-ranked S-Gini and applying the principle of population, then, will isolate a new class of measures.

For the family $\{\hat{\Xi}_{\mathscr{S}}^n\}$ we define the function h: $\{0, I\} \to \mathbb{R}$ by

$$\begin{aligned} h(m) &:= 0, & m &= 0, \\ &:= \sum_{i=1}^{m} b_i, & m &\in I. \end{aligned} \tag{40}$$

Thus

$$b_i = h(i) - h(i-1) \tag{41}$$

for all $i \in I$. As the proof of Theorem 1 does not utilize the inequality restrictions on the coefficients, one merely has to change the notation in Theorem 1 to discover the functional equation satisfied by h when $\{\hat{\Xi}_{\mathscr{S}}^n\}$ satisfies the principle of population.

We now solve this functional equation when $h(n)$ is required to satisfy the restrictions in (38), recalling the assumption that $b_i > 0$, all i.

THEOREM 5. *The function h, with h increasing,*[13] $h(0) = 0, h(1) = 1$, *and*

$$h(n+1) - h(n) \leqslant h(n) - h(n-1)^{14} \tag{42}$$

for all $n \in I$, satisfies the equation

$$h(qn) = h(q)h(n) \tag{43}$$

for all $q, n \in I$ if and only if

$$f(n) = n^\beta, \tag{44}$$

where $0 < \beta \leqslant 1$ is an arbitrary constant.

No proof of this proposition is provided as it is virtually identical to the proof of Theorem 2.[15] Choosing β determines a family of illfare-ranked S-Ginis, defined by

$$\hat{\Xi}_\beta^n(y) = \frac{\sum_{i=1}^n [i^\beta - (i-1)^\beta]\hat{y}_i}{n^\beta},$$

with $0 < \beta \leqslant 1$. Letting $\beta \to 1$ corresponds to the "utilitarian" rule while $\beta \to 0$ is the maximin rule.

The class of illfare-ranked S-Ginis has an important property—the ordering over the poor is strictly separable from (the incomes of) anyone who is richer. The social-evaluation function is, thus, strictly recursive (see [7, Chap. 6]). This property is important for the construction of ethical indices for the measurement of poverty (see [5]).

5. CONCLUSION

The generalized and single-series Ginis are simple and convenient extensions of the family of Gini social-evaluation functions and the corresponding families of inequality indices to a functional form capable of incorporating any (linear) distributional judgments. Unfortunately, families of indices which are members of these classes do not, in general, satisfy the principle of population, the requirement that exact replication will leave the value of an index of inequality (or the equally distributed equivalent income) unchanged. This important property can be imposed on these families of indices, providing a functional link between members of the same family defined for different population sizes. In the case of the single-series Ginis, the principle of population is only satisfied by families of indices which are single-parameter S-Ginis. They are characterized by a parameter δ which ranges from one to infinity. For each value of δ, the S-Gini is a separate family of (ordinal) social-evaluation functions with two families of inequality indices—a family of relative indices depending on income shares and a family of absolute indices depending on income differentials. "Utilitarianism," the ordinary Gini, and maximin are obtained as special cases by setting $\delta = 1$, 2, and ∞, respectively. Thus, subject to the restriction of its linear form, the class of S-Ginis takes on all possible distributional preferences.

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[13] This ensures $b_i > 0$. For Theorem 2 it was not necessary to assume f to be increasing, as this property is implied by the other assumptions.

[14] This is equivalent to (38), $b_n > b_{n+1}$.

[15] Specifically, h is assumed to be increasing. From (42), h is a concave function, so $0 \leqslant \beta \leqslant 1$. We eliminate $\beta = 0$ since h is increasing. The remainder of the proof is unchanged.

We hope that these new families of inequality indices and their implied families of social evaluation functions will be an important tool in the investigation of inequality. There is no index with all possible desirable properties, and that is, of course, true of the S-Ginis. They lack the convenient separability properties of the Atkinson and Kolm–Pollak indices but, on the other hand, the Atkinson index is only defined for positive incomes and the Kolm Pollak index has no corresponding relative index. Furthermore, the S-Ginis represent an improvement in ethical flexibility over the two families of Gini indices, and may be as useful as the Atkinson and Kolm–Pollak single-parameter families of indices.

APPENDIX

THEOREM 1. *The family* $\{\Xi_{\mathscr{S}}^n\}$, where

$$\Xi_{\mathscr{S}}^n(y) = \frac{\sum_{i=1}^n a_i \tilde{y}_i}{\sum_{i=1}^n a_i}$$

$$= \frac{\sum_{i=1}^n [f(i) - f(i-1)]\,\tilde{y}_i}{f(n)}, \tag{18}$$

$f(0) = 0, f(1) = 1$, *satisfies the principle of population if and only if*

$$f(qn) = f(q)f(n) \tag{19}$$

for all $q, n \in I$.

Proof. By the principle of population,

$$\Xi^{qn}(\underbrace{y,\dots,y}_{q}) = \Xi^{n}(y) \tag{A1}$$

or

$$\sum_{i=1}^{n}\left[\frac{a_{q(i-1)+1}+\cdots+a_{qi}}{f(qn)}\right]\tilde{y}_i = \sum_{i=1}^{n}\left[\frac{a_i}{f(n)}\right]\tilde{y}_i \tag{A2}$$

or

$$\sum_{i=1}^{n}\left[\frac{f(q^i)-f(q(i-1))}{f(qn)}\right]\tilde{y}_i = \sum_{i=1}^{n}\left[\frac{f(i)-f(i-1)}{f(n)}\right]\tilde{y}_i. \tag{A3}$$

This must hold for all $\tilde{y}$. Consequently, the coefficients of each $\tilde{y}_i$ must be equal. Setting $i = 1$, we have, given that $f(0) = 0$ and $f(1) = 1$,

$$f(qn) = f(q)f(n). \tag{A4}$$

This condition must hold for all $q, n \in I$.

To show sufficiency, the coefficient of $\tilde{y}_m$, $m < n$, in the left-hand side of (A3) is

$$\begin{aligned}\frac{f(qm)-f(q(m-1))}{f(qn)} &= \frac{f(q)f(m)-f(q)f(m-1)}{f(q)f(n)}\\ &= \frac{f(m)-f(m-1)}{f(n)},\end{aligned} \tag{A5}$$

the coefficient of $\tilde{y}_m$ in the right-hand side of (A3). ∎

LEMMA 1. *The function f satisfies the equation*

$$f(qn) = f(q)f(n) \tag{19}$$

for all $q, n \in I$ *if and only if*

$$f(n) = 0 \qquad \forall n \in I, \tag{A6}$$

or

$$f(n) = \prod_{i\in I} c_i^{n_i} \tag{A7}$$

where each $c_i \in \mathbb{R}$ *is an arbitrary constant and* n_i *is the number of times the* ith *prime number,* p_i, *occurs in the unique factorization of n into primes.*[16]

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[16] See Apostol [1, pp. 4–6] for a proof of the unique factorization theorem. The first prime number is 2, and thus $p_1 = 2$, $p_2 = 3$, $p_3 = 5$, and so on. We employ the convention that $c^0 = 1$ for all $c \in \mathbb{R}$.

Proof. Since $f(1) = f(1)f(1)$ by (19), $f(1) = 0$ or 1. If $f(1) = 0$, $f(n) = 0$ for all n yielding (A6). If $f(1) = 1$, write $n \in I$ as

$$n = \prod_{i \in I} p_i^{n_i}, \tag{A8}$$

where p_i is the ith prime number. Then by repeated applications of (19),

$$f(n) = \prod_{i \in I} f(p_i)^{n_i} \tag{A9}$$

Setting $f(p_i) = c_i$ yields (A7). Sufficiency is easily checked. ▌

THEOREM 2. *The function f, with $f(0) = 0$, $f(1) = 1$, and*

$$f(n+1) - f(n) \geqslant f(n) - f(n-1) \tag{20}$$

for all $n \in I$, satisfies the equation

$$f(qn) = f(q)f(n) \tag{19}$$

for all $q, n \in I$ if and only if

$$f(n) = n^{\delta}, \tag{21}$$

where $\delta \geqslant 1$ is an arbitrary constant.

Proof. Since $f(1) = 1$, we have, from Lemma 1,

$$f(n) = \prod_{i \in I} c_i^{n_i}. \tag{A7}$$

By (20), $f(n)$ is increasing so $f(n) > 0$ for all n, hence $c_i > 0$ for all $i \in I$. Writing

$$\delta_i = \frac{\ln c_i}{\ln p_i}, \tag{A10}$$

we have

$$f(n) = \prod_{i \in I} p_i^{\delta_i n_i} \tag{A11}$$

for all $n \in I$. We want to show $\delta_i = \delta$ for all $i \in I$. Suppose the contrary. Then, since I has a countable number of elements, there are two real numbers ρ, σ with $\sigma > \rho$ and a disjoint partition of $I = (I_1, I_2)$ such that

$$i \in I_1 \leftrightarrow \delta_i \leqslant \rho \tag{A12}$$

and

$$i \in I_2 \leftrightarrow \delta_i \geqslant \sigma. \tag{A13}$$

Now choose $\bar{n} > 1$ such that $\bar{n}_i = 0$, $i \in I_1$. Then

$$f(\bar{n}) \geqslant \bar{n}^{\sigma} \tag{A14}$$

and

$$f(\bar{n}^t) \geqslant \bar{n}^{t\sigma} \tag{A15}$$

for all $t \in I$. For all $i \in I_2$, no p_i is a prime factor of $(\bar{n}^t + 1)$, so

$$f(\bar{n}^t + 1) \leqslant (\bar{n}^t + 1)^{\rho}. \tag{A16}$$

Combining (A15) and (A16),

$$\begin{aligned} \frac{f(\bar{n}^t + 1)}{f(\bar{n}^t)} &\leqslant \frac{(\bar{n}^t + 1)^{\rho}}{(\bar{n}^t)^{\sigma}} \\ &= \left(\frac{\bar{n}^t + 1}{\bar{n}^t}\right)^{\rho} (\bar{n}^t)^{\rho-\sigma}. \end{aligned} \tag{A17}$$

Since $\sigma > \rho$,

$$\lim_{t \to \infty} \frac{f(\bar{n}^t + 1)}{f(\bar{n}^t)} = 0, \tag{A18}$$

contradicting increasingness of f. Hence $\delta_i = \delta$ for all $i \in I$ and

$$f(n) = \prod_{i \in I} (p_i^{n_i})^{\delta} = n^{\delta}. \tag{A19}$$

From (20), f is a convex function.[17] Hence, $\delta \geqslant 1$. ▮

THEOREM 3. *The family* $\{\Xi_{\mathscr{G}}^{n}\}$, where $\Xi_{\mathscr{G}}^{n}$ is defined in (28), $g(n, 0) = 0$, and $g(n, 1) = 1$ for all $n \in I$, satisfies the principle of population if and only if

$$g(qn, qm) = g(qn, q)g(n, m) \tag{29}$$

for all $q, n \in I$ *and for all* $m \leqslant n$, $m \in \{0, I\}$.

Proof. Using the method of Theorem 1, we require

$$\sum_{i=1}^{n} \left[\frac{g(qn, qi) - g(qn, q(i-1))}{g(qn, qn)}\right] \tilde{y}_i = \sum_{i=1}^{n} \left[\frac{g(n, i) - g(n, i-1)}{g(n, n)}\right] \tilde{y}_i \tag{A20}$$

for all $\tilde{y}$. Given that $g(n, 0) = 0$ and $g(n, 1) = 1$ for all $n \in I$, equality of the coefficients of $\tilde{y}_1$ requires

$$\frac{g(qn, q)}{g(qn, qn)} = \frac{1}{g(n, n)} \tag{A21}$$

---

[17] A function f is convex if and only if $f(\lambda x + (1 - \lambda)y) \leqslant \lambda f(x) + (1 - \lambda)f(y)$ for all x, y, $\lambda x + (1 - \lambda)y \in$ Domain f, $\lambda \in [0, 1]$.

or

$$g(qn, qn) = g(qn, q)g(n, n). \tag{A22}$$

Thus, (29) holds for all $q, n \in I$ when $m = n$. Equality of the coefficients of $\hat{y}_m$ requires

$$\frac{g(qn, qm) - g(qn, q(m-1))}{g(qn, qn)} = \frac{g(n, m) - g(n, m-1)}{g(n, n)}, \tag{A23}$$

or, using (A22),

$$g(qn, qm) - g(qn, q(m-1)) = g(qn, q)(g(n, m) - g(n, m-1)). \tag{A24}$$

Setting $m = 1$ we have (29) for $m = 1$. Now assume (29) is true for $(m-1)$ and (A24) yields (29). Therefore (29) is proved by induction. Sufficiency is easily demonstrated. ▮

THEOREM 4. *The function* $g\colon I \times \{0, I\} \to \mathbb{R}$, *with* $g(n, 0) = 0$ *and* $g(n, 1) = 1$ *for all* $n \in I$, *and with*

$$g(n, m+1) - g(n, m) \gtrless g(n, m) - g(n, m-1) \tag{30}$$

for all $m, n \in I$, *satisfies the equation*

$$g(qn, qm) = g(qn, q)g(n, m) \tag{29}$$

for all $q, n \in I$ *if and only if*

$$g(n, m) = \frac{\Psi(m/n)}{\Psi(1/n)}, \tag{31}$$

where Ψ *is defined on the nonnegative rational numbers with* $\Psi(0) = 0$ *and* (a) Ψ *is an arbitrary convex function with* $\Psi(x) > 0$ *for* $x > 0$ *or* (b) Ψ *is an arbitrary concave function with* $\Psi(x) < 0$ *for* $x > 0$.

Proof. Since $g(n, 0) = 0$ and $g(n, 1) = 1$ $\forall n \in I$, (30) implies $g(n, m) > 0$ for all $n, m \in I$. In (29), set $m = n$ obtaining

$$g(qn, qn) = g(qn, q)g(n, n). \tag{A25}$$

Since $g(n, n) > 0$, $n \in I$,

$$\begin{aligned} g(qn, q) &= \frac{g(qn, qn)}{g(n, n)} \\ &= \frac{cg(qn, qn)}{cg(n, n)} \\ &= \frac{\zeta(qn)}{\zeta(n)} \end{aligned} \tag{A26}$$

for all $n, q \in I$ where c is an arbitrary nonzero constant and $\zeta(m) := cg(m, m)$, $m \in I$. Define $\gamma = \Gamma(m, n)$ as the largest common divisor of m and n, m, $n \in I$. Then Γ is komogeneous of degree 1, $\Gamma(\lambda m, \lambda n) = \lambda\Gamma(m, n)$ for all $\lambda \in I$. From (29),

$$\begin{aligned} g(n, m) &= g\left(\gamma \frac{n}{\gamma}, \gamma \frac{m}{\gamma}\right) \\ &= g\left(\gamma \frac{n}{\gamma}, \gamma\right) g\left(\frac{n}{\gamma}, \frac{m}{\gamma}\right) \\ &= \frac{\zeta(n)}{\zeta(n/\gamma)}\, g\left(\frac{n}{\gamma}, \frac{m}{\gamma}\right) \\ &= \left(\frac{\zeta(n)}{\zeta(n/\Gamma(n, m))}\right) g\left(\frac{n}{\Gamma(n, m)}, \frac{m}{\Gamma(n, m)}\right) \qquad \text{(A27)} \end{aligned}$$

by (A26). Define the function A by

$$A(n, m) := \left(\frac{\zeta(n/\Gamma(m, n))}{g((n/\Gamma(n, m)), (m/\Gamma(n, m)))}\right), \qquad \text{(A28)}$$

n, $m \in I$. Since Γ is homogeneous of degree 1, A is homogeneous of degree zero, $A(\lambda n, \lambda m) = A(n, m)$ for all $\lambda \in I$. Furthermore

$$A(n, 1) = \frac{\zeta(n)}{g(n, 1)} = \zeta(n) \qquad \text{(A29)}$$

and

$$g(n, m) = \frac{A(n, 1)}{A(n, m)} \qquad \text{(A30)}$$

for all m, $n \in I$. Hence if g satisfies (29), then (A30) is a necessary condition.

To test for sufficiency of (A30),

$$\begin{aligned} g(qn, qm) &= \frac{A(qn, 1)}{A(qn, qm)} \\ &= \frac{A(qn, 1)}{A(qn, q)} \cdot \frac{A(n, 1)}{A(n, m)} \\ &= g(qn, q)\, g(n, m) \qquad \text{(A31)} \end{aligned}$$

for all q, $n \in I$ and all $m \lesssim n$, $m \in \{0, I\}$. Thus, (A30) is the general solution to (29). Since A is homogeneous of degree zero, we may define the function Ψ by

$$\begin{aligned} \Psi(x) &:= 0, & x &= 0, \\ &:= \frac{1}{A(n, m)}, & x &= \frac{m}{n}, \quad m, n \in I \qquad \text{(A32)} \end{aligned}$$

for all nonnegative rational x. Using (A32), (A30) may be rewritten as

$$g(n, m) = \frac{\Psi(m/n)}{\Psi(1/n)} \tag{A33}$$

for all $n, m \in I$. Since $\Psi(0) = 0$, we have $g(n, 0) = 0\ \forall n \in I$.

We now consider the sign and curvature properties of Ψ. For all $n \in I$, $\Psi(1/n) = cg(n, n)$, so $\Psi(1/n) > 0$ if and only if $c > 0$ since $g(n, n) > 0$. As $\Psi(1/n) = \Lambda(n, 1)$ and Λ is homogeneous of degree zero, $\Psi(x) > 0$ for $x > 0$ if and only if $c \gtrless 0$. Thus, for all positive rational x, $\Psi(x)$ is (a) always positive valued or (b) always negative valued.

We now demonstrate that Ψ is convex when $\Psi(x) > 0$ for $x > 0$. From (30), for all $q, n, m \in I$ and for all $p \leqslant m$, $p \in I$,

$$g(n, m + q) - g(n, m) \geqslant q[g(n, m + 1) - g(n, m)] \tag{A34}$$

and

$$g(n, m) - g(n, m - p) \leqslant p[g(n, m) - g(n, m - 1)]. \tag{A35}$$

Hence,

$$p[g(n, m + q) - g(n, m)] \geqslant q[g(n, m) - g(n, m - p)] \tag{A36}$$

or

$$pg(n, m + q) + qg(n, m - p) \geqslant (p + q)g(n, m). \tag{A37}$$

From (A33) and the fact that $\Psi(1/n) > 0$,

$$\left(\frac{p}{p + q}\right) \Psi\left(\frac{m + q}{n}\right) + \left(\frac{q}{p + q}\right) \Psi\left(\frac{m - p}{n}\right) \geqslant \Psi\left(\frac{m}{n}\right). \tag{A38}$$

Writing $(m + q) = a$ and $(m - p) = b$, $(p/p + q) = \lambda$ and $(q/p + q) = (1 - \lambda)$, we have

$$\lambda\Psi\left(\frac{a}{n}\right) + (1 - \lambda)\Psi\left(\frac{b}{n}\right) \geqslant \Psi\left(\lambda\frac{a}{n} + (1 - \lambda)\frac{b}{n}\right). \tag{A39}$$

Since m and $q \in I$ are arbitrary in (A38) and $p \leqslant m$ with $p \in I$, we may choose a, b, and λ arbitrarily in (A39) as long as $a, b \in \{0, I\}$ and λ is a rational number such that $0 < \lambda < 1$. As $n \in I$ is also arbitrary, Ψ is convex.

When $\Psi(x) < 0$ for $x > 0$, the inequalities in (A38) and (A39) are reversed, so Ψ is concave. ∎

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JOURNAL OF ECONOMIC THEORY **22**, 87–98 (1980)

Index Numbers and Consistency in Aggregation*

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Received August 10, 1978; revised June 26, 1979

This paper is an analysis of the concept of consistency in aggregation. This is the conjunction of three requirements: (i) for any given partition of the variables, two-stage aggregation is consistent, (ii) that each subaggregate and the overall aggregate have the same functional form, and (iii) that all partitions of the variables are feasible. We show that in the scalar case this results in a quasi-additive functional form. The generalization to many attributes is however shown to lead to few restrictions.

INTRODUCTION

In both empirical and theoretical economics the use of aggregates is ubiquitous. It is often useful, for both computational and analytical reasons, to be able to construct such aggregates from subaggregates, i.e., from aggregates of subsets of the relevant variables. To require that this two-stage calculation give the correct answer seems minimal indeed. In fact, several recent papers have been concerned in part with the two-stage consistency of various index number formulas. See, for example, [4, 6, 9–11].

A somewhat stronger notion has recently been proposed by Vartia [12–14] in the context of index numbers, and is called *consistency in aggregation.* Vartia's notion is really the conjunction of three logically independent requirements: (i) for any given partition of the variables, two-stage aggregation is consistent, (ii) that each subaggregate and the overall aggregate have the same functional form, and (iii) that all partitions of the variables are feasible. The first requirement is that the image of a function defined over the original set of variables must equal the image of the function defined over the values of the subaggregates, where each of the subaggregates is a function defined over some subset of the original set of variables. Requirements (ii) and (iii) are self-explanatory although the second is somewhat difficult to render precisely.

An example of an index which is consistent in aggregation is the Laspeyres

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* This research was partly supported by the Canada Council. An earlier version was presented at the University of Karlsruhe. We are indebted to them and to Erwin Diewert, Rolf Färe, Yäro Vartia, and the referee for helpful comments.

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price index. Suppose a set of consumer commodities is partitioned into two groups—"food" and "other." A Laspeyres food price subindex is a weighted average of price relatives (ratios of the current price to the base year price) where the weights are the expenditure shares of the food budget for each food commodity. A Laspeyres price subindex for "other" goods is similarly defined. A Laspeyres price index of the two Laspeyres price subindices is defined as a weighted average of the two subindices where the weights are the "food" and "other" expenditure shares of the total consumption budget. The reader may verify that a price index calculated by this two-stage procedure yields the same answer as a Laspeyres price index of the set of all prices.[1]

If a function is continuous and increasing, condition (i) *seems* to imply that the function be strictly separable[2] with respect to the given partition; the addition of condition (iii) makes it strictly separable in arbitrary partitions and hence completely additive;[3] the addition of condition (ii) renders the function (translated) quasi-additive.[4]

However, the Vartia I index [14] is *consistent in aggregation*, but is not separable in any partition of its variables. This puzzle is complicated by Diewert [6], who shows that the Vartia I index is exact[5] only for a Cobb–Douglas function which is, of course, completely additive and *consistent in aggregation*. This paper resolves the above dilemma and provides a general method for constructing indices which are consistent in aggregation. This procedure allows us to construct indices in an implicit manner which depend on attributes other than just the one over which we are aggregating. For example, we can construct an index of average family income which depends upon the size of the family, the number of employed members of the family, and the number of children and which is consistent in aggregation. In the case of price and quantity indices it is not as clear as in the above example just what should be treated as the attributes of prices and quantities. It turns out that this question is important for the consistency-in-aggregation issue. For example, using Vartia's definition [14], the Vartia I index is consistent in aggregation but the Vartia II index is not so. However, if functions of base and current period expenditure can be treated as attributes, then many indices are consistent in aggregation which were previously not thought to be so. We show—in Section 3—that this depends upon what one is allowed to interpret as an attribute. We do not pursue in this note the appropriate criteria for the choice of attributes but pursue instead the general structure of the consistency-in-aggregation problem. In Section 2 we provide necessary and sufficient conditions for an index number to be consistent in aggregation. In Section 3 we use this result to establish a rather general sufficient condition for consistency in aggregation when there are attributes associated with each observation and we present examples to show how our results apply to several well-known indices. In so doing, we examine the problems associated with choosing attributes.

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[1] See Section 3 for a formal demonstration.

[2] For the distinction between separability and strict separability see [2].

[3] See [8, Theorem 1] or [3, Theorem 4.].

[4] To be proved in Theorem 1 below. F is (translated) quasi-additive if $F(x) = \phi^{-1}/(\Sigma_{i=1}^{n} \phi(x_i) + (n - 1)b)$.

[5] See [5] for a definition of exactness.

2. SCALAR CONSISTENCY IN AGGREGATION

In this section we show that a continuous, increasing function is consistent in aggregation if and only if it is (translated) quasi-additive [1, p. 151]. In order to proceed it would seem that we first need to define the notion that the aggregator and marco functions have the same functional form. However, by first defining weak consistency in aggregation and solving the resulting functional equation, a definition of "same functional form" is automatically provided in the context of indices which are consistent in aggregation, continuous, and increasing. For ease of presentation the argument is divided into several lemmas.

Let $N = \{1, 2, \ldots, r, \ldots, R\}$ be the set of the first R positive integers, $\Omega^r = \{x \in E^r \mid x > 0\}$, and let $G_R = \{g_1, \ldots, g_R\}$ be a collection of R functions where $g_r\colon \Omega^r \to \Omega^1$ is a nonnegative real-valued function of r nonnegative real variables for each $r \in N$. Weak consistency in aggregation is first defined for $R = 3$. A nonnegative index of three nonnegative real numbers, $I_3\colon \Omega^3 \to \Omega^1$ is *weakly consistent in aggregation* if and only if there esists a collection of functions G_3 such that g_3 is symmetric and

$$\begin{aligned} I_3(x_1, x_2, x_3) &= g_3(g_1(x_1), g_1(x_2), g_1(x_3)) \\ &= g_2(g_2(x_1, x_2), g_1(x_3)) \\ &= g_2(g_1(x_1), g_2(x_2, x_3)) \end{aligned} \tag{1}$$

for all $(x_1, x_2, x_3) \in \Omega^3$.

LEMMA 1. *Assume that an index, $I_3\colon \Omega^3 \to \Omega^1$, is continuous and increasing. Then I_3 is weakly consistent in aggregation if and only if I_3 is quasi-linear, i.e., there exists a strictly monotonic function $f\colon \Omega^1 \to \Omega^1$ and constants α and β such that*

$$I_3(x) = f^{-1}(\alpha[f(x_1) + f(x_2) + f(x_3)] + \beta), \qquad \alpha > 0. \tag{2}$$

Proof. Symmetry of I_3 and the last two identities in (1) imply [3, Corollary 4.8.1; 8, Theorem 1] that there exist continuous, increasing functions $\overset{*}{g}$ and f such that

$$\begin{aligned} I_3(x_1, x_2, x_3) &= \overset{*}{g}(f(x_1) + f(x_2) + f(x_3)) \\ &= g_2(g_2(x_1, x_2), g_1(x_3)). \end{aligned} \tag{a}$$

Set $x_3 = c$ and invert g_2 in its first argument to get

$$g_2(x_1, x_2) = \overset{*}{g}_2(g_1(c), \overset{*}{g}(f(x_1) + f(x_2) + f(c)))$$
$$= \overset{**}{f}(f(x_1) + f(x_2)), \quad \text{say,} \tag{b}$$

where $\overset{**}{f}$ is defined by the last equality. Let $(x_1, x_2) = (a, b)$ and invert g_2 in (a) in its second argument to get

$$g_1(x_3) = \overset{*}{g}_1(g_2(a, b), \overset{*}{g}(f(a) + f(b) + f(x_3)))$$
$$= \overset{*}{f}(f(x_3)), \quad \text{say,} \tag{c}$$

where $\overset{*}{f}$ is defined by the last equality. Using (b) and (c) in (a) yields

$$\overset{*}{g}(f(x_1) + f(x_2) + f(x_3)) = g_2(\overset{**}{f}[f(x_1) + f(x_2)], \overset{*}{f}[f(x_3)])$$
$$= \overset{**}{f}(f(\overset{**}{f}[f(x_1) + f(x_2)]) + f(\overset{*}{f}[f(x_3)])), \tag{d}$$

where the last equality follows from using (b) again.

Since I is symmetric, permute the second and third variables in (d) to get

$$\overset{**}{f}[f(\overset{**}{f}[f(x_1) + f(x_2)]) + f(\overset{*}{f}[f(x_3)])]$$
$$= \overset{**}{f}[f(\overset{**}{f}[f(x_1) + f(x_3)]) + f(\overset{*}{f}[f(x_2)])]. \tag{e}$$

Equation (e) implies that

$$f(\overset{**}{f}[f(x_1) + f(x_2)]) - f(\overset{**}{f}[f(x_1) + f(x_3)])$$
$$= f(\overset{*}{f}[f(x_2)]) - f(\overset{*}{f}[f(x_3)]). \tag{f}$$

Equation (f) is the generalized Sincov equation whose solution is given by (see [1, Theorem 1, p. 303])

$$f(\overset{**}{f}[f(x_1) + f(x_2)]) = h(x_1) + h(x_2). \tag{g}$$

Let $x_2 = b$. Then,

$$h(x_1) = f(\overset{**}{f}[f(x_1) + f(b)]) - h(b) \equiv \phi(f(x_1)), \quad \text{say,} \tag{h}$$

where ϕ is defined by the last identity. Similarly $h(x_2) = \phi(f(x_2))$. Let $z_r = f(x_r)$, let $\bar{f} = f \circ \overset{**}{f}$ and put (h) into (g) to get

$$\bar{f}(z_1 + z_2) = \phi(z_1) + \phi(z_2).$$

This special case of Pexider's equation [1, Theorem 1, p. 142] has the solution:

$$\bar{f}(t) = at + 2d, \qquad \phi(t) = at + d.$$

Thus, $f(\overset{**}{f}(t)) = at + 2d$. Then Eq. (g) becomes

$$\begin{aligned} f(\overset{**}{f}[f(x_1) + f(x_2)]) &= \phi(f(x_1)) + \phi(f(x_2)) \\ &= a(f(x_1) + f(x_2)) + 2d \\ &= a(f(x_1) + f(x_2)) + b, \quad \text{say.} \qquad \text{(i)} \end{aligned}$$

In particular,

$$\overset{**}{f}[f(x_1) + f(x_2)] = f^{-1}(a(f(x_1) + f(x_2)) + b). \qquad \text{(j)}$$

Thus $\overset{**}{f}$ is symmetric and quasi-linear. Put (i) into (f) to get

$$a(f(x_2) - f(x_3)) = f(\overset{*}{f}[f(x_2)]) - f(\overset{*}{f}[f(x_3)]); \qquad \text{(k)}$$

letting $z_r = f(x_r)$ again yields

$$f(\overset{*}{f}(z_2)) - f(\overset{*}{f}(z_3)) = a(z_2 - z_3),$$

which has a general solution

$$f(\overset{*}{f}(t)) = at + c.$$

Solving for $\overset{*}{f}$ yields

$$\overset{*}{f}(f(x_r)) = f^{-1}(af(x_r) + c), \qquad r = 1, 2, 3. \qquad \text{(l)}$$

Putting (j) and (l) into (b) and (c) yields

$$g_2(x_1, x_2) = f^{-1}(af(x_1) + af(x_2) + b) \qquad (3)$$

and

$$g_1(x_3) = f^{-1}(af(x_3) + c). \qquad (4)$$

Substituting (3) and (4) into (a) yields

$$\begin{aligned} I_3(x_1, x_2, x_3) &= f^{-1}[af(f^{-1}[af(x_1) + af(x_2) + b]) + af(f^{-1}[af(x_3) + c]) + b] \\ &= f^{-1}(a^2 f(x_1) + a^2 f(x_2) + a^2 f(x_3) + ab + ac + b), \qquad (5) \end{aligned}$$

and hence

$$I_3(x_1, x_2, x_3) = f^{-1}(\alpha[f(x_1) + f(x_2) + f(x_3)] + \beta, \qquad \alpha \geqslant 0, \qquad (2)$$

where $\alpha = a^2$ and $\beta = ab + ac + b$, completing necessity. Conversely suppose

$$I_3(x) = f^{-1}(\alpha[f(x_1) + f(x_2) + f(x_3)] + \beta), \qquad \alpha \geqslant 0.$$

Define the functions g_1, g_2, and g_3 by

$$\begin{aligned} g_1(x_1) &= f^{-1}(af(x_1) + c) \\ g_2(x_1, x_2) &= f^{-1}(af(x_1) + af(x_2) + b) \\ g_3(x_1, x_2, x_3) &= f^{-1}(af(x_1) + af(x_2) + af(x_3) + ab + b - 2ac), \end{aligned}$$

where a, b, c, satisfy (i) $\alpha = a^2$ and (ii) $\beta = ab + ac + b$. Then

$$\begin{aligned} g_3(g_1(x_1), g_1(x_2), g_1(x_3)) &= f^{-1}\left(a \sum_{r=1}^{3} f[f^{-1}(af(x_r) + c)] + ab + b - 2ac\right) \\ &= f^{-1}\left(\sum_{r=1}^{3} a^2 f(x_r) + 3ac + ab + b - 2ac\right) \\ &= f^{-1}\left(\alpha \sum_{r=1}^{3} f(x_r) + \beta\right) \\ &= I_3(x). \end{aligned}$$

Next calculate

$$\begin{aligned} &g_2(g_2(x_1, x_2), g_1(x_3)) \\ &\quad = f^{-1}(af[f^{-1}(af(x_1) + af(x_2) + b)] + af[f^{-1}(af(x_3) + c)] + b) \\ &\quad = f^{-1}(a^2 f(x_1) + a^2 f(x_2) + a^2 f(xt) + ab + ac + b) \\ &\quad = f^{-1}(\alpha[f(x_1) + f(x_2) + f(x_3)] + \beta) \\ &\quad = I_3(x). \end{aligned}$$

A similar argument establishes that

$$g_2(g_1(x_1), g_2(x_2, x_3)) = I_3(x). \quad \blacksquare$$

The notion of weak consistency in aggregation can be slightly specialized to include a "nestedness" property. As indicated by Lemma 2 this has the effect of normalizing the function g_1 so that $g_1(x_r) = x_r$.

The index I_3 is said to exhibit *weak, nested, consistency-in-aggregation* if in addition to (1) it satisfies

$$I_3(x_1, x_2, x_3) = g_2(g_1(g_2(x_1, x_2)), g_1(g_1(x_3))), \tag{6}$$

for all $(x_1, x_2, x_3) \in \Omega^3$.

LEMMA 2. *Assume that I_3: $\Omega^3 \rightarrow \Omega^1$ is continuous and increasing in its arguments. Then I_3 exhibits* weak, nested, consistency-in-aggregation *if and only if I_3 is a (translated) quasi-additive function, i.e., there exist f: $\Omega^1 \rightarrow \Omega^1$ and a constant b (Eq. (3)) such that*

$$I_3(x_1, x_2, x_3) = f^{-1}(f(x_1) + f(x_2) + f(x_3) + 2b). \tag{7}$$

Proof. From (5) and (6) we know that

$$\begin{aligned} &f^{-1}(a^2[f(x_1) + f(x_2) + f(x_3)] + ab + ac + b) \\ &\quad = g_2(g_1(g_2(x_1, x_2)), g_1(g_1(x_3))). \end{aligned} \tag{a}$$

Substituting (3) and (4) into (a) yields

$$\begin{aligned} &f^{-1}\left(a^2\left[\sum_{r=1}^{3} f(x_r)\right] + ab + ac + b\right) \\ &\quad = f^{-1}\{a[f(f^{-1}\{af[f^{-1}(a[f(x_1) + f(x_2)] + b)] + c\}) \\ &\qquad + f(f^{-1}[af\{f^{-1}(af(x_3) + c)\} + c])] + b\} \\ &\quad = f^{-1}\left(a^3\left[\sum_{r=1}^{3} f(x_r)\right] + a^2b + 2ac + b\right). \end{aligned} \tag{b}$$

This must, however, hold for all $(x_1, x_2, x_3) \in \Omega^3$, and therefore $a^2 = a^3$ and $2b + c = 2b + 2c$. Hence $a = 1$ and $c = 0$ which yields (7). ∎

Note that this implies that (3) and (4) can be written as

$$g_2(x_1, x_2) = f^{-1}(f(x_1) + f(x_2) + b) \tag{3'}$$

and

$$g_1(x_3) = x_3. \tag{4'}$$

An index I_R: $\Omega^R \rightarrow \Omega$ exhibits *weak, nested, consistency-in-aggregation* if and only if there exists a family of function G_R such that each g_r, $r = 1,\ldots, R$, exhibits *weak, nested, consistency-in-aggregation* and

$$I_R(x_1, \ldots, x_R) = g_M(g_{n_1}(x^1), \ldots, g_{n_M}(x^M)), \tag{8}$$

where $\hat{N} = (N^1, \ldots, N^M)$ is any partition of N, the cardinality of N^m is n_m, $(x_1, \ldots, x_R) = (x^1, \ldots, x^M)$, where x^m is the n_m-tuple corresponding to N^m, and $M \leqslant \sum_{m=1}^{M} n_m = R$.

THEOREM 1. *Assume that I_R: $\Omega^R \rightarrow \Omega$ is continuous and increasing in its arguments, $R \geqslant 3$. Then, I_R exhibits* weak, nested, consistency-in-aggregation *if and only if* I_R *is a (translated) quasi-additive function, i.e., there exists* $f\colon \Omega^1 \to \Omega^1$ *and a constant b such that*

$$I_R(x_1,\dots,x_r) = f^{-1}\left(\sum_{r=1}^{R} f(x_r) + (R-1)\,b\right). \tag{9}$$

Proof. Lemmas 1 and 2 prove this for $R = 3$. Proceed by induction: Suppose the theorem holds for $R - 1 \geq 3$. Then, by hypothesis

$$\begin{aligned} I_R(x_1,\dots,x_R) &= g_2(g_1(x_1), g_{R-1}(x_2,\dots,x_R)) \\ &= f^{-1}\{f(x_1) + f[g_{R-1}(x_2,\dots,x_R)] + b\} \\ &= f^{-1}\left\{f(x_1) + f\left[f^{-1}\left(\sum_{r=2}^{R} f(x_r) + (R-2)\,b\right)\right] + b\right\} \\ &= f^{-1}\left(\sum_{r=1}^{R} f(x_r) + (R-1)\,b\right), \end{aligned}$$

which establishes necessity. To show sufficiency note that (9) can be written as in (8), where each

$$g_{n_m}(x^m) = f^{-1}\left(\sum_{r\in N^m} f(x_r) + (n_m - 1)\,b\right)$$

and

$$g_M(g_{n_1}(x^1),\dots,g_{n_M}(x^M)) = f^{-1}\left(\sum_{m=1}^{M} f(g_{n_m}(x^m)) + (M-1)\,b\right).$$

Given that *weak consistency-in-aggregation* implies *consistency-in-aggregation* and that (9) is *consistent in aggregation* we get

COROLLARY 1.1. *Under the maintained hypotheses of Theorem* 1, *an index* I_R *is* consistent in aggregation *if and only if it can be written as in* (9).

3. CONSISTENCY-IN-AGGREGATION WITH ATTRIBUTES[6]

Many price indices are weighted averages of price relatives and, as such, are not symmetric functions of the price relatives. Viewed in this way, they are not consistent in aggregation within the narrow framework of the last section. The arithmetic mean is an elementary example of this phenomenon. It is a symmetric quasi-linear function but is not consistent in aggregation since the mean of a set of numbers is not in general equal to the mean of group means. An additional piece of information is required in order to compute the overall mean using group means—the cardinality of each group. That is, each observation is associated with a single characteristic or attribute. We now generalize to the case of many attributes. For example, an index of income inequality could depend on a number of household variables, which, in addition to household income, might include location, the number of children, or the number of working adults.

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[6] This is a generalization of Vartia's [13, 14] characterization of this problem.

Suppose that each observation x_r, $r = 1,\dots,R$, is associated with a vector of A attributes, $z_{r\cdot} = (z_{r1},\dots,z_{ra},\dots,z_{rA})$. The associated matrix is

$$(Z, x) = \begin{bmatrix} z_{11} & \cdots & z_{1A} & x_1 \\ z_{R1} & \cdots & z_{RA} & x_R \end{bmatrix}.$$

The column vector of attribute a is $z_{\cdot a} = (z_{1a},\dots,z_{Ra})^T$. Again, let $\tilde{N} = \{N^1,\dots,N^M\}$ be a partition of $N = \{1,\dots,R\}$ which induces a partition of (Z, x) so that

$$(Z, x) = \begin{bmatrix} Z^1 & x^1 \\ \vdots & \vdots \\ Z^m & x^m \\ \vdots & \vdots \\ Z^M & x^M \end{bmatrix}.$$

The index of x^m for group m will depend on (Z^m, x^m) and is denoted by $I^m(Z^m, x^m)$. In addition, we wish to construct an index for each attribute in group m; these are denoted by $h^{m1},\dots,h^{ma},\dots,h^{mA}$. Then, the overall index is to be computed using each group index $I^m(Z^m, x^m)$ and its associated vector of attribute indices $(h^{m1},\dots,h^{ma},\dots,h^{mA})$.

Assume that there exists a family of quasi-additive attribute aggregators $\{H^1,\dots,H^A\}$ whose images are $h^a = H^a(z_{\cdot a}) = \phi_a^{-1}(\sum_{r=1}^{R} \phi_a(z_{ra}))$, $a = 1,\dots,A$. Each of these computes the aggregate amount of an attribute. The amount of attribute a in group m is (using quasi additivity) $h^{ma} = \phi_a^{-1}(\sum_{r\in N^m} \phi_a(z_{ra}))$. Having aggregated elements within the partition $\tilde{N}$ leaves us with a new set of observations,

$$G = \begin{bmatrix} h^{11} & \cdots & h^{1a} & \cdots & h^{1A} & I^1(Z^1, x^1) \\ \vdots & & & & & \vdots \\ h^{m1} & \cdots & h^{ma} & \cdots & h^{mA} & I^m(Z^m, z^m) \\ \vdots & & & & & \vdots \\ h^{M1} & \cdots & h^{Ma} & \cdots & h^{MA} & I^M(Z^M, x^M) \end{bmatrix},$$

where each $I^m(Z^m, x^m)$ is implicitly defined by

$$f(h^{m1},\dots,h^{mA}), I^m(Z^m, x^m)) = \sum_{r\in N^m} f(z_{r\cdot}, x_r).$$

We define an index to be consistent-in-aggregation given attributes if there exists a function f such that

$$f(h^1,\ldots,h^A, I(Z,x)) = \sum_{r=1}^{R} f(z_r, x_r),$$

$$f(\bar{h}^1,\ldots,\bar{h}^A, \hat{I}(G)) = \sum_{m=1}^{M} f(h^{m1},\ldots,h^{mA}, I^m(Z^m, x^m)),$$

and

$$I(Z,x) = \hat{I}(G),$$

where

$$\bar{h}^a = \phi_a^{-1}\left(\sum_{m=1}^{M} \phi_a(h^{ma})\right).$$

One may interpret $\bar{h}^a$ as the aggregate amount of attribute a computed from the M group aggregates, $h^{1a},\ldots,h^{Ma}$. Quasi additivity of the attribute aggregators implies that $\bar{h}^a = h^a$, $a = 1,\ldots,A$. If quasi additivity of the attribute aggregators is assumed then any function f defined on Ω^{A+1} which is increasing in its last argument will satisfy the condition of consistency-in-aggregation given attributes given in the above definition.

We now give some examples of indices which are consistent-in-aggregation given attributes. The simplest is the Laspeyres price index for which the required attribute is expenditure on the commodity (or group of commodities) and the function f is defined by $f(a,b) = ab$. The Laspeyres price index, $L = I_L(P^0, X^0, P^1, X^1)$, may be implicitly defined by

$$(P^0\cdot X^0)\,L = \sum_{r=1}^{R} (p_r^0 x_r^0)\left(\frac{p_r^1}{p_r^0}\right)$$

where $P^0\cdot X^0 = \sum_{r=1}^{R} p_r^0 x_r^0$. The Laspeyres index is an index of price relatives p_r^1/p_r^0, $r = 1,\ldots,R$. Associated with each price relative is the base period expenditure, $p_r^0 x_r^0$. The Laspeyres index, L^m, for group m is calculated implicitly by

$$(P_m^0 X_m^0)\,L^m = \sum_{r\in N^m} (p_r^0 x_r^0)\left(\frac{p_r^1}{p_r^0}\right),$$

where

$$P_m^0 X_m^0 = \sum_{r\in N^m} p_r^0 x_r^0.$$

At the second stage, $\hat{L}$ is implicitly defined by

$$(P^0\cdot X^0)\,\hat{L} = \sum_{m=1}^{M} (P_m^0 X_m^0)\,L^m,$$

where $P^0 X^0 = \sum_{m=1}^{M} P_m^0 X_m^0$. Clearly $L = \hat{L}$.

Although it seems quite natural to think of expenditure on a group as being an attribute, it is not always so clear. We illustrate this by examining the Walsh [15] and the Walsh–Vartia indices; Vartia [13, 14] has shown that the latter is consistent in aggregation while the former is not. Let

$$v_r^1 = p_r^1 x_r^1, \qquad v_r^0 = p_r^0 x_r^0, \qquad V^0 = \sum v_r^0, \qquad V^1 = \sum v_r^1.$$

The Walsh index is

$$\log I_w(P^1, P^0) = \sum_{r=1}^{R} \frac{(v_r^0 v_r^1)^{1/2}}{\sum_{r=1}^{n} (v_r^0 v_r^1)^{1/2}} \log\left(\frac{p_r^1}{p_r^0}\right)$$

and the Walsh–Vartia index is

$$\log I_{wv}(P^1, P^0) = \sum_{r=1}^{R} \frac{(v_r^0 \cdot v_r^1)^{1/2}}{(V^0 \cdot V^1)^{1/2}} \log\left(\frac{p_r^1}{p_r^0}\right).$$

We can rewrite these implicitly as

$$f_w\left(\sum_{r=1}^{R} (v_r^0 \cdot v_r^1)^{1/2}, I_w(P^1, P^0)\right) = \sum_{r=1}^{R} f_w\left((v_r^0 \cdot v_r^1)^{1/2}, \frac{p_r^1}{p_r^0}\right)$$

and

$$f_{wv}(V^0, V^1, I_{wv}(P^1, P^0)) = \sum_{r=1}^{R} f_{wv}\left(v_r^0, v_r^1, \frac{p_r^1}{p_r^0}\right),$$

where

$$f_w(a, b) = a \log b$$

and

$$f_{wv}(a, b, c) = (ab)^{1/2} \log c.$$

If the geometric mean of expenditures in base and current period is viewed as an attribute then I_w is consistent while I_{wv} is not. On the other hand if base and current expenditures are themselves attributes then I_{wv} is consistent while I_w is not. A similar distinction can be made between the Vartia I and II indices. See [14, pp. 124]. Thus, unless there is some a priori notion of how the attributes are defined, this generalized consistency-in-aggregation notion does not seem helpful.

4. CONCLUDING REMARKS

If an index is a function of n observations and if each observation is a scalar then the index satisfies the condition of scalar consistency-in-aggregation if and only if the index is a (translated) quasi-additive function of the n real variables. In this simple case, the index is completely additive and the *notion of "same functional form" is a direct consequence of the other two consistency requirements.*

However, most index number formulas in economics are more complicated than the above simple case. Even the simple Laspeyres price index falls into a more general category since it is a function of both price relatives and of an associated attribute, viz., base period commodity expenditures. We have generalized the consistency-in-aggregation definition of Vartia using the notion of attributes associated with each observation. This does not in general place many restrictions on the acceptable class of index numbers. Consistency in aggregation, however, is only one possible condition that might be imposed on an index. The addition of other desirable properties would delimit the class of acceptable indices.[7]

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[7] For example, Eichhorn [7] has shown that quasi-linearity and linear homogeneity imply a CES or Cobb–Douglas functional form.

JOURNAL OF ECONOMIC THEORY **22**, 99-106 (1980)

Notes, Comments, and Letters to the Editor

Income Distribution and Distortion of Preferences: The ℓ Commodity Case*

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Received July 28, 1978; revised May 8, 1979

1. INTRODUCTION

In an earlier paper [2] I analyzed a distortion game in which agents can misrepresent their utility functions in order to gain a strategic advantage. The underlying game is the Aumann-Kurz [1] income redistribution game in which agents pretend to have characteristics which will maximize their payoff in terms of the true utility. That is, if u_t is the true utility function of agent t, he may exhibit a utility function v_t in order to end up with a better consumption bundle relative to u_t. The strategy space U from which an agent selects his strategy v_t is defined in the following way: $v_t(x) \in U$ if

(i) $v_t(x)$ is increasing, concave, continuously differentiable at positive values of x, and continuous on the boundary,

(ii) $v_t(x) > 0$ for all $x \geq 0$, $x \neq 0$,

(iii) $v_t(0) = 0$ for all t.

Now restricting x to be a one-dimensional real variable which is interpreted as "income," we have the following result:

THEOREM (Kurz [2]). *If x is one dimensional then $v_t^*(x) = \gamma x$, $\gamma > 0$ for all t, is the unique dominant strategy in U resulting in a unique Nash Equilibrium in the distortion game. The implied income tax function is linear with a marginal tax rate of* 50%.

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* This work was supported by the National Science Foundation Grant SOC75-21820 at the Institute for Mathematical Studies in the Social Sciences, Stanford University. I wish to thank Professor Sergiu Hart of Stanford University and my students Haruo Imai, Martin Osborne, and Akira Tomioka for stimulating conversations on the subject of this paper.

0022-0531/80/010099-08$02.00/0

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The generalizability of this result was left open. Thus, suppose the economy has l commodities with u_t defined over R_+^l; what are the strategies in U that would constitute a Nash equilibrium in the distortion game? Also, what are the characteristics of the equilibrium strategies? The results here are as follows:

(i) In l dimensions the distortion game can be defined in two different ways.

(ii) The dominant strategy of all agents in the distortion game is to select *appropriate* utility functions which are always homogeneous of degree 1 in commodity bundles and result in indirect utility functions which are linear in income for each fixed commodity price vector p.

(iii) The implied income tax remains linear with a marginal rate of 50%.

2. THE MODEL

A market M consists of:

(i) A measurable space $(T, \mathscr{C})$ of agents with a σ-additive, nonnegative measure μ on $\mathscr{C}$ with $\mu(T) = 1$ (the population measure).

(ii) An integrable function $\mathbf{e}$ from T, the set of agents, to Ω, the consumption set which is taken to be R_+^l. ($\mathbf{e}(t)$ is the initial endowment density of t.) We shall assume $\mathbf{e}(t) \leq e$ for some vector e for all t.

(iii) For each t a utility function on Ω where $u_t \in U$, the admissible set.

Given a market as above, Aumann and Kurz [1] introduce a strategic game $\Gamma(M)$ called a redistribution game which is defined as follows: T, $\mathscr{C}$, and μ are the same as in the market but the strategy spaces are defined with respect to coalitions. Thus for any coalition S there exists a strategy space X^S and for each strategy $\sigma \in X^S$ and τ in $X^{T/S}$ there exist payoffs to S and $T\backslash S$. In this game critical assumptions are made about certain strategies contained in the sets X^S. These can now be stated:

(i) If S is a majority with (i.e., $\mu(S) > \frac{1}{2}$) then there exists $\sigma \in X^S$ which will threaten to expropriate any part of the endowment of members of the minority coalition $T\backslash S$.

(ii) The counterthreat of the minority is to be found in the minority's ability to "strike" against the majority and refuse to make its endowment abailable for taxation by simply destroying it.

The bargaining between any majority and miiority coalitions results in a compromise defined by the Nash bargaining solution and leads to a valuation of each coalition S. Aumann and Kurz [1] then select the Harsanyi–Shapley nontransferable utility value as a solution concept. They show that the characterization of the solution $\mathbf{x}$ can be made in two different ways:

a. *Commodity Redistribution Scheme*

There exists a pair (λ, p) of a measurable function λ from $(T, \mathscr{C})$ to R_+ and a price vector p such that $\mathbf{x}$ is a solution to the redistribution game if and only if it is efficient with (λ, p) as the associated efficiency pair and

$$\lambda(t)\, u_t(\mathbf{x}(t)) - \int_T \lambda(\tau)\, u_\tau(\mathbf{x}(\tau))\, \mu(d\tau) = p(\mathbf{e}(t) - \mathbf{x}(t)). \tag{1}$$

b. *Income Redistribution Scheme*

There exists a price system p and an income redistribution scheme $\mathbf{y}$ such that if u_t^p is the indirect utility function of agent t at price p then

$$\frac{u_t^p(\mathbf{y}(t))}{u_t^{p\prime}(\mathbf{y}(t))} + \mathbf{y}(t) = p\mathbf{e}(t) + \int_T \frac{u_\tau^p(\mathbf{y}(\tau))}{u_\tau^{p\prime}(\mathbf{y}(\tau))}\, \mu(d\tau). \tag{2a}$$

(2b) $(p, \mathbf{x})$ is a competitive equilibrium relative to the distribution $\mathbf{y}$. Theorem B in [1] shows that the two schemes result in the same allocation. In the development here I use this theorem in an essential way.[1]

I turn now to the distortion game. When an agent selects the strategy $v_t \in U$ it results in two possible outcomes depending upon which scheme one considers: (i) an Aumann-Kurz Commodity vector $\mathbf{x}(t)$ satisfying (1) with v_t replacing u_t, (ii) an income $\mathbf{y}(t)$ relative to the equilibrium price p and a vector $\mathbf{z}(t)$ which is optimal consumer allocation.

If there is no distortion of preferences then by Theorem B of Aumann and Kurz [1] it follows that $\mathbf{z}(t) = \mathbf{x}(t)$. However, given that the main thrust of the distortion game in a market economy is to gain income, one way of defining the distortion game is to let the strategy v_t determine the income redistribution $\mathbf{y}$ but then permit agents to select their consumption bundles $\mathbf{z}(t)$ to be optimal relative to the true u_t rather than the distorted v_t.

The opposite view insists that in selecting a false utility function v_t an agent should be required to select his demand to be consistent with v_t and the associated budget constraint. The argument in support of this view is based on the idea that the opponents can "audit" the consumption pattern of the agent and by inference decide if the strategy v_t is credible.

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[1] One notes that in [1] condition (2b) is required to hold a.e. In the present paper I assume this to hold for all t.

The distinction made above gives rise to different games: A distortion game *without consumption audit* and a game *with consumption audit*. I give formal definitions now.

A distortion game consists of a measurable space $(T, \mathscr{C})$ with a population measure μ as in the market M. The strategy spaces of the agents are X^t and they are all the same:

$$X^t = U \qquad \text{all} \quad t \in T.$$

To define the payoff of the game let the *true* demand correspondence of agent t be defined by

$$\gamma_t(p, y) = \{x \in \Omega \mid x \text{ maximizes } u_t(x) \text{ s.t. } px \leq y\} \tag{3}$$

and the distorted demand correspondence be defined by

$$\gamma_t^v(p, y) = \{x \in \Omega \mid x \text{ maximizes } v_t(x) \text{ s.t. } px \leq y\}. \tag{4}$$

Obviously $\gamma_t(p, y) = \gamma_t^u(p, y)$.

The final allocation $\mathbf{x}$ *in a distortion game with an audit is defined* in terms of an equilibrium price system p^* and income distribution scheme $\mathbf{y}^*$ such that if $v_t \in U$ is a strategy of t then

$$\mathbf{x}(t) \in \gamma_t^v(p^*, \mathbf{y}^*(t)), \tag{5a}$$

$$\frac{v_t^p(\mathbf{y}^*(t))}{v_t^{p'}(\mathbf{y}^*(t))} + \mathbf{y}^*(t) = p^*\mathbf{e}(t) + \int_T \frac{v_\tau^p(\mathbf{y}^*(\tau))}{v_\tau^{p'}(\mathbf{y}^*(\tau))}\, \mu(d\tau), \tag{5b}$$

$$\int_T \mathbf{x}(\tau)\, \mu(d\tau) = \int_T \mathbf{e}(\tau)\, \mu(d\tau). \tag{5c}$$

The final allocation $\mathbf{x}$ *in a distortion game without an audit is* defined in terms of an equilibrium price vector p^* and income distribution scheme $\mathbf{y}^*$ such that if $v_t \in U$ is a strategy of t then

$$\mathbf{x}(t) \in \gamma_t(p^*, \mathbf{y}^*(t)), \tag{6a}$$

$$\frac{v_t^p(\mathbf{y}^*(t))}{v_t^{p'}(\mathbf{y}^*(t))} + \mathbf{y}^*(t) = p^*\mathbf{e}(t) + \int_T \frac{v_\tau^p(\mathbf{y}^*(\tau))}{v_\tau^{p'}(\mathbf{y}^*(\tau))}\, \mu(d\tau), \tag{6b}$$

$$\int_T \mathbf{x}(\tau)\, \mu(d\tau) = \int_T \mathbf{e}(\tau)\, \mu(d\tau). \tag{6c}$$

The difference between (5a)–(5c) and (6a)–(6c) is found in (5a) and (6a).

Note that for any set of strategies $v_t \in U$, $t \in T$ an allocation $\mathbf{x}$ can result from the application of the Aumann–Kurz Commodity Redistribution scheme defined earlier without reference to prices or markets. However, since this redistribution is attained relative to the false utility functions v_t, it can be shown that it results in an allocation which is equivalent to the one in the distortion game with an audit. The proof of this result is omitted.

3. NASH EQUILIBRIA

The two theorems here will cover the two cases presented in Section 2.

THEOREM 1. *Let the distortion game be played without an audit. Then* $v_t = H_t$ *is a dominant strategy leading to a Nash equilibrium in the distortion game, where* H_t *is an arbitrary function which is homogeneous of degree* 1 *with* $H_t \subset U$ *all* $t \in T$.

Proof. Let v_t be a Nash equilibrium strategy in the distortion game without an audit. Since there is no audit, consider only the income tax equilibrium. To do this let v_t^p be the indirect utility function induced by v_t. By [1, 9.16] and the definition of equilibrium, a solution $(\mathbf{x}^*, p^*, \mathbf{y}^*)$ must satisfy

$$\frac{v_t^p(\mathbf{y}^*(t))}{v_t^{p\prime}(\mathbf{y}^*(t))} + \mathbf{y}^*(t) = p^*\mathbf{e}(t) + \int_T \frac{v_\tau^p(\mathbf{y}^*(\tau))}{v_\tau^{p\prime}(\mathbf{y}^*(\tau))}\,\mu(d\tau). \tag{7a}$$

$$\mathbf{x}^*(t) \in \gamma_t(p^*, \mathbf{y}^*(t)), \qquad t \in T, \tag{7b}$$

$$\int_T \mathbf{x}^*(\tau)\,\mu(d\tau) = \int_T \mathbf{e}(\tau)\,\mu(d\tau). \tag{7c}$$

But since v_t is a Nash equilibrium in the distortion game it follows from the "no audit" provision that v_t^p is a Nash equilibrium strategy in the one-dimensional distortion game. This means that the basic theorem of Kurz [2] applies and one can conclude that in the one-dimensional game the dominant strategy in a Nash equilibrium calls for a linear indirect utility function, i.e.,

$$v_t^p(y) = h(p)y, \qquad h(p) > 0.$$

Note now that indirect utility functions are homogeneous of degree 0 in prices and income, so that

$$h(kp)\,ky = h(p)y \qquad \text{for all} \quad k > 0.$$

Thus

$$h(kp) = k^{-1}h(p).$$

We thus find that $h(p)$ is homogeneous of degree -1. It follows from [3, Theorem 1] that

$$v_t(x) = H_t(x)$$

for some function H_t which is homogeneous of degree 1. The equilibrium price vector p^* is not unique but the Nash equilibrium strategires are independent of the value of p^*.

COROLLARY 1. *In a Nash equilibrium of the distortion game without audit, the income tax function is linear with a marginal tax rate of* 50%.

Proof. If $v_t = H_t$ is homogeneous of degree 1 then the indirect utility function is $h(p)y$ and therefore

$$\frac{v_t{}^p(\mathrm{y}^*(t))}{v_t^{p'}(\mathrm{y}^*(t))} = \frac{h_t(p)\,\mathrm{y}^*(t)}{h_t(p)} = \mathrm{y}^*(t).$$

On the other hand (6b) implies that

$$2\mathrm{y}^*(t) = p^*\mathrm{e}(t) + \int_T \mathrm{y}^*(\tau)\,\mu(d\tau) = p^*(\mathrm{e}(t) + \bar{e})$$

or

$$\mathrm{y}^*(t) = \tfrac{1}{2}p^*(\mathrm{e}(t) + \bar{\mathrm{e}}),$$

and since the tax density of t is $\mathrm{Tax}_t = p^*\mathrm{e}(t) - \mathrm{y}^*(t)$ we have

$$\mathrm{Tax}_t = \tfrac{1}{2}p^*\mathrm{e}(t) - \tfrac{1}{2}p^*\bar{\mathrm{e}}$$

and thus the marginal tax rate on endowment income is 50%.

I turn now to the analysis of equilibrium with audit. I shall use below the notation $(p^*, \mathrm{x}^*, \mathrm{y}^*)$ for price vector and commodity and income allocations resulting from the distortion game *without* audit.

Now define a "homogeneous fit of u_t at x^*" in the following way:

$$H_t^{p^*} = \begin{cases} \text{A function in } U \text{ which satisfies} \\ (1)\quad H_t^{p^*} \text{ is homogeneous of degree 1,} \\ (2)\quad \nabla H_t^{p^*}\big|_{x=x^*} = \nabla u_t\big|_{x=x^*}. \end{cases}$$

Figure 1 shows how to construct $H_t^{p^*}$. At $\mathrm{x}^*(t)$ the functions u_t and $H_t^{p^*}$ are tangent and $H_t^{p^*}$ represents homothetic preferences. However, at z the gradients of $H_t^{p^*}$ and u_t are not the same since u_t may not represent homothetic preferences. Note also that it follows from the construction that at $(p^*, y^*(t))$ the demands $x^*(t)$ induced by u_t and $H_t^{p^*}$ are the same. Thus

$$x^*(t) \in \gamma_t(p^*, \mathrm{y}^*(t)) = \gamma_t^{H^{p^*}}(p^*, \mathrm{y}^*(t)).$$

THEOREM 2. *Let the distortion game be played with an audit. Then* $v_t = H_t^{p^*}$ *is a dominant strategy leading to a Nash equilibrium where* $H_t^{p^*}$ *is a homogeneous fit of* u_t *at* $\mathbf{x}^*(t)$.

Proof. Since $H_t^{p^*}$, a function in U, is homogeneous of degree 1 it follows from Theorem 1 that if equilibrium prices p^* prevail then the outcome of the income redistribution game is $\mathbf{y}^*$. However, since at $(p^*, \mathbf{y}^*(t))$ by construction $\mathbf{x}^*(t) \in \gamma_t^{H^{p^*}}(p^*, \mathbf{y}^*(t))$ it follows that $(p^*, \mathbf{y}^*, \mathbf{x}^*)$ does constitute a competitive equilibrium.

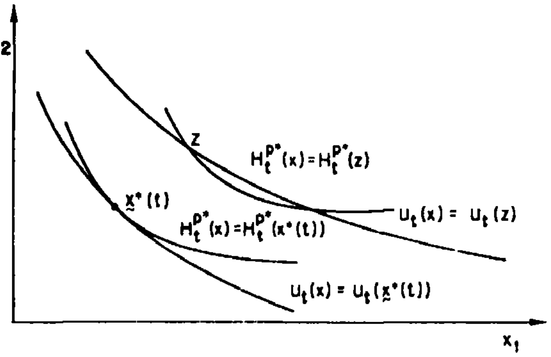
x_2
z
$H_t^{p^*}(x) = H_t^{p^*}(z)$
$\underset{\sim}{x}^*(t)$
$H_t^{p^*}(x) = H_t^{p^*}(x^*(t))$
$u_t(x) = u_t(z)$
$u_t(x) = u_t(\underset{\sim}{x}^*(t))$
x_1

FIGURE 1.

COROLLARY 2. *In a Nash equilibrium of the distortion game with audit the income tax function is linear with a marginal tax rate of* 50%.

Proof. The same as in Corollary 1 since the indirect utility functions are linear in income in both games.

4. FINAL REMARK

In my earlier paper [2] I argued that the main interest in the distortion model is to be found in its striking prediction that all agents will distort their preferences in the same way and thus the tax function will apply uniformly to all. The two theorems presented in this paper demonstrate that the Nash equilibrium strategy of exhibiting linear utility functions *does not* hold in the l-dimensional case but the results regarding the equilibrium tax functions *do* hold in the multidimensional case. This I find as an important supporting evidence in favor of the proposed theory. It appears to me that revelation of preferences in social bargaining rarely occurs and whenever revelation matters, the bargaining results in the distortion of preferences. Here I have exploited the distortion to explain why income tax schedules are uniform rather than personal.

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JOURNAL OF ECONOMIC THEORY 22, 107-112 (1980)

Keynesian Multipliers in Temporary Equilibrium with Consumer Credit Rationing

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Received January 1, 1979; Revised June 5, 1979

I

The simplest version of the multiplier shows changes in income and output that follow changes in, say, government purchases. A feature of the literature is that as the level of complexity increases, the multiplier is usually, though not always, lowered. For example, when the price of output is allowed to vary freely, Dixit [1] has shown that the output multiplier may not necessarily exceed unity. He writes, "The problem is that the increased government demand raises the price of commodities, and this decreases the quantity demanded by the consumers" [1, p. 252].

On the other hand, consumer credit rationing has almost always been ignored (for an exception, see Flemming [2]). Consumer credit rationing has the important feature that, unlike the life-cycle planner of the Dixit model (see Section II), the credit-rationed consumer behaves myopically. This type of behavior affects the size of various multipliers and therefore ought to be incorporated into the analysis.

In this paper, consumer credit rationing is incorporated into the analysis by adapting the temporary equilibrium model employed by Dixit. We introduce into the analysis a consumer who is credit rationed and examine his plans together with those of other agents. This examination of micro-behavior that is consistent with temporary equilibrium is carried out in Section II whereas in Section III the multipliers are derived and are compared with those originally obtained by Dixit. The last section contains a few concluding remarks.

II

In Dixit's model, there is one representative consumer, one representative firm, and a government. There are two periods and a temporary equilibrum

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is assumed to exist for the first period. Agents therefore, form expectations concerning the state of second-period markets and these expectations are held with subjective certainty. The goods market clears by price adjustment and the labor market clears by rationing.

We shall consider an economy with a single consumer, one firm, and a government, but in two possible states, the one without and the other with consumer credit rationing. We shall then compare the two states.

Micro behavior: The Dixit Consumer

The Dixit Consumer is endowed with m_{11} stocks of money and his employment is constrained at e_{11} in the first period and is expected to be constrained at e_{21} in the second period (the first subscript denotes the time period and the second denotes consumer type, 1 being the Dixit Consumer and 2 the credit-rationed consumer). He faces price q_1 for consumption good x and after tax wage rate v_1 for labor e in the first period and prices q_2 and v_2, respectively, in the second period. We shall retain the Dixit assumption that the rate of interest is equal to zero and also the assumption that m_{11}, v_1e_{11}, and v_2e_{21} are such that he expects $(m_{11} + v_1e_{11}) > v_2e_{21}$. Assuming that the direct utility function, u, is strictly concave, the consumer saves a positive amount in the first period for second-period consumption purposes. The consumer's utility level and his first-period demand for commodities is also defined by the expenditure function,

$$E(q_1, q_2, e_{11}, e_{21}, u) = m_{11} + v_1e_{11} + v_2e_{21}. \tag{1}$$

Then the first-period consumption, x_{11}, is given by

$$x_{11} = E_1(q_1, q_2, e_{11}, e_{21}, u), \tag{2}$$

where E_i denotes the partial derivative of E with respect to its ith argument, $i = 1, 2, \ldots, 5$. Total differentiation of (1) and (2) yields

$$dx_{11} = -A_{11}\, dq_1 + B_{11}\, de_{11} + \Gamma_{11}\, dv_1 + \Delta_{11}\, dm_{11}, \tag{3}$$

where

$$-A_{11} = \left(E_{11} - E_1 \frac{E_{15}}{E_5}\right) + \left(E_{12} - E_2 \frac{E_{15}}{E_5}\right) q_2'(q_1), \tag{4}$$

$$B_{11} = \left(E_{13} + (v_1 - E_3) \frac{E_{15}}{E_5}\right) + \left(E_{14} + (v_2 - E_4) \frac{E_{15}}{E_5}\right) e_{21}'(e_{11}), \tag{5}$$

$$\Gamma_{11} = \frac{E_{15}}{E_5} (e_{11} + e_{21} v_2'(v_1)), \tag{6}$$

$$\Delta_{11} = \frac{E_{15}}{E_5}. \tag{7}$$

The derivatives $q_2'(q_1)$, $e_{21}'(e_{11})$, and $v_2'(v_1)$ denote the assumption that expectations are based on respective current market experiences. The Dixit assumptions on signs of various terms are $\Delta > 0$, $\Gamma_{11} > 0$, $E_{11} \leqslant 0$, $E_{12} < 0$, $E_{13} > 0$, $E_{14} > 0$, $(v_1 - E_3) > 0$, $(v_2 - E_4) > 0$, $B_{11} > 0$, $A_{11} > 0$, $q_2(q_1) \geqslant 0$, $e_{21}'(e_{11}) \geqslant 0$, and $v_2(v_1) \geqslant 0$.

Microbehavior: The Credit-Rationed Consumer

We assume that the preferences of this consumer are the same as that of the Dixit consumer and also that they both expect the same present-value income (earned income plus money holdings). We shall assume that this consumer borrows in order to smooth out his consumption and given the rest of the assumptions of the model, this is tantamount to assuming that $e_{12} < e_{11}$, $e_{22} > e_{21}$ and $m_{12} < m_{11}$. As a crude approximation to the real-life feature that lending to consumers is usually myopic, we shall write the debt limit that consumer 2 faces as

$$\alpha v_1 e_{12}, \qquad \alpha \leqslant 0, \quad \alpha, \text{ a constant,} \tag{8}$$

so that the debt limit depends on current earned income only. More realistic debt limits can be introduced but the one we have adopted is simple and captures the feature that lending is virtually not carried out with reference to present-value income.

The first-period demand of consumer 2 is given by

$$q_1 x_{12} = v_1 e_{12}(1 - \alpha) + m_{12}, \tag{9}$$

so that

$$dx_{12} = -A_{12}\,dq_1 + B_{12}\,de_{12} + \Gamma_{12}\,dv_1 + \Delta_{12}\,dm_{12} - C_{12}\,d\alpha \tag{10}$$

where

$$\begin{gathered} -A_{12} = -\frac{x_{12}}{q_1}, \qquad B_{12} = \frac{v_1(1-\alpha)}{q_1}, \qquad \Gamma_{12} = \frac{e_{12}(1-\alpha)}{q_1}, \\ \Delta_{12} = \frac{1}{q_1}, \qquad -C_{12} = -\frac{v_1 e_{12}}{q_1}. \end{gathered} \tag{11}$$

Obviously, A_{12}, B_{12}, Γ_{12}, Δ_{12}, and C_{12} are all positive.

Microbehavior: Firms and the Government

There is one representative firm. We shall adhere to the Dixit assumption that if the first-period price of the consumption good increases, then the resultant changes in the firm's demand for this input is given by $-\Phi < 0$, and in the demand for labor by $\psi > 0$. Finally, the government, among other activities, purchases amount x_g of commodities and l_g of labor in the first period markets.

Equilibrium

In state 1 (without credit rationing) the equilibrium conditions are

$$p_1 = q_1 - t_1, \tag{12}$$

$$v_1 = w_1 - r_1, \tag{13}$$

$$x_{11} + x_g + y_1 = Y_1, \tag{14}$$

$$e_{11} = l_{11} + l_{g1}, \tag{15}$$

where p_1 is the price faced by the producer for commodities in period 1, t_1 is the tax rate on commodities in period 1, w_1 is the wage rate paid to labor by the firm in the first period, r_1 is the tax rate on labor, Y_1 is the commodities endowment of the producer, and y_1 is the commodities currently demanded by the producer, l_{11} is the producer's current labor demand and l_g is given by l_{g1}.

In state 2 (with credit rationing), the equilibrium conditions are (12), (13), and

$$x_{12} + x_g + y_1 = Y_1, \tag{16}$$

$$e_{12} = l_{12} + l_{g2}, \tag{17}$$

where the producer's current labor demand is given by l_{12} and l_g is given by l_{g2}.

III

In order to draw meaningful comparisons and isolate the role of credit rationing, we follow Dixit [1, pp. 250-251] and derive the following multipliers:

In State 1 (without credit rationing),

$$de_{11} = \frac{A_{11} + \Phi}{A_{11} + \Phi - \Psi B_{11}}, \tag{18}$$

$$d(x_{11} + x_g) = \frac{\Phi}{A_{11} + \Phi - \Psi B_{11}}, \tag{19}$$

and in State 2 (with credit rationing),

$$de_{12} = \frac{A_{12} + \Phi}{A_{12} + \Phi - \Psi B_{12}}, \tag{20}$$

$$d(x_{12} + x_g) = \frac{\Phi}{A_{12} + \Phi - \Psi B_{12}}. \tag{21}$$

Clearly, therefore, the difference that credit rationing makes to the multipliers depends on the differences between B_{11} and B_{12}, and A_{11} and A_{12}, and if $A_{11} > A_{12}$ and $B_{11} < B_{12}$ then (20) and (21) are respectively greater than (18) and (19).

First, in order to show that $A_{12} < A_{11}$, from (4) and (11) we need to demonstrate that

$$x_{11}\frac{E_{15}}{E_5} - \frac{x_{12}}{q_1} > E_{11} + \left(E_{12} - E_2\frac{E_{15}}{E_5}\right) q_2'(q_1).$$

But since each of the two terms on the right-hand side is nonpositve, $x_{11} > (x_{12}/q_1)(E_5/E_{15})$ will suffice. This implies that we require $x_{11} > x_{12}$ as by homogeneity of E and $E_{25} > 0$ we have $q_1E_{15} < E_5$. Now given our assumption that present-value incomes are the same for both types of consumers, it must be true that $x_{11} > x_{12}$. Thus we have good reasons to assert that $A_{12} < A_{11}$. Also note that as α increases, $x_{11} - x_{12}$ becomes larger as well.

In order to show that $B_{12} > B_{11}$, from (5) and (11) we need to demonstrate that

$$\frac{v_1(1-\alpha)}{q_1} > \left(E_{13} + (v_1 - E_3)\frac{E_{15}}{E_5}\right) + \left(E_{14} + (v_2 - E_4)\frac{E_{15}}{E_5}\right) e_{21}'(e_{11}).$$

When $e_{21}'(e_{11}) = 0$, using $q_1E_{15} < E_5$, our requirement becomes

$$(1-\alpha)\, v_1\frac{E_{15}}{E_5} > \left(E_{13} + (v_1 - E_3)\frac{E_{15}}{E_5}\right).$$

But since $\alpha \leqslant 0$, $E_{13} < E_3(E_{15}/E_5)$ will suffice. In other words, if the substitution effect of e_{11} on x_{11} is smaller than the income effect and the elasticity of work expectations is zero then $B_{12} > B_{11}$. However, note that as the elasticity of work expectations rises, the positive difference, $B_{12} - B_{11}$, becomes smaller.

Thus, provided the economically meaningful conditions that we have isolated are satisfied, $A_{12} < A_{11}$ and $B_{12} > B_{11}$ so that credit rationing increases the Keynesian output and employment multipliers.

IV

Dixit has integrated the general equilibrium with the income expenditure approach in order to study the theory of public finance. However, his model abstracts from a number of real-life features, incorporation of some of which must be a priority task. In this paper, we have introduced consumer credit rationing into the analysis and shown how this affects the size of the two multipliers. In particular, we have isolated the conditions under which the multipliers turn out to be larger than the original Dixit multipliers. Thus we hope to have given the model another element of "Keynesianism" and thereby to have complemented the work of Dixit.

ACKNOWLEDGMENT

An associate editor of this Journal has made valuable comments and his help is gratefully acknowledged. I am, of course, solely responsible for any remaining errors.

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JOURNAL OF ECONOMIC THEORY 22, 113-114 (1980)

Utility Theory in Unfair Lotteries

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Received June 4, 1979

Readers of Patel and Subrahmanyam's recent note [4] may be interested in two articles apparently unknown to them. While the first duplicates their note's thesis entirely and adds a bit more, the second is a significant extension.

Kwang's article [3], though written 13 years earlier, repeats Patel and Subrahmanyam's theorem that indivisibilities can lead to convexity in the utility of wealth function even when the utility function is concave in commodity space. Additionally, he extends this analysis by suggesting that the greater prevalence of gambling among the lower classes may be due to the possibility that they face more indivisibilities with their low incomes than do wealthier classes. (Note of course that Kwang's derived utility curve and the original Friedman–Savage utility curve [2] imply that the very poorest do not gamble, a thesis supported by Pryor's discussion [5] of the strong empirical evidence for Friedman–Savage utility curves.)

Even more interesting is an article by Flemming [1], who argues that while indivisible goods are exceedingly rare when time is taken into account, a similar result can be obtained for the utility of windfalls (e.g., from gambling) when the buying and selling prices of durable goods differ. Moreover, Flemming provides a number of valuable extensions, discussing the effects of time horizons, the possibility of a number of "dents" in the utility function, and the implications for cost curves.

This last point, suggested but not developed by Flemming, is of particular interest. If transactions costs, the prestige associated with (for example) a new building, or labor training costs lead to a divergence between the buying and selling prices of factors, then a firm would be reluctant to make a temporary large change in output. An increase in output would be made by using existing factors more intensively until the extra cost from depreciation and overtime was greater than the price divergence involved in purchasing more production factors. An unexpected permanent change in demand would enable the firm to avoid these inefficient periods just as a windfall gain enables an individual to jump over transactions costs to better consumer durables and higher utility. The firm might therefore engage in "gambling."

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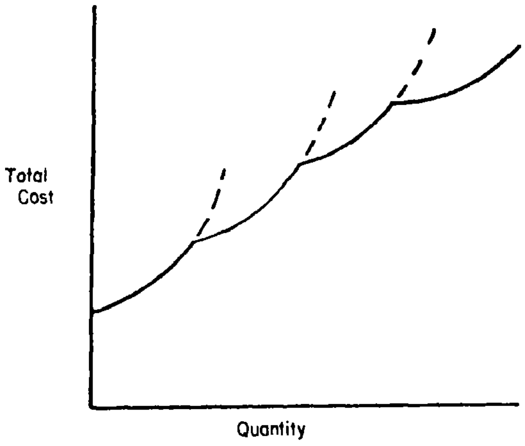

FIGURE 1

Figure 1 shows what these cost curves might look like. As Flemming explains [1, p. 66] these curves do not have "the normal short-run/long-run distinction; first there is nothing temporal about it [the distinction they do have], second it does not depend on the existence of one fixed and one flexible good in addition to financial assets."

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JOURNAL OF ECONOMIC THEORY 22, 115–117 (1980)

The ALEP Definition of Complementarity and Least Concave Utility Functions*

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Received July 4, 1979

The use of least concave utility functions describing a given concavifiable preference relation is suggested for determining the complementary vis-à-vis substitute nature of a pair of commodities.

Let $\gtrsim$ be a preference relation defined on an open convex subset K of R^n such that $\gtrsim$ is representable by a twice differentiable concave utility function u. The commodities i and j, $i \neq j$, are said to be ALEP complementary at x if $\partial^2 u(x)/\partial x_i \partial x_j > 0$. (This definition is due to Auspitz and Lieben [2, p. 482], and was adopted by Edgeworth [6, p. 117n] and Pareto [11, Chap. IV and Appendix, Sects. 12–13, pp. 505–507]—hence the term ALEP definition.) It is well known that this definition was criticized by Allen [1, p. 171n], Hicks and Allen [9, p. 60n], and Hicks [8, pp. 42–45] on the grounds that the condition $\partial^2 u(x)/\partial x_i \partial x_j$ is not invariant under monotone (increasing) transformations of the utility function u and so is not an intrinsic (ordinal) property of the preference ordering $\gtrsim$. In fact, this condition is not invariant even under the more restricted class of strictly monotone transformations preserving the concavity (or even the strict concavity) of the utility function. Quite the other way around: the more concave the utility function representing the given preference relation $\gtrsim$, the less apt are i and j to be ALEP complementary at x, at least if $\gtrsim$ is monotone (compare also [12]). For, let F be a twice-differentiable function of a single variable with $F'(t) > 0$ for all t, and set $v(x) = F(u(x))$. Denoting partial derivatives by subindices, we find that

$$v_{ij}(x) = F'(u) \left[u_{ij}(x) + \frac{F''(u)}{F'(u)} u_i(x) \, u_j(x) \right]. \tag{1}$$

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* This work was partially supported by National Science Foundation Grant SOC-7825734 at the University of Minnesota, Minneapolis, Minn., and Office of Naval Research Grant N00014-77-C-0518, at Cowles Foundation in Yale University, New Haven, Conn.

0022-0531/80/010115-03$02.00/0

If u is assumed to be sufficiently monotone so as to satisfy $u_i(x) > 0$ for all $1 \leq i \leq n$, then, choosing $F''(u(x))$ sufficiently negative, we can make $v_{ij}(x)$ negative even if $u_{ij}(x) > 0$.

We have seen thus that in the interesting cases every pair of commodities can be made to be ALEP substitutes at any given point x, by a suitable choice of a concave utility function. On the other hand, it was proved recently by Chipman [3] that ALEP complementarity does affect, in an invariant way, the behavior of the demand function (it was observed earlier by Georgeseu-Roegen [7] that complementarity influences the shape of the indifference curves of $\succsim$). In view of all this, we are led to consider the class of least concave utility functions for the given preference ordering $\succsim$. Least concave (alias minimally concave) utility functions were introduced by de Finetti [5]; their existence was proved by Debreu [4], and several ways of computing them were given by the author in [10]. The concave utility function $u(x)$ representing the preference relation $\succsim$ is said to be least concave if every concave utility function $v(x)$ representing $\succsim$ is given by $v(x) = F(u(x))$, where F is a strictly monotone and concave function of a single variable. (Note that least concave utility functions are unique up to translations and multiplications by positive constants.) We read off (1) that if a least concave utility function u is twice differentiable, if i and j are (ALEP) complementary at a certain x when a certain twice-differentiable utility v is considered, and if the preference relation $\succsim$ is monotone, then $u_{ij}(x)$ must also be positive. If, however, $u_{ij}(x) < 0$, then for no concave utility function (representing $\succsim$) is it going to happen that $v_{ij}(x) > 0$. (Note that the sign of $u_{ij}(x)$ is the same for all least concave utility functions representing $\succsim$.)

These facts suggest that we consider the sign of the mixed derivatives of least concave utility functions (if the latter are sufficiently smooth) in defining complementarity, and adopt the following.

DEFINITION. The commodities i and j, $i \neq j$, are said to be complementary at x if $u_{ij}(x) > 0$ for a least concave utility function u representing the preference relation $\succsim$ of the consumer, and are said to be substitutes if $u_{ij}(x) < 0$.

Remarks. (i) If the least concave utility function is not twice differentiable at x, one could still define i and j to be complementary if

$$\liminf_{h,k \to 0} [u(x + he_i + ke_j) + u(x) - u(x + he_i) - u(x + ke_j)]/(hk) > 0, \quad (2)$$

where e_i and e_j are the unit vectors in the ith and jth directions, respectively. Similarly, i and j are substitutes if

$$\limsup_{h,k \to 0} [u(x + he_i + ke_j) + u(x) - u(x + he_i) - u(x + ke_j)]/(hk) < 0. \quad (3)$$

If $u_{ij}(x) = 0$ then i and j are said to be (ALEP) independent. Note that if u is not twice differentiable, then it might happen that the lim inf in (2) is negative while the lim sup in (3) is positive. Such a case may be termed indeterminate. It is doubtful, however, whether this case is of importance in the economic applications.

(ii) The following example might illustrate the ideas presented here. Let $K = \{(x_1, x_2): x_1 > 0, x_2 > 0\}$ and let $(x_1, x_2) \gtrsim (y_1, y_2)$ if and only if $x_1x_2 \geq y_1y_2$. Commodities 1 and 2 appear to be independent (everywhere) if one chooses the separable (concave) utility function $\ln x_1 + \ln x_2$, and appear to be substitutes everywhere if the concave utility function $-1/(x_1x_2)$ is chosen instead. Considering the least concave utility function $(x_1x_2)^{1/2}$, we see that commodities 1 and 2 are indeed complementary everywhere, according to the definition suggested here.

(iii) The argument of this note lends further credibility to the selection of least concave utility functions as cardinal utilities (compare [4, 10]).

ACKNOWLEDGMENTS

I am very much indebted to Professors J. Chipman and L. Hurwicz for suggesting this problem to me and for a critical reading of the manuscript.

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JOURNAL OF ECONOMIC THEORY 22, 118-120 (1980)

Acknowledgment

The *Journal of Economic Theory* is indebted to many people for the editorial advice which they have given. We take this opportunity to thank the scholars who refereed manuscripts during 1979.

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JOURNAL OF ECONOMIC THEORY

Volume 22, Number 2, April 1980

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Published bimonthly at 37 Tempelhof, Bruges, Belgium
by Academic Press, Inc., 111 Fifth Avenue, New York, N.Y. 10003
1980: Volumes 22–23. Price per volume: $61.00 U.S.A.;
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Printed in Bruges, Belgium, by the St. Catherine Press, Ltd.

JOURNAL OF ECONOMIC THEORY 22, 121–135 (1980)

Noncooperative Approaches to the Theory of Perfect Competition: Presentation

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Received December 28, 1979

1. INTRODUCTION

Modern Walrasian economics (as presented, for example, in Debreu's "Theory of Value" or Arrow and Hahn's "General Competitive Analysis") provides an analysis of the decentralized economic coordination problem under the hypothesis that prices are publicly quoted and are viewed by economic agents as exogenously given. If we regard this as the Hypothesis of Perfect Competition, then modern Walrasian economics is a theory of perfect competition only in the sense of examining the consequences of the Hypothesis but not in that of giving a theoretical explanation of the Hypothesis itself. It is the latter line of enquiry which will interest us here.

The received body of economic doctrine abounds with theories of perfect competition (see Stigler [19]). The 19th-century contributions of Cournot [2] and Edgeworth [4] are remarkable cases in point. It was precisely in order to explain perfect competition (in terms of the insignificance of individual economic agents) that they, respectively, invented the concepts that modern game theorists call Noncooperative Equilibrium and the Core.

Under the impetus of the rediscovery of the Core concept by game theorists, Edgeworth's theory of perfect competition has been subject to extensive and rigorous elaboration in the last two decades (see Hildenbrand [9] for a presentation of the current state of the theory). Nothing of this sort has happened with Cournot's equilibrium concept. Although his partial-equilibrium example is widely known and highly attuned to the ways economists informally think about competition, it has proven resistant to generalization. This situation is beginning to change. In the last few years, several papers have laid the groundwork for what we shall call the Noncooperative Theory of Perfect Competition (e.g., Gabszewicz and Vial [5], Hart [7], Novshek and Sonnenschein [11], Shubik [18], Shapley [17], Dubey and Shapley [3], Jaynes *et al.* [12]), and to them we shall now add the contributions to this Symposium issue of the *Journal of Economic Theory.*

The Symposium was conceived as a vehicle for the joint publication of a number of papers already extant. It has not therefore developed from an a priori specified research agenda. The papers presented here are nonetheless all related and bear upon one another in diverse and interesting ways. Thus it was hoped that a useful service would be provided by bringing them out together.

One could distinguish three progressively more specific senses in which the notion of noncooperative equilibrium is understood when applied to the analysis of markets. At the first, most general level, noncooperation means simply that economic agents cannot rely upon any agreement being binding on other agents. A concomitant notion of noncooperative equilibrium follows. The second level is associated with noncooperative game theory. The trading process is formalized as a game where economic agents have well-defined strategies and trading outcomes depend on those strategies in a precisely specified manner. Then a noncooperative equilibrium (sometimes called a *conjectural equilibrium) is a combination of strategies with the property* that no agent has an incentive to modify the given conjectures he has on the reaction of other agents. The conjectures of primary interest to us are the stationary ones, i.e., every agent takes the strategies played by others as given. They were used by Cournot and are of great theoretical importance. (In fact, they are the sole meaningful ones in noncooperative game theory.) It has been suggested (by Shapley) that a good term for the noncooperative equilibrium in this second sense (i.e., when a game is explicit) would be strategic equilibrium. At the third level of specificity, it is understood that the noncooperative strategies are of the same nature as in Cournot's own work, i.e., quantities demanded or supplied.

The game theoretic notion of Noncooperative Equilibrium (see Harsanyi [6]) is rich in possibilities. In particular, it allows for the explicit analysis of cooperation. In a static model the contrast between cooperation and noncooperation is sharp. As indicated in the last paragraph, noncooperation (resp. cooperation) means that binding agreements are not (resp. are) possible. However, in a dynamic situation (an infinite repetition of the static model, for example) the distinction is blurred. Dynamic strategies allow for actions contingent upon past histories and therefore it may be possible to enforce agreements noncooperatively by the use of deterrence strategies. Thus, the noncooperative equilibrium of the repeated situation provides a useful analytical tool to determine the prevalence of the cooperative or noncooperative mode of behavior in the static situation.

As might be expected, the size of the market relative to individual agents will be a key explanatory variable for the tendency of noncooperative behavior to approximate perfect competition. Directly or indirectly, every paper in the Symposium bears upon this central theme. It could also be argued that the number of agents plays an even more fundamental role. Just as Walrasian economics demands an explanation of its basic hypothesis and noncooperative theory provides a framework for it, the latter demands an explanation of the maintained noncooperative hypothesis. It accords with intuition that the number of agents will be a crucial consideration. As a research project, this goes well beyond Cournot although, as was pointed out in the last paragraph, not necessarily beyond general noncooperative theory. The papers by Green and Radner in the Symposium represent a beginning in this direction.

At least as instructive as the positive theoretical results obtained are the new cases of market failures identified in various Symposium contributions (e.g., Roberts, Hart, Makowski). Indeed, a familiar characteristic of perfect competition research does not fail to appear here: the closer one examines the concept of perfect competition the stronger are found to be the conditions for its validity. The horizon of perfect-competition prerequisites seems to be ever receding. Surely, the central position the concept of perfect competition has enjoyed since the 18th century is not about to falter. But the emphasis should be on its centrality as an organizing concept and certainly not as an innocuous descriptive category for the real world.

Almost every paper in the Symposium looks explicitly at perfect competition as a limit case of imperfect, or monopolistic, competition. Unfortunately, a powerful theory of the latter subject is still missing, but perhaps one can hope that some of the methods developed in this Symposium will take us a step closer.

A traditional requirement of perfect competition is the "transparency of markets," i.e., some appropriate form of perfect information about market conditions. The general tenor of the Symposium papers is to accept this requirement without close scrutiny and to embody it at the modeling stage. Imperfect information as a source of imperfect competition has been extensively investigated (Rothschild [16] is still a useful reference).

The Symposium papers can be classified into three general categories:

(i) The papers by Green and Radner analyze the noncooperative equilibria of repeated games for partial equilibrium models which closely resemble Cournot's original example. Thus, these papers study the robustness of Cournot's result when the choice of cooperation vs noncooperation is made endogenous.

(ii) The papers by Dubey, Dubey, Mas-Colell, and Shubik, Gabszewicz and Thisse, Hart, Novshek, Novshek and Sonnenschein, and Roberts are concerned with *static* noncooperative equilibria in general equilibrium models with large numbers of small agents.

(iii) The contribution of Ostroy and the two by Makowski seek to characterize the domain of validity of the Hypothesis of Perfect Competition by a deepening of the complementary notion that monopoly power is absent. They provide a noncooperative theory of perfect competition which does not rely on the number of agents in any explicit way and which has its intellectual roots more in the marginal productivity theory of distribution than in Cournot. In spite, or because, of this their papers interact in subtle and pervasive ways with the other contributions to the Symposium.

2. COOPERATION VERSUS NONCOOPERATION

Suppose that a number N of firms produce a homogeneous good and face a given demand function. With the quantity produced as the strategic variable, the static cooperative (i.e., cartel) solution involves producing a quantity that maximizes the sum of profits, while the noncooperative solution is the classical one described by Cournot. Which outcome is more likely will depend upon many particulars of the market being studied, but surely two relevant circumstances are: (i) the number of firms, and (ii) the coordination possibilities open to firms. The theory of repeated games provides a method for modeling their impact. Indeed, suppose that the market game will be repeated M times. Then firm production strategies for the M periods are specified in advance and can be made contingent on past histories. Thus, a potentially large (depending on M) amount of bargaining moves can be embodied in the independent specification of strategies.

Radner and Green each discuss instances of such repeated market games. Of course, there are many differences between their models. A particularly important one is that the payoff criterion of Radner is the average profit over the M periods, while Green adopts a discounted sum of profits (payoffs), the discount rate being positive but perhaps very small.

In the context of a linear example, Radner examines the following intuition: for a fixed number of firms N the larger is M the more numerous are the coordination possibilities available and therefore the more stable the cartel agreement will be. On the other hand, for a fixed $M < \infty$ the larger is N the more difficult it is to coordinate, and, therefore, the more likely it is that noncooperation prevails and that the outcome will resemble the Cournot equilibrium which, being N large, will in turn be close to the Walrasian Equilibrium. The first thing needed for a rigorous analysis is a solution concept. Suppose we were to adopt as such the (perfect) noncooperative equilibrium of the repeated game. It turns out then that the first part of the intuition is incorrect. Only if $M = \infty$ can the cartel agreement be noncooperatively enforced. If $M < \infty$, a familiar backward recursion argument yields the conclusion that the only possible equilibrium outcome is the repeated sequence of stationary Cournot productions. Thus, the coordination possibilities embodied in M come to naught. While the discontinuity at $M = \infty$ is rather disconcerting it is not at all clear that it is of economic importance since it crucially depends upon individual agents' sharp perception of a last period. Radner establishes that if we instead adopt as a solution concept his notion of ϵ (perfect) noncooperative equilibrium (which he has developed in the context of a wider research program), then the discontinuity disappears and, in a precise sense, the informal intuition given above holds true.

In Radner's approach, if $M = \infty$ then the cartel agreement can be enforced, whatever the number N of firms. If we leave aside a subtle point (i.e., the issue of perfectness), this enforcement can be accomplished by the adoption by firms of the following reprisal strategy: Every firm will produce the cartel output as long as all other firms do, but if any deviates, each firm will increase production to the Cournot output and maintain it thereafter.

Green reexamines this conclusion when account is taken of a realistic characteristic of markets, namely, the imperfection of information. In particular: (i) markets are anonymous in the sense that only a summary statistic (for example, price) of the firms' strategies is publicly known, and (ii) markets are noisy in the sense that demand, at a given price, is random. Green exhibits an example where the cartel agreement of an infinitely repeated anonymous market is still enforceable for any finite number of firms (if there were a continuum of firms, then anonymity suffices for the collapse of the cartel). The equilibrium strategies are of the following form: If any departure of market prices from cartel prices is observed, then cooperation stops. These are rather farfetched strategies. They require that agents have the ability to discern prices perfectly and that they react discontinuously. Green proceeds to show that if in addition demand is noisy a different result emerges. When the number of firms is large, the noncooperative equilibria of the infinitely repeated noisy markets he considers yield production levels which are close to the stationary Cournot ones (and are, therefore, almost Walrasian). Strictly speaking, this result is established for noncooperative equilibria with stationary strategies, but it seems plausible that the result is true more generally. Also, discounting is important here. If, as in Radner, the payoff criterion were the long range average profit, then it would appear that the cartel outcome would be enforceable even with noise.

3. NONCOOPERATIVE EQUILIBRIA IN ECONOMIES WITH A LARGE NUMBER OF AGENTS

Consider a noncooperative game superimposed upon a given market economy. If the game is intended to model market transactions, then it should be expected that, with a relatively small number of economic agents (consumers and firms), the Walrasian solution will not generally emerge as a noncooperative equilibrium of the game. Indeed, at such solutions agents will have incentives to exploit monopoly power. Even more, if a noncooperative equilibrium exists, we will not expect it to be Pareto Optimal at all. In order to properly relate a specific noncooperative theory to Walrasian economics and the Hypothesis of Perfect Competition, we should ask if the noncooperative equilibria are Walrasian (or almost so), and therefore Pareto Optimal, when agents have no monopoly power. If this is not the case then we have a market failure (which, for reasons to be made clear later on, we shall designate Type I). A classical embodiment of the notion that monopoly power is absent is the hypothesis that individual agents are negligible or, more specifically, that a very large number of them, in the limit of a continuum, are present in the marketplace.

Two broad lines can be distinguished in the current research in Cournotian general equilibrium theory:

(i) the first, inagurated by the quantity setting model of Shubik [18] (see also Shapley [17], Jaynes, Okuno and Schmeidler [12], and Dubey and Shapley [3]), emphasizes trade and treats all agents (buyers or sellers) symmetrically. In this Symposium it is represented by the Dubey *et al.* [20] and Dubey [21] papers.

(ii) the second, inagurated by the quantity setting model of Gabszewicz and Vial [5], stays closer to Cournot's original partial-equilibrium example. It maintains the distinction between firms and consumers and views the noncooperative game as being played among firms, the consumers adapting passively (i.e., as price takers) along an endogenously determined demand curve. The papers of Hart, Gabszewicz and Thisse, Novshek and Sonnenschein, Novshek, and Roberts fall broadly into this category. The models of Hart, Roberts, and Novshek and Sonnenschein have quantities as strategies while the model of Gabszewicz and Thisse has prices and that of Novshek has location and prices.

The approach taken in the Dubey *et al.* [20] paper is axiomatic and it is developed primarily within the framework of a continuum of agents. It deals with a class of abstractly given market allocation mechanisms which satisfy a number of axioms (convexity of the strategy set, anonymity, aggregation of strategies, continuity of payoffs) intended to be characteristic of well-functioning markets. The conclusion can be phrased as asserting that, under the axioms, every noncooperative equilibrium is Walrasian relative to the set of "open" markets, where "open" means that, as far as the market is concerned, it is possible for individual agents to change slightly their trade. This set of open markets is endogenously determined. In fact, in some of the prototypical examples, the set of open and active (i.e., those where transactions take place) markets coincide, and it is possible (for somewhat degenerate reasons) to obtain equilibria for any a priori specification of nonactive markets. Thus, in those examples market failure (of Type I) is not only possible but even typical.

It is also shown in the Dubey *et al.* [20] paper that (under the axioms), the noncooperative equilibria (with all markets open) of market games with a finite number of agents may fail to be Pareto Optimal for some specification of agents' characteristics. One may feel that it should be possible to go much further, i.e., to assert that except for coincidental cases, they never will be Pareto Optimal. This is accomplished in Dubey's paper for a particular quantity-setting example. But note that the issue is more subtle than it appears at first sight. Indeed, the incentive compatibility literature (see the recent *Review of Economic Studies* [13] Symposium) has taught us that it is possible to design mechanisms yielding Pareto Optimal noncooperative equilibria. In essence, Dubey establishes that, if to the axioms we add a key dimensionality condition on the set of strategies, then generic Pareto inefficiency obtains.

The Symposium contribution of Novshek and Sonnenschein is a continuing step in the research initiated in their previous paper [11]. Here they work out the Walrasian Equilibrium Theory and the Welfare-theoretic propositions of the limit continuum model whose finite number of traders approximations are the basic models of their noncooperative theory. The economy they consider has a continuum of firms, each with bounded (but "nonzero") nonconvexities in their production sets (i.e., relatively small efficiency scale). From the Novshek–Sonnenschein point of view, small nonzero efficiency scale is the cornerstone of the theory of perfect competition. They pay special attention to the existence problem and to the two fundamental propositions of Welfare economics. They emphasize that the validity of the second (on the sustainability of Pareto Optimal states as equilibria, modulo an appropriate redistribution of wealth) is not affected by the presence of externalities and is therefore unrelated to the convexity or not of the aggregate production set (individual production sets are always nonconvex but because the nonconvexities are bounded and there is a continuum of firms, the aggregate production set will be nonconvex only if externalities are present). A complete perspective of the status of the two fundamental propositions in the Novshek and Sonnenschein noncooperative theory of competition will be obtained by combining the results of this paper (on the relation between Pareto Optimality and Walrasian Equilibrium) with the conclusions of the earlier one (on the relation between Walrasian Equilibrium and Noncooperative Equilibrium).

The papers by Hart and Roberts can be regarded as studies in market failure for related quantity setting models. Suppose there is a finite number of firms and a consumption sector. The actions or strategies of a firm consist in choosing a feasible input–output vector. Equilibrium prices are determined so that the aggregate input–output vector is absorbed by consumers acting as price takers (there is a well-defined profit distribution rule). For matters to be well defined, there should be available a priori a selection from the correspondence which assigns equilibrium prices to aggregate input–output vectors. Once actions have been taken, the profits of firms are determined. Under the hypothesis that firms are aware of the dependence of price on actions and that they maximize profits (see Hart [8] for a justification in a similar model) we have a notion of noncooperative equilibrium and we can analyze its properties. It is then relevant to ask if there is an absence of (Type I) market failure, i.e., if the noncooperative equilibria tend to become Walrasian as the size of the economy relative to individual firms increases. The technique used by Hart and Roberts to expand the economy is replication of the number of firms *and* consumers without limit.

Roberts proves that there will not be market failure if (under other standard conditions) the price equilibrium selected depends continuously on the (mean) aggregate input-output vector (Gabszewicz and Vial [5] and Novshek and Sonnenschein [11] also have results of this type). He exhibits examples where price discontinuity leads to market failure. It is possible to give an intuitive justification of his conclusions. Absence of market failure is tantamount to every agent (firms as well as consumers) becoming a price taker as the size of the economy increases. Now, it is reasonably clear that this will happen if individual production sets are suitably bounded and prices depend continuously on the mean aggregate input-output vector. On the contrary, at a discontinuity point of the price function it is very likely that the isoprofit surfaces of individual firms will not become hyperplanes and so the economy may remain stable at one such point even if it is very far from being Walrasian. This is a significant case of market failure because the price continuity hypothesis is by no means a mild one. It essentially imposes two strong requirements: (i) a price equilibrium exists for every feasible aggregate input–output vector, (ii) it is unique.

An important instance of this variety of market failure appears in the paper by Hart. He shows that, unless we resort to economically quite strong smoothness and convexity hypotheses (which imply a form of substitutability among commodities), it is possible that no continuous selection of equilibrium prices exists at points with zero production of *several* goods (the existence of such a selection is an implication of his Simultaneous Reservation Price Property). For example, homogeneous production and utility functions which are not linear have discontinuous gradients at the origin. The economically significant reason for this lack of continuity is the presence of complementarities. Thus, think of goods that should be consumed (or used as inputs) in fixed proportions or, more generally, of goods which are indispensable in the sense that a positive amount is required in order get any output (or utility). The consequences are quite damaging. For example, if there are several indispensable goods, there is no hope of guaranteeing an efficient outcome in a completely decentralized manner, since any of them, produced alone, will command a zero price. Matters should be so coordinated that all goods are produced at once. Otherwise, it may well happen that the economy gets locked up in a situation with the "wrong" set of active markets. (Interestingly, the market failures in the Dubey *et al.* [20] examples are also of this sort but the failure here is less degenerate.) Hart discusses a number of instructive and quite unsuspected examples. It should be emphasized that these market failures do not arise because of divergence of private and social incentives. In Hart's model, the cost-benefit analysis associated with a *single* firm is (asymptotically) identical whether we take the private or the social point of view.

The message of Hart's paper is to emphasize a crucial qualitative difference between active and nonactive markets which does not come through in conventional Walrasian economics. What we have called the Hypothesis of Perfect Competition has two parts: price quoting and price taking. For active markets prices are embodied in transactions and so, there is no particular mystery about their origins. But for nonactive markets, Walrasian prices are more in the nature of a theoretical construct. It is indeed a very strong hypothesis to postulate that prices are universally quoted even for nonactive markets. It implies, for example, that every agent is perfectly informed about the set of possible commodities. If, as Hart does, we take the limited point of view that quoted prices are available only in active markets, then the sort of decentralized coordination that Walrasian prices are supposed to achieve will come about only if the theoretical clearing prices are well approximated by the prices obtained in a decentralized manner. Or, in other words, only if the Simultaneous Reservation Price Property (i.e., the possibility of continuous extension of the price function to multiple zero-production levels) holds. Otherwise, the usual Walrasian picture may be quite misleading. The Invisible Hand may have to work hard and coordinate extensively if the right set of commodities is to be made available. It is interesting to observe how entirely parallel are Hart's conclusions to the ones obtained by Makowski in his Symposium paper on innovation, itself based on a different approach to perfect competition.

Up to now, market failure has meant that noncooperative equilibrium does not yield full Walrasian equilibrium. There is a second type (Type II) of market failure, namely, when a Walrasian equilibrium of a limit continuum economy fails to be the limit of a sequence of noncooperative equilibria for approximating sequences of economies with a finite number of agents (or, more generally, when a Walrasian equilibrium is not a noncooperative equilibrium in some appropriate sense). The interpretation of such failure is clear: that particular Walrasian equilibrium should not properly be regarded as a decentralized equilibrium.

Type II market failures are also studied by Roberts. Under smoothness and convexity hypotheses on preferences and production sets, he is able to rule them out for Walrasian equilibria which are regular in a strong sense. By regular he means that some relevant Jacobian determinants do not vanish.

A crucial implication of regularity is the existence of a locally continuous price equilibrium selection function (a property which was already exploited with a similar objective in Roberts and Postlewaite [14]). In the context of his model, the standard result in the theory of regular economies holds, i.e., regular economies are generic (in other words, nonregularity corresponds to pathology). This is a satisfactory and reasonably general result. It would be hasty, however, to dismiss the significance of Type II failures. There are important directions in which Roberts result does not extend. Suppose there are some purely intermediate goods. Then, market failure seems to occur generically. Indeed, if demand equals supply it will always pay for a demander of the intermediate good to understate his demand slightly, thus creating excess supply and bringing the price to zero. Note that no locally continuous selection of equilibrium prices exists in this case. In fact, the presence of purely intermediate goods seems to present an as yet untackled modeling problem. In another direction Novshek and Sonnenschein [11] have established that in the presence of setup cost (even if very small), the availability of a locally continuous selection of equilibrium prices does not even generically suffice to rule out Type II failures. One also needs a multidimensional analog of the downward sloping demand curve condition.

As formulated in the previous paragraph, the absence of Type II failure implies the existence of noncooperative equilibria for all large number of agents' economies near the considered limit. It is, of course, well known that existence (i.e., consistency) failures are a major difficulty for the use of the noncooperative equilibrium concept in the analysis of market situations with a fixed finite number of traders (see, for example, Roberts and Sonnenschein [15]). The results of Roberts and Novshek and Sonnenschein [11] indicate that the problem becomes more tractable as the economy nears its continuum limit, but even then it is quite subtle. Thus it may be worthwhile to explore the usefulness of approximate noncooperative equilibrium concepts (the plural is important—there is more than one and which one is picked may make a difference). A beginning in this direction is to be found in Hart's paper. This may be a good point to remark that the existence question also obscures somewhat the analysis of Type I failures. Indeed, a given model may be "free" of them simply because the noncooperative equilibria fail to exist!

The paper by Novshek carries out a careful and penetrating analysis of a prototypical simple differentiated commodity model in the hereditary line of Hotelling [10]. No useful service would be performed here by a detailed description of the model and results. The Abstract and first pages of his paper are quite self-explanatory. It should be emphasized however that it contains a full treatment of the existence question (so, narrowing down the analysis to a simple, but still rich, model pays off in this case) and that a number of interesting and instructive points arise: (i) in order to get existence of noncooperative equilibrium (in location and price strategies) the Cournot stationary conjectures must be modified in a most sensible way, firms will no longer expect others to remain put if an effect of their strategies is to take away all the demand of a competitor; (ii) with no set-up cost for firms (and, of course, a number of other assumptions) a suitably defined noncooperative equilibrium will exist; (iii) with setup costs and the possibility of exit and entry there may be no equilibrium, but if the setup cost is small and an unlimited number of firms can, in principle, enter, then a noncooperative equilibrium will exist and, moreover, will closely approximate the Walrasian equilibrium. The Novshek and Sonnenschein [11] paper provides a far reaching generalization of this latter feature.

In their contribution Gabszewicz and Thisse analyze a numerically specified example of *price* (rather than quantity) competition among firms offering a spectrum of differentiated products. In the example there are two free parameters: the number of firms and the degree of substitutability between commodities. Roughly described, the result states that, for a fixed degree of substitutability, the noncooperative equilibrium approaches the Walrasian equilibrium as the number of (potential) firms increases. But this also happens if the number of firms stays fixed (but is larger than one) and the degree of substitutability increases. Of course, this is in line with Bertrand's classic counterexample [1], showing that if strategies are prices, then the perfectly competitive duopoly is a possible noncooperative outcome. Does this mean that the role of the number of agents as a fundamental explanatory variable of perfect competition is limited to quantity setting models? The answer is no to the extent that the number of agents may explain the prevalence of noncooperation over cooperation.

A final but basic observation is that, ideally, the choice of strategy space (price, quantities, supply functions,...) or conjectures should not be exogenously given but should be determined from economic considerations. This appears to be a wide open field of research. Intuitively, it seems clear that factors such as the timing and the irreversibility of decisions regarding location, choice of technique, or scale of production are relevant as is, more generally, any factor which affects the strategic precommitment possibilities of firms. It may be that a static timeless approach is too limited for this analysis and that a multistage specification is required if conceptual fuzziness is to be avoided.

4. PERFECT COMPETITION AS THE ABSENCE OF MONOPOLY POWER

Until now the emphasis has been on justifying the Perfect Competition Hypothesis. The contributions of Ostroy and Makowski can be viewed as analyses of its inner structure. Ostroy's paper concentrates on the exchange case, while the first by Makowski extends the theory to a production, firm-oriented setting.

Ostroy and Makowski both begin by defining a Walrasian equilibrium as perfectly competitive if agents are justified in treating prices as given in the specific sense that the Walrasian prices are still equilibrium prices if the individual agent is dropped from the economy. This test is devised to capture the notion of individual agents facing an infinitely elastic demand curve and therefore enjoying no monopoly power. Observe that, under the standard general equilibrium hypotheses on preferences and production sets, perfectly competitive Walrasian allocations have these two properties: (i) they are Pareto Optimal or, in their words, every consumer obtains maximum utility given the utility level of the other consumers, (ii) every group formed by all agents but one is able to protect their utility levels in the sense that no loss of utility need occur for any remaining agent as a result of dropping a single agent from the economy. In other words, at the given allocation no single agent can possibly succeed in bargaining for a higher level of utility with the rest of the economy since he has already succeeded in extracting all "surplus" accruing to the other agents as a result of his participation in trade. Ostroy defines allocations satisfying properties (i) and (ii) as no-surplus allocations and proves a simple but basic result, namely that an allocation is a no-surplus one if and only if it is Walrasian and perfectly competitive. Thus, a price-free characterization of the latter concept is obtained.

Ostroy's theory provides a general equilibrium and purely ordinal extension of the classical characterization of competitive equilibrium in terms of "marginal productivities." Indeed, an allocation is perfectly competitive (i.e., no-surplus) if and only if every agent receives exactly its "marginal contribution." It might also be pointed out that there are interesting relationships between the no-surplus concept and some of the cooperative game theoretic notions that have been used to analyze perfect competition (i.e., Core, Shapley Value). But the theory remains very definitively a non-cooperative one. It is a basic hypothesis that every agent bargains for surplus in isolation and that no collusion for that purpose is possible. The idea of a no-surplus equilibrium does not require the explicit structure of a game for its formulation. Nevertheless, it makes some sense to say that the idea is not based on stationary conjectures. Rather the contrary, every agent expects that the rest of the economy can and will, as a group, do everything possible to protect their levels of utility.

In his first paper Makowski shows that this no-surplus theory includes a theory of profits. Indeed, he sets up a model of individual production where the role of agents as producers (i.e., entrepreneurs) and consumers can be separated, and he proves that, in a precise sense, at a perfectly competitive outcome, profits measure the entrepreneurial contribution to "total surplus." Thus, for example, zero profit is equivalent to the redundancy of the entrepreneur. One of the contributions of his second paper is an extension of the theory to the joint ownership case. It takes this form: At a perfectly competitive equilibrium, (i) every owner of the firm agrees on the profit maximization criterion, and (ii) profits measure the surplus contributed by the firm.

It is possible for no-surplus allocations to exist for economies with a finite number of agents. For example, if the initial endowments allocation is Pareto Optimal it is also no-surplus (and, one might add, there are no gains from trade). As another instance, presented by Makowski, suppose that at the Walrasian Equilibrium every active firm has an inactive double. Then firms have no surplus to extract and are, therefore, perfect competitors (thus, in a market with two identical firms and constant average cost no-surplus theory agrees with the Bertrand solution). But such situations are rare. Typically, if there is only a finite number of agents, no surplus allocations will fail to exist, a reflection of the all-pervasive presence of at least some extent of monopoly power. In contrast, it accords with general intuition that in a continuum of agents situation monopoly power vanishes and we should expect Walrasian allocations to be no-surplus ones. Thus, consider the exchange case and assume all the necessary differentiability. Then prices represent precisely the welfare contribution of the marginal unit of commodities brought and sold, but if individual agents are infinitesimally small, all their transactions take place at the infinitesimal margin, and so, they contribute no surplus. Ostroy investigates this issue rigorously in the context of a replicated economy. There it takes the form of determining the conditions under which the Walrasian allocations will become approximately no-surplus as the number of agents increases (i.e., asymptotically no-surplus). He shows that the key condition is the existence of a unique (normalized) vector of efficiency prices for the given Walrasian allocation. This is a very weak condition since it can be violated only with extreme forms of (what else?) complementarities. If these are present, then Walrasian allocations may not be asymptotically no-surplus (i.e., we have a kind of Type II failure). Examples can be found in Ostroy's paper.

The second paper by Makowski, which is also firm oriented, explores the ramifications of a sensible ammendation to the perfect-competition concept as it has been discussed so far. The latter captures well the notion of agents not being able to influence prices. But it says nothing about prices quotation. As Hart does in his paper, Makowski hypothesizes that prices are quoted only in active markets. Therefore, when a firm is considered unable to influence prices, this is in reference only to prices that would be quoted at the feasible production plans of the firm or, for short, the prices "faced" by the firm. Note that, since they are unobservable, nothing is said here about the possibility to affect prices faced by other firms. Thus, as Makowski notes, a type of pecunary externality is compatible with perfect competition.

For Makowski, an allocation is perfectly competitive if: (i) there are prices for marketed commodities which put the consumers in price taking equilibrium, (ii) firms cannot influence the prices they face. The precise definition of this is analogous to Ostroy's and to the first Makowski paper. Essentially, the economy can do as well without the particular firm and at the same prices. Makowski is able to show that, in a precise sense, profits, at a perfectly competitive allocation, measure the contribution to social welfare of the particular firm, firms profit maximize and there is unanimity among joint owners on the profit motive.

Perfect competition in the sense of Makowski provides a theory of innovation and endogenous determination of marketed commodities. Indeed, finding the "right" set of commodities becomes a real issue. In particular, complementarities (Makowski emphasizes the ones connected with chains of intermediate products) may lead to inefficient perfectly competitive equilibria where there is no private or social incentive (remember that here they coincide) for any *single* firm to change its production plan (offering a new product, for example). The analytical conclusions are entirely analogous to Hart's. In fact, Makowski's model, which is more general, provides a useful perspective on Hart's analysis. The role of a large number of agents in the latter is simply to make the noncooperative equilibrium approximately perfectly competitive in the sense of the former.

ACKNOWLEDGMENT

I would like to acknowledge the help I have received in the preparation of these pages from the Institute for Advanced Studies at the Hebrew University and from Mr. D. Wells.

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JOURNAL OF ECONOMIC THEORY 22, 136–154 (1980)

Collusive Behavior in Noncooperative Epsilon-Equilibria of Oligopolies with Long but Finite Lives

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Received March 9, 1979

In a game of a finite number of repetitions of a Cournot-type model of an industry, if firms are satisfied to get close to (but not necessarily achieve) their optimal responses to other firms' sequential strategies, then in the resulting noncooperative "equilibria" of the sequential market game, (1) if the lifetime of the industry is large compared to the number of firms, there are equilibria corresponding to any given duration of the cartel, whereas (2) if the number of firms is large compared to the industry's lifetime, all equilibria will be close (in some sense) to the competitive equilibrium.

1. INTRODUCTION

In 1838 Augustin Cournot introduced his model of market equilibrium, which has become known in modern game theory as a noncooperative (or Nash) equilibrium [1]. Cournot's model was intended to describe an industry with a fixed number of firms with convex cost functions, producing a homogeneous product, in which each firm's action was to choose an output (or rate of output), and in which the market price was determined by the total industry output and the market demand function. A Cournot–Nash equilibrium is a combination of outputs, one for each firm, such that no firm can increase its profit by changing its output alone.

Cournot thought of his model as describing "competition" among firms; this corresponds to what we call today the "noncooperative" character of the equilibrium. He showed that, if the number of firms is regarded as a parameter of the market, a larger number of firms would lead to a larger industry output and a lower price (in equilibrium), and that as the number of firms increased without limit, the corresponding equilibria would converge to the situation he called "unlimited competition," in which marginal cost equaled price.

If there are at least two firms, then they can make more profit than in the Cournot–Nash equilibrium (CNE) by forming a cartel in which the total industry output is chosen to maximize total industry profit, and this profit

0022-0531/80/020136-19$02.00/0

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is shared equally among the firms. This corresponds to what would be called a cooperative solution of the game.

What determines whether there will be a cooperative rather than a non-cooperative outcome in the market situation? If the market situation were repeated a number of periods, then, even in the absence of some institution (such as regulation) to enforce cooperation, it would seem that the firms would have opportunities to signal their willingness to cooperate. Furthermore, the larger the number of periods, the greater would be the relative loss due to defection from the cartel and a reversion to the CNE outputs and profits. On the other hand, the larger the number of firms, the greater would be the difficulty of holding the cartel together, at least according to conventional wisdom.

If the market situation is repeated, say T times, then this gives rise to a game in which the strategies available to the players (firms) are sequential. A sequential strategy is a sequence of functions, one for each period, each of which determines the output in that period as a function of the outputs of all firms in the previous periods. One can show that every perfect Cournot–Nash equilibrium of the T-period game results in each firm producing its one-period CNE output in each period.[1] (For this result, one assumes that each firm's objective is to maximize its average, or total discounted, profit over the T periods.) I should emphasize that this property of T-period perfect CNE's is satisfied for any finite T, no matter how large. On the other hand, one can show that if T is infinite, and each firm's objective is to maximize its long-run-average profit, then there is a perfect CNE of the (infinite-period) game that results in indefinite survival of the cartel.[2] The result for infinite T goes part of the way towards confirming our intuition about the determinants of cooperation, but has the unsatisfactory feature that the infinite case is not well approximated by the case of large but finite T. Another unsatisfactory feature of the result is that it holds for any number of firms, no matter how large.

In this paper I explore the consequences of weakening the strict rationality of the Cournot–Nash equilibrium concept, so that each player is satisfied to get close to (but not necessarily achieve) his best response to the other players' strategies. Formally, an *epsilon-equilibrium* is a combination of strategies, one for each player, such that each player's strategy is within epsilon in utility (e.g., average profit) of the maximum possible against the other players' strategies. I shall show that, for any fixed positive epsilon, any given number of firms, and any integer k, there is an integer T_0 such that, for all T exceeding T_0, there is a perfect epsilon-equilibrium of the T-period game in which the cartel lasts exactly k periods. In choosing T_0, it is sufficient to make $(T_0 - k)$ larger than some number that depends on epsilon and the number of firms. In particular, one can (approximately) achieve any desired fraction, k/T, by taking T large enough.

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[1] A perfect equilibrium of a sequential game is defined in Section 3. As will be seen there, the restriction that the equilibrium be perfect excludes equilibria based on unconvincing threats. The statement in the text is no longer correct if one eliminates the condition that equilibria be perfect.

[2] This is a special case of a more general theorem on perfect equilibria of infinite supergames, due to Aumann and Shapley (unpublished) and Rubinstein [5, 6].

The effect on perfect epsilon-equilibria of varying the number of firms depends on the relationship between the industry demand function and the number of firms. (The latter is a parameter of the game.) Suppose first that, as we compare markets with different numbers of firms, the demand price is the same function of the *average* industry output *per firm*; call this the *replication case*. This would be the situation that would obtain if, when we doubled the number of firms, we also duplicated the population of consumers. In the replication case, one can show that, for every fixed positive epsilon and T, as the number of firms increases without limit all perfect epsilon-equilibria approach competitive equilibrium (Cournot's "unlimited competition") in the following sense. For every positive ϵ, T, and N (>1), there is a number $B(\epsilon, T, N)$ such that, in every perfect ϵ-equilibrium of the T-period game, each firm's output and total industry output at each date are all within $B(\epsilon, T, N)$ of their corresponding one-period CNE values. The bounds $B(\epsilon, T, N)$ can be chosen so that, (1) for every ϵ and T, the ratios $B(\epsilon, T, N)/N^{1/2}$ are uniformly bounded in N, and (2) for every T and N, $B(\epsilon, T, N)$ approaches zero as ϵ approaches zero. One can further show that, for fixed ϵ and T, as N increases without limit, at every date (1) average output per firm approaches the competitive equilibrium output per firm, (2) market price approaches marginal cost, and (3) every firm's relative share of total industry output approaches zero. Finally, under the same conditions, the difference between each firm's profit and its one-period CNE profit is uniformly bounded in N, and approaches zero as ϵ approaches zero.

Leaving the replication case, suppose now that the industry demand function is fixed as we compare markets with different numbers of firms; call this the *fixed-demand* case. In this case, one can show that, for any fixed number of periods, T, there is a number B (depending on epsilon and T) such that, in all perfect epsilon-equilibria of the T-period game, the outputs and profits of all firms, total industry output and profit, and market price are all within B of their corresponding one-period CNE values, *uniformly in the number of firms*. Furthermore, the bound B tends to zero as epsilon tends to zero, with T fixed. This implies that, in the fixed-demand case, for any fixed number of periods and *small* epsilon, all perfect epsilon-equilibria are "close" to competitive equilibrium in terms of outputs, profits, and prices, for a sufficiently large number of firms.

Note that the effect of increasing the number of firms on reducing the possibilities for cooperation is observed in the replication case, but not in the fixed-demand case. In both cases, reducing epsilon reduces the possibilities for cooperation (i.e., keeps all perfect epsilon-equilibria closer to the CNE), and increasing the number of firms brings the CNE closer, of course, to the competitive equilibrium. Thus, for the replication case, one can paraphrase the results of this paper as follows: if firms are satisfied to get close to (but not necessarily achieve) their optimal responses to other firms' strategies, then in the resulting noncooperative "equilibria" of the sequential market game, (1) if the lifetime of the industry is large compared to the number of firms, there are equilibria corresponding to any given duration of the cartel, whereas (2) if the number of firms is large compared to the lifetime of the industry, all equilibria will be close (in some sense) to the competitive equilibrium.

Although the replication case is the one of central interest here, I shall use the fixed demand case (Section 5) as a stepping-stone in the analysis of the replication case (Section 6).

As will be seen below, several alternative definitions of perfect epsilon-equilibria may be reasonably considered. I shall begin the analysis with the simplest one (Section 4). A more satisfactory definition, is introduced in Section 7. An important behavioral implication of this second definition is that cooperation will tend to break down as the industry approaches the horizon T. In Section 8, I discuss some alternative interpretations of epsilon.

The entire analysis in the present paper is carried out only for a special model, in which the market demand function is linear, and all firms have the same linear homogeneous cost function (average and marginal costs are equal and constant). However, the analysis can be extended easily to the case in which each firm has a fixed (setup) cost of production, and in which there is free entry. In this case, the number of firms in the industry is endogenous.[3] This extension will be discussed in a forthcoming paper.

2. THE ONE-PERIOD COURNOT GAME

Consider an industry producing a single homogeneous product, with N firms. The cost to a single firm of producing a quantity Q is γQ. If firm j produces quantity Q_j, the market-clearing price P is determined by the industry demand function

$$P = \alpha - \beta \sum_{j=1}^{N} Q_j, \tag{2.1}$$

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[3] For this situation, Cournot-Nash equilibria in the one-period game have been studied by Novshek [2] and by Novshek and Sonnenschein [3].

if this is positive; otherwise it is zero. The profit to firm i is therefore

$$PQ_i - \gamma Q_i = \left(\alpha - \beta \sum_{j=1}^{N} Q_j\right) Q_i - \gamma Q_i$$
$$= \left(\alpha - \gamma - \beta \sum_{j \neq i} Q_j\right) Q_i - \beta Q_i^2.$$

Assume that $\alpha > \gamma$, and define

$$Q_i' = \sum_{j \neq i} Q_j, \qquad \delta = \alpha - \gamma; \tag{2.2}$$

then firm i's profit can be expressed as

$$p(Q_i, Q_i') = (\delta - \beta Q_i') Q_i - \beta Q_i^2. \tag{2.3}$$

Equations (2) and (3) define a game with N players, in which the pure strategy of player i is a nonnegative number Q_i, and his utility is $p(Q_i, Q_i')$.

It is easily verified that, given Q_i', the Q_i that maximizes i's profit is

$$r(Q_i') = \begin{cases} \dfrac{\delta - \beta Q_i'}{2\beta}, & \text{if \textit{this is nonnegative},} \\ 0, & \textit{otherwise}. \end{cases} \tag{2.4}$$

I shall call $r(Q_i')$ firm i's *best response* to Q_i'. If the best response is positive, firm i's corresponding maximum profit is

$$g(Q_i') = \frac{(\delta - \beta Q_i')^2}{4\beta}; \tag{2.5}$$

otherwise it is zero. A *Cournot–Nash equilibrium* (CNE) is an N-tuple (Q_i) of outputs such that, for each i, Q_i is the best response to $\sum_{j \neq i} Q_j$. In other words, a CNE is a solution (Q_i) of

$$r\left(\sum_{j \neq i} Q_j\right) = Q_i, \qquad i = i, \ldots, N.$$

It is easily verified that the unique CNE is given by

$$Q_i = Q_N^* = \frac{\delta}{(N + 1)\beta}, \qquad i = 1, \ldots, N, \tag{2.6}$$

and the corresponding CNE profit per firm is

$$\frac{\delta^2}{\beta(N + 1)^2}. \tag{2.7}$$

Note that the total industry CNE output is

$$\frac{N\delta}{(N+1)\beta}, \tag{2.8}$$

and the total industry CNE profit is

$$\frac{N\delta^2}{(N+1)^2\beta}. \tag{2.9}$$

Therefore, as N increases without bound, total CNE industry output approaches δ/β, total industry CNE profit approaches zero, and CNE price approaches $\alpha - \delta - \gamma$ (i.e., marginal cost), all of which conditions characterize a competitive equilibrium.

If the industry acts as a cartel to maximize total industry profit then the cartel output is $(\delta/2\beta)$, and the corresponding cartel profit is $(\delta^2/4\beta)$. If the cartel output and profit were divided equally among the firms, then the corresponding cartel output per firm would be

$$\hat{Q}_N = \frac{\delta}{2\beta N}, \tag{2.10}$$

and the cartel profit per firm would be

$$\frac{\delta^2}{4\beta N}. \tag{2.11}$$

Note that if $N > 1$, then the cartel profit per firm is strictly greater than the CNE profit per firm, so that (from the point of view of the firms) the CNE is not Pareto optimal.

If the capacities of the firms are sufficiently large, then no coalition of fewer than N firms can guarantee itself more than a zero profit. That is to say, for any output of the coalition, there is an output of the other firms such that the coalition's profit is not greater than zero. Hence in this case, the core is the set of all nonnegative allocations of the cartel profit among the firms.[4] Given the symmetry among the firms, the equal division of the cartel profits is a "reasonable" target for cooperation, and in any case is the one to which attention will be given in this paper.

In what follows, I shall simplify the formulas by taking $\beta = 1$ and $\delta = 1$, unless notice is given to the contrary. This normalization will not entail any essential loss of generality.

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[4] For a characterization of the core with arbitrary capacities, see Radner [4].

3. THE SEVERAL-PERIOD COURNOT GAME

Consider now a sequential, *T-period*, game in which the one-period game is repeated T times (T finite). The resulting utility to a firm is assumed to be the average of the T one-period profits. Let Q_{it} denote the output of firm i at date $t(1 \leq t \leq T)$, i.e., during the ith one-period game. A pure strategy for firm i is a sequence of functions, σ_{it}, one for each date t; the function for date t determines i's output at t as a function of the outputs of all firms at all previous dates. A Cournot-Nash equilibrium of the T-period game is a combination of strategies, one for each firm, such that each firm's strategy is a best response to the combination of the other firms' strategies.

The concept of *perfect* equilibrium of the T-period game has been introduced by Selten (1975) to rule out equilibria in which players use threats that are not credible. For every data t, let H_t denote the *history* of all the firms' outputs through t, i.e., the array of outputs Q_{ik}, $i = 1,..., N$, $k = 1,..., t$. For any sequential strategy σ_i for firm i, any date t and any history H_{t-1} let $\sigma_i[t, H_{t-1}]$ denote the *continuation* of σ_i from date t on, given the history H_{t-1}. A strategy combination (σ_i) is a *perfect* CNE if, for every date t and history H_{t-1}, the strategy combination $(\sigma_i[t, H_{t-1}])$ is a CNE of the sequential game corresponding to the remaining dates $t,..., T$. Note that in the definition of a perfect CNE one must test, for each t, whether the combination of continuations is a CNE for *all* possible histories H_{t-1}, not just the history that would be produced by the strategy combination (σ_i).

It is easy to verify that in every perfect CNE of the T-period game, each firm produces output Q^* at each date, where Q^* is given by (2.6). This can be seen by "working backwards," since at the end of period t the firms face a $(T - t)$- period game. The resulting utility to each firm is, from (2.7), $1/(N + 1)^2$. (Recall that $\beta = \delta = 1$.)

On the other hand, if each firm were to produce its cartel output $\hat{Q}$ at each date (see (2.10)), then the resulting utility to each player would be $1/4N$. Since there are several periods, the firms have the opportunity to react differently to cooperative and noncooperative moves by the other firms. For example, consider the following strategy: firm i produces output $\hat{Q}$ in each period as long as every other firm has been doing the same; thereafter firm i produces Q^* in each period. Call this strategy C_T. Formally, define D_i as follows:

$$D_i \begin{array}{ll} = \infty, & \text{if } Q_{jt} = \hat{Q} \text{ for all } t \text{ and all } j \neq i, \\ = \min\{t\colon Q_{jt} \neq \hat{Q} \text{ for some } j \neq i\}, & \text{otherwise.} \end{array} \tag{3.1}$$

The pure strategy C_T is defined by:

$$Q_{it} \begin{array}{ll} = \hat{Q} & \text{if } t \leqslant D_i, \\ = Q^* & \text{if } t > D_i. \end{array} \tag{3.2}$$

More generally, for any integer k between 0 and T define the pure strategy C_k by:

$$Q_{it} \begin{cases} = \hat{Q} & \text{if } t \leqslant \min(D_i, k), \\ = Q^* & \text{if } t > \min(D_i, k). \end{cases} \tag{3.3}$$

Note that if i uses the strategy C_0, then he always produces the CNE output Q^*.

The strategy C_k is a special case of a slightly more general class, which I shall call *trigger strategies of order k*. Let D_i be defined again as in (3.1) and let Q^D be some output. If $D_i \geqslant k$, then

$$\begin{aligned} &= \hat{Q} && \text{if } t \leqslant k, \\ Q_{it} &= Q^D && \text{if } t = k + 1, \\ &= Q^* && \text{if } t \geqslant k + 2. \end{aligned} \tag{3.4}$$

If $D_i \leqslant k$, then

$$Q_{it} \begin{cases} = \hat{Q} & \text{if } t \leqslant D_i, \\ = Q^* & \text{if } t > D_i. \end{cases} \tag{3.5}$$

One might call Q^D the *defection output*, which i uses once only if all other firms have stayed with the cartel for (at least) k periods.

Suppose now that all firms other than i use the same trigger strategy of order $k > 0$, with some defection output $Q^D > \hat{Q}$. I shall show that i's best response is a trigger strategy of order $(k - 1)$, with a defection output equal to

$$\tilde{Q} = r[(N - 1)\,\hat{Q}] = \frac{N + 1}{4N}. \tag{3.6}$$

Note that $\tilde{Q}$ is the best *one-period* response to a total output of $(N - 1)\hat{Q}$ by all the other firms, and yields a one-period profit of

$$\frac{(N + 1)^2}{16N^2}. \tag{3.7}$$

To prove this, first observe that if at some date t any firm i produces an output different from $\hat{Q}$, then at all subsequent dates all firms other than i will produce Q^*, and it will be optimal for firm i to do the same. Hence firm i's best response has the property that

$$\begin{aligned} Q_{jt} &\neq \hat{Q} && \textit{for some j implies} \\ Q_{it'} &= Q^* && \textit{for all } t' > t. \end{aligned} \tag{3.8}$$

It follows that firm i's best response to the given trigger strategies of order k is a trigger strategy of some order n. It is straightforward to verify that if $n < k$ then the optimal defection output is $\tilde{Q}$, with resulting *total* profit

$$\frac{n}{4N} + \frac{(N+1)^2}{16N^2} + \frac{T-n-1}{(N+1)^2}. \tag{3.9}$$

If $n = k$, then the optimal defection output is $r[(N-1)Q\tilde{Q}^D]$, and the corresponding total profit is

$$\frac{k}{4N} + g[(N-1)\,Q^D] + \frac{T-k-1}{(N+1)^2}. \tag{3.10}$$

Finally, if $n > k$, then i's defection output is irrelevant and the corresponding profit is

$$\frac{k}{4N} + p[\hat{Q}, (N-1)\,Q^D] + \frac{T-k-1}{(N+1)^2}. \tag{3.11}$$

In (3.10) and (3.11), if $k = T$, then it is to be understood that the final term in the equation is zero. Since $Q^D > \hat{Q}$,

$$\left(\frac{N+1}{4N}\right)^2 = g[(N-1)\,\hat{Q}] > g[(N-1)\,Q^D] \geqslant p[\hat{Q}, (N-1)\,Q^D].$$

Hence since (3.9) is increasing in n, it follows that i's optimal response has $n = (k-1)$. This completes the proof that i's *optimal response to a trigger strategy of order* $k > 0$ *with defection output* $Q^D > \hat{Q}$ *is a trigger stragegy of order* $k-1$ *with defection output* $\tilde{Q}$. *The resulting average* profit for i is

$$\left(\frac{1}{T}\right)\left[\frac{k-1}{4N} + \left(\frac{N+1}{4N}\right)^2 + \frac{T-k}{(N+1)^2}\right]. \tag{3.12}$$

Note that neither i's optimal response nor his resulting average profit depend on the other firms' defection output.

In particular, if all firms $j(\neq i)$ use the trigger strategy C_T, then i's best response gives him an average profit of

$$\frac{1}{T}\left[\frac{T-1}{4N} + \left(\frac{N+1}{4N}\right)^2\right], \tag{3.13}$$

whereas if *all* firms use C_T, then every firm's average profit is $(1/4N)$, which is the cartel profit (per firm). The difference between (3.13) and the cartel profit per firm is

$$\left(\frac{1}{T}\right)\left(\frac{N-1}{4N}\right)^2. \tag{3.14}$$

Hence, as T increases without limit, the advantage to any one firm of defecting from the cartel one period before the end of the game approaches zero.

4. Epsilon-Equilibria

In the previous section it was noted that, in the T-period game, all perfect Cournot–Nash equilibria have the property that each firm produces the one-period CNE output in each period. On the other hand, the advantage to any one firm of defecting from the cartel approaches zero as T gets large, provided the other firms use trigger strategies of order T.

These considerations suggest a weakened form of the Cournot–Nash equilibrium concept. For any positive number ϵ, an ***ϵ-equilibrium*** is an N-tuple of strategies, one for each firm, such that each fiirm's average profit is within ϵ of the maximum average profit it could obtain against the other firms' strategies. In this and the following sections I shall explore some of the properties of epsilon-equilibria in the T-period Cournot game. I do not, however, have a complete characterization of epsilon-equilibria in this game.

The first candidate for an epsilon-equilibrium is the situation in which each firm uses the trigger strategy C_k (with defection output Q^*). From (3.10) and (3.12) we see that the difference in average profit between the best response and C_k, for an individual firm, is

$$\left(\frac{1}{T}\right)\left[\left(\frac{N+1}{4N}\right)^2 - \frac{1}{4N}\right] = \left(\frac{1}{T}\right)\left(\frac{N-1}{4N}\right)^2. \tag{4.1}$$

Hence the N-tuple (C_k) is an ϵ-equilibrium with $k > 0$ if and only if

$$\left(\frac{1}{T}\right)\left(\frac{N-1}{4N}\right)^2 \leqslant \epsilon \quad \text{or} \quad T \geqslant \left(\frac{1}{\epsilon}\right)\left(\frac{N-1}{4N}\right)^2. \tag{4.2}$$

Note that (4.2) is independent of k, so that either (C_k) is an ϵ-equilibrium for all $k = 1,\dots, T$, or for none. Note, too, that for fixed ϵ, (4.2) is satisfied uniformly in N for sufficiently large T. Of course, (C_0) is a Cournot–Nash equilibrium, so it is an ϵ-equilibrium for all ϵ.

The concept of perfect CNE can be extended to epsilon-equilibria as follows. A strategy combination (σ_j) is a *perfect* ϵ-equilibrium if for every date t, every history H_{t-1}, and every firm i, the continuation of i's strategy from date t on, given the history H_{t-1}, is within ϵ of being the best response to the corresponding continuations of the other firms' strategies. In this definition, the utility of a continuation of a strategy is the average of the profits in *all* T periods. (For an alternative definition, see Section 7.)

It is easy to verify that (4.2) is a necessary and sufficient condition for the combination (C_k) of trigger strategies to be a perfect ϵ-equilibrium. Hence, *for any $\epsilon > 0$ there is a T_ϵ such that, for all $T \geqslant T_\epsilon$ and all $k = 0,\dots, T$, there is a perfect ϵ-equilibrium in which each firm produces its cartel output for* *exactly k periods*. Furthermore, one can take T_ϵ to be independent of the number of firms.

An examination of (3.9)–(3.11) shows that one can get similar results for N-tuples of trigger strategies that use defection outputs other than Q^*, and even for N-tuples of trigger strategies that differ among firms in both the orders of the trigger strategies and the defection outputs. No attempt will be made here to characterize all such perfect epsilon-equilibria, but it is clear that for fixed ϵ, the larger T is the larger, in some sense, is the set of perfect ϵ-equilibria.

In the rest of this paper, all epsilon-equilibria are to be understood as perfect, unless specific notice is given to the contrary.

5. LARGE NUMBER OF FIRMS: THE FIXED-DEMAND CASE

In the last section I considered how the set of (perfect) ϵ-equilibria varied with T, the number of periods, with the number of firms fixed. In this section I consider the effect of increasing N, the number of firms, with the horizon, T, fixed. I first analyze the case in which the total industry demand function remains fixed as the number of firms varies. The results for this case will be used to analyze the more interesting, but slightly more complicated, replication case (Section 6).

To begin the analysis of the fixed-demand case, first observe that, by (4.2), for fixed T, there is no (C_k) ϵ-equilibrium ($k \geqslant 1$) for ϵ sufficiently small.

I shall prove, in addition, a result that is in some ways stronger. Roughly speaking, I shall show that if ϵ is small then all ϵ-equilibria are close to the CNE, *uniformly in N*. To be precise I shall show: for every $\epsilon > 0$ and $T \geq 1$ there is a number $B(\epsilon, T)$ such that for every $N \geq 1$ and every ϵ-equilibrium the following are all bounded by $B(\epsilon, T)$:

$$\begin{gathered} | Q_{it} - Q_N^* |, \\ \left| \sum_{i=1}^{N} Q_{it} - NQ_N^* \right|, \\ \left| p\left(Q_{it}, \sum_{j\neq i} Q_{jt}\right) - \frac{1}{(N+1)^2} \right|, \end{gathered} \tag{5.1}$$

for $i = 1,\ldots, N$, $t = 1,\ldots, T$. In addition, for every T

$$\lim_{\epsilon\to 0} B(\epsilon, T) = 0. \tag{5.2}$$

The first line of (5.1) is the difference between firm i's output in period t and CNE output; the second line is the difference between total industry output in period t and industry CNE output; and the third line is the difference between firm i's profit in period t and CNE profit per firm. It follows that, for any positive number d, and any horizon T, there is a positive ϵ such that, for all N and all ϵ-equilibria of the T-period game with N firms, industry outputs and market prices in all periods $t = 1,\ldots, T$ will be within d of their corresponding one-period CNE values (note that ϵ does not depend on N). In particular, as N increases without limit, the corresponding one-period CNE values will converge to their respective competitive equilibrium values, and hence any corresponding sequence of ϵ-equilibria will approach the "neighborhood" of competitive equilibrium defined by the distance d, with respect to total industry output and price.

To prove (5.1) and (5.2), I shall first do the one-period case, and then make an induction on T. For the one-period case, an ϵ-equilibrium is an N-tuple $(Q_1 ,\ldots, Q_N)$ satisfying

$$p\Big(Q_i , \sum_{j\neq i} Q_j\Big) \geqslant g\Big(\sum_{j\neq i} Q_j\Big) - \epsilon, \qquad i = 1,\ldots, N. \tag{5.3}$$

Define new variables x_i and x_i' by

$$\begin{aligned} x_i &= Q_i - Q_N^* = Q_i - \frac{1}{N+1}, \\ x_i' &= \sum_{j\neq i} x_j . \end{aligned} \tag{5.4}$$

Using (2.3) and (2.5) one easily verifies that (5.3) is equivalent to

$$(x_i + (x_i'/2))^2 \leqslant \epsilon, \qquad i = 1,\ldots, N,$$

or to

$$| x_i + x. | \leqslant 2h, \qquad i = 1,\ldots, N,$$

where

$$x. = \sum_{j=1}^{N} x_j , \qquad h = (\epsilon)^{1/2}. \tag{5.5}$$

Summing the inequalities (5.5) one gets

$$| x. | \leqslant \frac{2Nh}{N+1} ; \tag{5.6}$$

this and (5.5) imply

$$| x_i | \leqslant \frac{2(2N + 1)\, h}{N + 1} , \tag{5.7}$$

$$| x_i' | \leqslant \frac{2(3N + 1)\, h}{N + 1} . \tag{5.8}$$

LEMMA. *If* $|x| \leq b$ and $|x'| \leq b$, then

$$|p[Q_N^* + x, (N-1)Q_N^* + x'] - p[Q_N^*, (N-1)Q_N^*]| \leq \frac{b}{N+1} + 2b^2. \tag{5.9}$$

Proof. The inequality can be verified using (2.3), (2.6), and (2.7).

Note that the right-hand sides of inequalities (5.6)–(5.8) are all dominated by $6h$. Hence, by the lemma, one can take

$$B(\epsilon, 1) = \max\left[6h, \frac{6h}{N+1} + 2(6h)^2\right]. \tag{5.10}$$

For ϵ sufficiently small $((\epsilon)^{1/2} \leq 1/18)$, the right side of (5.10) is $6h = 6(\epsilon)^{1/2}$.

Now make the induction hypothesis that the main result is true for any number of periods up to and including T. Fix ϵ, and consider any $(T+1)$-period ϵ-equilibrium. Given the initial outputs Q_{i1}, $i = 1, \ldots, N$, the remaining T-period strategies constitute, *a fortiori*, a T-period, ϵ'-equilibrium, where

$$\epsilon' = \frac{(T+1)\,\epsilon}{T}. \tag{5.11}$$

Let i be any particular firm, which will remain fixed for the time being, let $Q = Q_{i1}$ be i's output in period 1, and let S denote the strategy that i follows during periods 2 through $(T+1)$. Given the strategies of the other firms, one may denote i's average profit as

$$\frac{p(Q, Q')}{T+1} + V(Q, S) = \pi(Q, S), \tag{5.12}$$

where Q' is the total output in period one of all the firms other than i.

The induction hypothesis is that all of the quantities in (5.1), for $i = 1, \ldots, N$ and $t = 2, \ldots, T+1$, are bounded by $b = B(\epsilon', T)$. Actually, I shall strengthen the induction hypothesis to add the following inequalities:

$$\left|\sum_{j \neq i} Q_{jt} - (N-1)\,Q_N^*\right| \leq b, \qquad i = 1, \ldots, N, \quad t = 2, \ldots, T+1. \tag{5.13}$$

The definition of epsilon-equilibrium implies that

$$\pi(Q, S) \geq M - \epsilon,$$

where (5.14)

$$M = \max_{q, s} \pi(q, s).$$

(In the definition of M, (q, s) ranges over all $(T + 1)$-period strategies for firm i, holding constant the ϵ-equilibrium strategies of the other firms.) This last is equivalent to

$$\frac{p(Q, Q')}{T+1} + V(Q, S) \geq M - \epsilon,$$

or

$$p(Q, Q') \geq (T + 1)[M - \epsilon - V(Q, S)]. \tag{5.15}$$

By the induction hypothesis,

$$\frac{(T + 1)\, V(Q, S)}{T} - \frac{1}{(N + 1)^2} \leq b. \tag{5.16}$$

To get a bound on M, note that

$$(T + 1)M \geq g(Q') + Tm,$$

where

$$m = \max_X \min_Y \{p(X, Y) : |Y - (N - 1)\, Q_N^*| \leq b\}. \tag{5.17}$$

One easily verifies that

$$m = g[(N - 1)\, Q_N^* + b] = \frac{1}{(N + 1)^2} - \frac{b}{(N + 1)} + \frac{b^2}{4}. \tag{5.18}$$

Putting together (5.15)–(5.18), one has

$$p(Q, Q') \geq g(Q') - \epsilon'',$$

where

$$\epsilon'' = \frac{T(N + 2)\, b}{(N + 1)} + (T + 1)\, \epsilon, \tag{5.19}$$

The inequality (5.19) holds for all firms i. In other words, in a $(T + 1)$-period ϵ-equilibrium, the first-period outputs constitute a one-period ϵ''-equilibrium, where ϵ'' is given by (5.19) and

$$b = B\left[\frac{(T + 1)}{T}\, \epsilon, T\right]. \tag{5.20}$$

Hence, by the one-period case, the quantities in (5.1) and (5.13), with $t = 1$, are all bounded by

$$b' = 6(\epsilon'')^{1/2} = 6\left[\frac{T(N + 2)\, b}{(N + 1)} + (T + 1)\, \epsilon\right]^{1/2}, \tag{5.21}$$

provided that $\epsilon''^{1/2} < 1/18$. If we take

$$B(\epsilon, T+1) = \max\{b', B(\epsilon, T)\}, \qquad (5.22).$$

then the induction step is completed, as far as the inequalities (5.1) and (5.13) are concerned. Furthermore, (5.20)–(5.22) determine a recursion formula for the bounds $B(\epsilon, T)$, which with the formulas (5.5) and (5.10) for $B(\epsilon, 1)$ prove (5.2), i.e., that $B(\epsilon, T)$ tends to zero as ϵ tends to zero, for fiyed T.

6. LARGE NUMBERS OF FIRMS: THE REPLICATION CASE

In the replication case, the demand function depends on the number of firms, so that corresponding to (2.1) we have

$$P = \alpha - \frac{\beta_1 Q}{N}, \qquad (6.1)$$

where Q is the total industry output, P is the demand price, and β_1 is a parameter that is independent of N, the number of firms. One may motivate this formulation in terms of replicating an industry. Consider an industry with 1 firm and a given population of M potential buyers, whose demand function is given by

$$Q = \frac{\alpha - P}{\beta_1}. \qquad (6.2)$$

The N-fold replication of this industry is made up of N firms, together with a population of NM potential buyers with the same per capita demand at any price as the original population. The resulting demand function is then

$$\frac{Q}{N} = \frac{\alpha - P}{\beta_1},$$

which is equivalent to (6.1). Without essential loss of generality, I shall take $\beta_1 = 1$ and $\delta = \alpha - \gamma = 1$.

Corresponding to the formulas of Section 2, one has the following formulas, obtained by everywhere replacing β by $1/N$. Firm i's profit function is

$$p(Q_i, Q_i') = \left(1 - \frac{Q_i'}{N}\right) Q_i - \frac{Q_i^2}{N}. \qquad (6.3)$$

Its optimal response to Q_i' is

$$r(Q_i') = \begin{cases} \dfrac{N - Q_i'}{2N}, & \text{if this is positive,} \\ 0, & \text{otherwise,} \end{cases} \qquad (6.4)$$

and its corresponding profit is

$$g(Q_i') = \frac{(N - Q_i')^2}{4N}, \qquad (6.5)$$

if this is nonnegative, and zero otherwise.

The one-period CNE output and profit per firm are given by

$$Q_N^* = \frac{N}{N+1}, \qquad (6.6)$$

$$g[(N-1)\,Q_N^*] = \frac{N}{(N+1)^2}. \qquad (6.7)$$

The one-period cartel output and profit per firm are

$$\hat{Q}_N = \tfrac{1}{2}, \qquad (6.8)$$

$$p[\hat{Q}_N, (N-1)\hat{Q}_N] = \tfrac{1}{4}. \qquad (6.9)$$

Firm i's best response if each other firm produces the cartel output is

$$r[(N-1)\,\hat{Q}_N] = \frac{N+1}{4N}, \qquad (6.10)$$

with corresponding profit

$$g[(N-1)\,\hat{Q}_N] = \frac{(N+1)^2}{16N}. \qquad (6.11)$$

Turning to the T-period game, one easily verifies that the strategy combination in which each firm has the trigger strategy C_k (cf. Section 4) is an ϵ-equilibrium if and only if

$$T \geqslant \left(\frac{N}{2}\right)\left(\frac{N-1}{4N}\right)^2; \qquad (6.12)$$

compare this with (4.2). Hence, as in the fixed-demand case, for every ϵ and N, every (C_k) strategy combination is an ϵ-equilibrium for all sufficiently large T. However, in this case one gets the result that, for fixed ϵ and T, no (C_k) strategy combination $(k > 0)$ is an ϵ-equilibrium for sufficiently large N. In other words, *for any fixed ϵ and number of periods, the cartel cannot survive at all if the number of firms is large enough.*

Corresponding to the rest of the analysis in Section 5, one has the following results. For every ϵ, T, and $N(>1)$, there is a number $B(\epsilon, T, N)$ such that, in every ϵ-equilibrium, the following quantities are bounded by $B(\epsilon, T, N)$:

$$\begin{gathered} | Q_{it} - Q_N^* |, \\ \left| \sum_j Q_{it} - NQ_N^* \right|, \end{gathered} \qquad (6.13)$$

for $i = 1,\ldots, N$ and $t = 1,\ldots, T$; the bounds $B(\epsilon, T, N)$ may be chosen so that, for every ϵ and T,

$$\frac{B(\epsilon, T, N)}{N^{1/2}} \text{ is uniformly bounded in } N, \tag{6.14}$$

and for every T and N,

$$\lim_{\epsilon \to 0} B(\epsilon, T, N) = 0. \tag{6.15}$$

It follows from (6.6) that average output per firm approaches 1, and market price approaches $1 - \alpha = \gamma$ (marginal cost) as N increases without limit. In addition, in every period every firm i's *relative* share of total industry output, which is

$$\frac{Q_{it}}{\sum_j Q_{jt}} = \frac{(1/N)\, Q_{it}}{(1/N) \sum_j Q_{jt}},$$

converges to zero as N gets large. Finally, one can show that, in every period, every firm i's profit is within

$$\frac{B(\epsilon, T, N)}{N+1} + \frac{2[B(\epsilon, T, N)]^2}{N} \tag{6.16}$$

of the one-period CNE profit, which is $N/(N+1)^2$, and this bound is uniformly bounded in N, and goes to zero with ϵ.

Thus, in these various ways, for large N, ϵ-equilibria are close to competitive equilibrium.

I shall omit the proof of these results, which parallels the argument in Section 5. The key facts are that, for the one-period case, Eqs. (5.5)–(5.8) are still valid, but with

$$h = (N\epsilon)^{1/2}, \tag{6.17}$$

and in the Lemma of Section 5, the right side of (5.9) must be replaced by

$$\frac{b}{N+1} + \frac{2b^2}{N}, \tag{6.18}$$

which is less than b for

$$b < \frac{N^2}{2(N+1)}. \tag{6.19}$$

In particular, for the one-period case one can take

$$B(\epsilon, 1, N) = (N\epsilon)^{1/2}, \tag{6.20}$$

provided that

$$(\epsilon)^{1/2} \leqslant \frac{N}{12(N+1)}, \tag{6.21}$$

7. AN ALTERNATIVE DEFINITION OF PERFECT EPSILON-EQUILIBRIUM

As an alternative to the definition of perfect epsilon-equilibrium given in Section 4, one can take the utility of the continuation of a strategy to be the average of the profits in the remaining periods, rather than the average of the profits in all T periods.[5] This change leads to a definition of perfect epsilon-equilibrium that is more restrictive, in the sense that, for every positive epsilon, the set of perfect epsilon-equilibria is smaller. Results analogous to those of Sections 5 and 6 can be derived for this definition; I omit the details. However, the results in Section 4, on trigger strategies, are changed in an interesting way. For a fixed positive epsilon, and a fixed number of firms, the combination (C_k) of trigger strategies is a perfect epsilon-equilibrium for all sufficiently large horizons T and all k not too close to T, namely, if and only if

$$T - k \geqslant \left(\frac{1}{\epsilon}\right)\left(\frac{N-1}{4N}\right)^2 - 1.$$

Thus, for this alternative definition of epsilon-equilibrium, *a cartel held together by trigger strategies will break down as the industry approaches the horizon T.*

8. INTERPRETATIONS OF EPSILON

Why should a firm be satisfied with a less-than-optimal response to the strategies of other firms? One type of answer refers to the various costs of discovering and using alternative strategies, and alludes to the possibility that a truly optimal response might be more costly to discover and use than some alternative, "nearly optimal" strategy. In this interpretation, the "epsilon" for a particular firm represents a judgement of the firm that the additional benefits from improving its strategy would be outweighed by the additional costs. (In the present analysis, all firms were assumed to have the same epsilon, but this simplification is not strictly needed for the results.) It would be consistent with the spirit of the model for this judgement to be in part subjective, rather than necessarily based on some precise calculation of benefits and costs.

A second interpretation of epsilon might be based on the supposition that the firms realize that strict optimization of each firm's response to the other firms' strategies would lead to a breakdown of the cartel. This approach is intuitively appealing, but I am not aware of any satisfactory formal model of rational behavior on which it could be based.

Recall that, in the model of the present paper, it is assumed that each firm uses the criterion of average profit per period to compare strategies; thus epsilon is measured in "dollars per period." This scale of measurement would be consistent with the cost-of-decision interpretation of epsilon-equilibrium. In a more general treatment, epsilon would be measured in units of utility. However, epsilon-equilibria would not be invariant under transformations of the utility functions of the firms that change the unit of measurement of utility. For example, if the preferences of a firm were scaled in terms of a von Neumann-Morgenstern utility function, then epsilon-equilibria would not be invariant under all transformations of the utility function that leave the firm's preferences invariant. A solution to this last problem would be to adopt a "canonical" utility representation for each firm, and then to interpret epsilon with reference to those canonical utility functions. This would be equivalent, for each firm, to interpreting epsilon as a given *percentage* of the difference in utility between two reference profits. Within this framework, the interpretation of the results of the present paper is straightforward, with the proviso that, for those situations in which the number of firms increases without limit, some condition of "similarity" of the epsilons of the different firms would have to be satisfied (as would naturally occur in the replication case).

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[5] This alternative was suggested to me by Sanford Grossman and Robert Rosenthal.

ACKNOWLEDGMENTS

I am grateful to R. Rosenthal, C. Futia, and A. Mas-Colell for helpful discussions of this problem. Preliminary versions of this paper were presented at the North American regional meeting of the Econometric Society, Boulder, Colo., June 1978, and at the IMSSS, Stanford University, July 1978.

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JOURNAL OF ECONOMIC THEORY 22, 155–182 (1980)

Noncooperative Price Taking in Large Dynamic Markets

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Received March 21, 1979

1. INTRODUCTION

In "Researches into the Mathematical Principles of the Theory of Wealth" [1], Cournot founded the theory of oligopoly. His contribution can be divided into two parts. First, he proposed a static noncooperative equilibrium concept for a situation of dynamic inter-firm conflict. Second, he showed that this equilibrium solution coincides with perfect competition in markets where there are many agents.

Cournot's solution for oligopoly has come under heavy criticism for its static nature. Is it really plausible, the critics ask, that the commitment of each firm to a long-term level of production brings an end to strategic interaction in a market? A strongly opposing view has been put forth by Stigler, who suggests (in [9]) that firms in an oligopoly maintain a monopolistic price by threatening defectors with retaliatory actions. Such retaliation presupposes a dynamic equilibrium in which each firm observes the ongoing behavior of its rivals and makes decisions over time contingent on that behavior. The prerequisite for collusive equilibrium of this type is not necessarily that each individual firm should be capable of significantly affecting aggregate output, but only that each firm's action (whatever its scale relative to the industry) should be visible to its competitors. On this account, it has been suggested that even some very large markets may fail to be competitive.[1]

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* Financial support for this research was provided through a grant from the Sloan Foundation to the Princeton University Economics Department. I would like to express my appreciation to Edward Prescott, who drew my attention to the problem studied here. I am grateful to Robert Anderson and Hugo Sonnenschein for discussions and suggestions which have greatly improved the paper.

[1] For instance, it has been suggested that markets for the services of doctors and other professionals are not competitive, despite the presence (at least in large cities) of many sellers. The presence of legal barriers to entry in these markets may explain why professionals earn more than they would in alternative employment, but it would not by itself account for the observation of price discrimination by professionals. An equal-profit condition would be satisfied in static equilibrium, because professionals would bid away from their colleagues those types of client whom it was particularly profitable to serve.

0022-0531/80/020155-28$02.00/0

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In the present paper, Cournot's proposition that large markets are competitive will be reconciled with the possibility of equilibrium involving strategic interaction among firms. It will be shown that, except when extraordinarily precise information is available, firms in a market with many agents are not visible enough for the equilibrium described by Stigler to be sustained. This result will be proved by studying two formal representations of markets with many agents.

One of these representations involves initially choosing a particular market (with finitely many participants), and then forming a sequence of increasingly large markets by adjoining at each stage a new copy of the original one. The result of this procedure is called a *sequence of replica markets*. All of the markets in the sequence have the same statistical characteristics (e.g., the ratio of firms to consumers is constant), and they all share the same competitive market-clearing prices. Thus, it makes sense to ask whether the noncooperative equilibria of the markets in the sequence converge to competition. We will say that the *limit principle* is satisfied in situations where this convergence takes place.

The other representation of markets with many agents exploits the fact that, if counting measure on the set of agents in a market is normalized to make the set into a probability space, then a description of agents' characteristics and actions is mathematically a random vector defined on the sample space of agents. An analogous random vector defined on a nonatomic sample space may be substituted for a large finite market as an object of investigation. Such a random vector, called a *nonatomic market*, is natural to study because one occurs as the limit of a replica sequence of markets. We will say that the *inclusion principle* is satisfied in a situation where every noncooperative equilibrium of a nonatomic market is competitive.[2]

The outline of the paper is as follows. A dynamic noncooperative equilibrium concept will be developed in the next section. Then, in Section 3, a sequence of replica markets will be defined, each of which has a noncooperative equilibrium which supports the monopoly allocation at every time. This example refutes the limit principle for dynamic equilibrium. An example is also given in which, if each firm can observe the output level of every other firm, then a collusive noncooperative equilibrium exists for a market with a continuum of firms. Thus the most general form of the inclusion principle is refuted as well. Together these examples suggest that, even when there are many firms, the potential for collusion envisioned by Stigler may still exist. In the remaining sections of the paper, the special nature of such potentially collusive markets will be shown.

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[2] Representations of markets with many agents have been used extensively to study the core of an economy. A standard reference for this work is [5]. Although results for the core are not very sensitive to which representation is used, there are substantial differences between the conditions which are sufficient to ensure the limit principle and the inclusion principle for noncooperative equilibrium. This matter is discussed in [2]. The term "inclusion principle," rather than "equivalence principle," is used here because of the issues raised in [6].

In the counterexample to the inclusion principle, firms are required to have exhaustive non-price information about their competitors. In Section 4, it is proved that all equilibria which violate the principle require such information. This is done by interpreting dynamic markets within an abstract theory of repeated games, and providing a sufficient condition for all Nash equilibria of a repeated game to be constructible from Nash equilibria of the static game on which it is based. This condition yields an inclusion theorem for dynamic markets in which agents have information only about price and other market aggregates.

The inclusion principle derived in Section 4 sheds light on the failure of the limit principle. The principle must fail because there is information about individual producers which the price mechanism conveys in every finite market, but not in a nonatomic one. In fact, in a finite market, price information can reveal to a firm that a competitor has disregarded its strategic threat, although the identity of the violator is not revealed. If the price varies because of random demand, its informativeness about the supply side of the market should be reduced. As the scale of an individual producer becomes arbitrarily small relative to market demand in a sequence of replica dynamic markets, any stochastic demand disturbance should make strategic threats unenforceable in the limit. This is proved in Section 5, again in an abstract game-theoretic context. It yields a limit theorem for dynamic markets with demand uncertainty.

In Section 6 some concluding remarks are made concerning the relevance of the present results to industrial organization.

2. NONCOOPERATIVE EQUILIBRIA WHICH SUPPORT THE MONOPOLY ALLOCATION

The first inclusion and limit theorems were derived by Cournot in a static partial equilibrium setting. These will be reviewed in this section, and counterexamples to their dynamic versions will be given in the next. These results may be reformulated in a general equilibrium setting. The positive results of this paper are immediate consequences of game theoretic results which apply straightforwardly to the general equilibrium version.

First, static and dynamic markets are defined. Intuitively, a static market consists of a set of firms, each of which supplies the market good at a total cost which depends on the quantity it produces, and an inverse demand function which determines the price as a function of mean supply. Implicitly, this inverse demand function is determined by the actions of price-taking consumers. The question to be investigated is: are profit-maximizing firms also price-takers when there are many firms?

A (stationary) dynamic market is a static market which operates repeatedly, its times of operation being indexed by the natural numbers. Firms maximize the discounted present value of their profits.[3] These definitions are now presented formally:

DEFINITION. A *cost function* is an element of the set $Y = \{f \mid f\colon R_+ \to R_+ \cup \{\infty\}, f \text{ is continuous}\}$. A *normal inverse demand function* is a continuous function $D\colon R_+ \cup \{\infty\} \to R_+$ with $D(\infty) = 0$. A *static market* is an ordered 4-tuple $\langle K, \mu, y, D\rangle$ where K is an abstract set of firms, μ is a probability measure on K, $y\colon K \to Y$ is a measurable function (Y may be considered discrete) which specified the cost function of each firm, and D is a normal inverse demand function. A (stationary) *dynamic market* is an ordered 5-tuple $\langle K, \mu, y, D, \beta\rangle$, of which the first four components specify a static market and $\beta \in (0, 1)$ is the market discount factor. Let M be a market (static or dynamic) with measure space $\langle K, \mu\rangle$ of firms. A *supply vector* for M is a bounded μ-measurable function $q\colon K \to R_+$.

Given a supply vector q for M, mean supply will be $\int_K q(k)\, d\mu$ and the market price will $D(\int_K q(k)\, d\mu)$. If a firm $k \in K$ has cost function $y(k) = f$, then a supply vector determines a net profit level for k. This net profit will be denoted by $\pi_k(q)$, and satisfies the equation

$$\pi_k(q) = q(k)\, D\left(\int_K q(i)\, d\mu\right) - f(q(k)). \tag{1}$$

In selecting its output level, either a firm may accept the market price as an exogenous parameter of its decision problem, or it may recognize its own influence on the market price. A price-taking equilibrium results when all firms act in the former way, and a noncooperative equilibrium results when they act in the latter way.

DEFINITION. A *price-taking equilibrium* of the static market $\langle K, \mu, y, D\rangle$ is a supply vector q such that, for almost all (w.r.t. μ) $k \in K$,

$$\pi_k(q) = \max_{r \in R_+} \left[rD\left(\int_K q(i)\, d\mu\right) - (y(k))(r)\right]. \tag{2}$$

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[3] The present formulation of dynamic markets is restrictive. For instance, it does not allow for the holding of inventories. The inclusion theorem (Theorem 5) to be proved here generalizes straightforwardly to apply to richer models. I am confident that the limit theorem (Theorem 6) is also robust.

A *price-taking equilibrium* of the dynamic market $\langle K, \mu, y, D, \beta \rangle$ is a sequence $\langle q_t \rangle_{t \in N}$ of supply vectors such that, for almost all $k \in K$,

$$\sum_{t \in N} \beta^t \pi_k(q_t) = \max \left\{ \sum_{t \in N} \beta^t \left[r_t D \left(\int_K q_t(i)\, d\mu \right) - (y(k))(r_t) \right) \,\middle|\, \langle r_t \rangle_{t \in N} \in R_+{}^N \right\}. \tag{3}$$

A *noncooperative equilibrium* of the static market $\langle K, \mu, y, D \rangle$ is a supply vector q^* juch that, for almost all $k \in K$,

$$\pi_k(q^*) = \max_{q \in Q} \left[q(k)\, D \left(\int_K q(i)\, d\mu \right) - (y(k))(q(k)) \right], \tag{4}$$

where $Q = \{q \mid q \text{ is a supply vector and } q(i) = q^*(i) \text{ for all } i \neq k\}$.

To define noncooperative equilibrium for a dynamic market, the notion of a strategy must be introduced. A strategy of a firm is a rule which, at each time, determines an output level for the firm as a function of information which is available to the firm at that time. A strategy vector is an assignment of strategies to firms. Thus, if S is the space of strategies, a strategy vector is a measurable function $f\colon X \to S$. A noncooperative equilibrium of a dynamic market is a strategy vector f such that

(a) for almost all $k \in K$, $f(k)$ yields as high a discounted present value of returns (given that the other firms are using the strategies assigned to them by f) as would any other strategy $s \in S$, and

(b) Clause (a) will continue to hold at every future time for almost all $k \in K$, regardless of firms' information at that time.[4]

This rather informal definition of dynamic noncooperative equilibrium will be adequate to verify that the strategy vectors to be discussed in this section are equilibria. A more explicit definition will be given in Section 4. In the remainder of this section, straightforward generalizations of Cournot's theorems for static markets will be stated. The statement of the inclusion theorem requires no further preparation.

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[4] Clause (b) ensures that a firm will not base its initial output decision on an expectation that, at a later time, it will produce at a level which from the perspective of that time will seem suboptimal. In the next section, equilibria will be studied in which firms in a market maintain collusion by means of mutual strategic threats. Although these threats will not be exercised in equilibrium, they will not have a deterrent effect unless it would be in the firms' interest to exercise them in the event that deviation from the cartel did occur. This explains why (a) must hold even conditional on firms having received information which they in fact will not receive in equilibrium. Condition (b) was introduced (in [8]) by Selten, whose concept of perfect Nash equilibrium is equivalent to the dynamic NCE defined here.

THEOREM 1. *Every noncooperative equilibrium of a nonatomic static market is a price-taking equilibrium.*

To facilitate the statement of Cournot's limit theorem, the notion of a sequence of replica markets is introduced. Intuitively, such a series is formed by starting with a finite market, and at each stage adding a new "twin" of each of the original market agents (firms and consumers). This is done explicitly for firms, but must be done implicitly for consumers. Since there are n times as many consumers of each type in the nth market as in the first, aggregate demand in the nth market at any price is n times what it is in the first. However, since the inverse demand function is defined formally in terms of mean quantity rather than of aggregate quantity, and since there are n firms in the nth market for every one in the first, all markets in the sequence should share the same inverse demand function. As replication continues, the members of the sequence should resemble the nonatomic representation of the market in their equilibrium behavior. The sequences are now defined formally, after which Cournot's limit theorem is immediately stated.

DEFINITION. A *sequence of replica (static or dynamic) markets* is an infinite sequence $\langle M_n \rangle_{n \in N}$ of markets such that the following hold for some finite set K, $y\colon K \to Y$, $\beta \in (0, 1)$ and normal inverse demand function D. For all n, $M_n = \langle K_n, \mu_n, y_n, D, \beta \rangle$, where

$$K_n = K \times \{0 \cdots n\} \tag{5}$$

$$\mu_n(B) = \#B/\#K_n \qquad \text{for all } B \subseteq K_n . \tag{6}$$

(That is, μ_n is normalized counting measure.)

$$y_n(\langle k, m \rangle) = y(k) \qquad \text{for} \quad k \in K, m \leqslant n. \tag{7}$$

The market $M_\infty = \langle K \times [0, 1], \mu_\infty, y_\infty, D, \beta \rangle$ is the *nonatomic representation* of the sequence if

$$\mu_\infty(B) = \sum_{k \in K} \lambda(\{r \in [0, 1] \mid \langle k, r \rangle \in B\}) \qquad \text{for all} \quad B \subseteq K \times [0, 1]. \tag{8}$$

(That is, μ_∞ is Lebesgue measure on the copies of [0, 1] in K_∞.)

$$y_\infty(\langle k, r \rangle) = y(k) \qquad \text{for all} \quad \langle k, r \rangle \in K_\infty . \tag{9}$$

THEOREM 2. *Suppose that $\langle M_n \rangle_{n \in N}$ is a sequence of replica static markets, that the supply vector q_n is a noncooperative equilibrium of M_n for every n,* *and that some subsequence of* $\langle q_n, y_n \rangle_{n \in N}$ converges in distribution:[5] Then the subsequence has a limit $\langle q_\infty, y_\infty \rangle$, and q_∞ is a price-taking equilibrium of the nonatomic representation M_∞.

3. NONCOOPERATIVE EQUILIBRIA WHICH SUPPORT THE MONOPOLY ALLOCATION *(continued)*

Counterexamples to the analogs of Theorems 1 and 2 for dynamic markets are constructed by exhibiting noncooperative equilibria of large markets, in which the monopoly price is maintained by mutual strategic threats among the firms. The existence of these equilibria in markets with finitely many firms was first pointed out by Friedman [4]. A counterexample is provided to the analog of Theorem 2, simply by noting that such an equilibrium may exist for every market in a sequence of replica dynamic markets. Such a sequence is easy to construct. It requires only one type of firm (K is a singleton $\{k\}$) with a production function $y(k) = f$ described by

$$\begin{aligned} f(r) &= 0, && \text{if } r \leq \tfrac{2}{3} \\ &= r - \tfrac{2}{3}, && \text{if } r > \tfrac{2}{3}. \end{aligned} \tag{10}$$

Price is specified by the inverse demand function

$$\begin{aligned} D(r) &= 1 - r, && \text{if } r \leq 1 \\ &= 0, && \text{if } r > 1. \end{aligned} \tag{11}$$

The discount factor for the sequence is $\beta = 0.9$. The counterexample to the limit principle is now presented.

THEOREM 3. *There exist a sequence of replica dynamic economies* $\langle M_n \rangle_{n \in N}$ *and a double sequence of supply vectors* $\langle q_{nt} \rangle_{n,t \in N}$ *such that*

(a) *For each* n, *the sequence of supply vectors* $\langle q_{nt} \rangle_{t \in N}$ *is determined (in the sense that* q_{nt} *specifies firms' output levels at time* t*) by a noncooperative equilibrium of* M_n, *and*

(b) *For each* t *there is a supply vector* $q_{\infty t}$ *such that* $\lim_{n \to \infty} q_{nt} = q_{\infty t}$ *in distribution, but*

(c) $\langle q_{\infty t} \rangle_{t \in N}$ *is not a price-taking equilibrium of the nonatomic representation* M_∞ *of the sequence.*

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[5] Let Z be a metric space. A sequence $\langle \eta_n \rangle_{n \in N}$ of probability measures converges weakly to a probability measure η if, for every bounded continuous function f: $Z \to R$, $\lim_{n \to \infty} \int z f d\eta_n \to \int z f d\eta$. If $\langle \Omega_n, \mu_n \rangle_{n \in N}$ is a sequence of probability spaces and $\langle \Omega, \mu \rangle$ is a probability space, and if $z_n : \Omega_n \to Z$ for every n, then $\langle z_n \rangle_{n \in N} \to z : \Omega \to Z$ in distribution if, when the measures η_n are defined by $\eta_n(B) = \mu_n(\{z_n \in B\})$ for all Borel sets $B \subseteq Z$ (and η is defined analogously for z), then $\langle \eta_n \rangle \to \eta$ weakly.

Proof. Let $\langle M_n\rangle_{n\in N}$ be determined by the conditions that $K = \{k\}$, $y(k) = f$ defined by (10), D is determined by Eq. (11), and $\beta = 0.9$. Recall that a strategy for a firm in a dynamic market is a rule which, at each time, determines an output level on the basis of information then available to the firm. The double sequence $\langle q_{nt}\rangle_{n,t\in N}$ will specify the output decisions of firms which follow the strategy

(d) Produce at level $\frac{1}{2}$ (the monopoly level) at time 0.

(e) If $D(\int_{K_n} q_{n0}\,d\mu_n) = D(\int_{K_n} q_{n(t-1)}\,d\mu_n) = \frac{1}{2}$. Then produce at level $\frac{1}{2}$ at time t.

(f) If the condition of (e) is not satisfied at t, then produce at level $\frac{2}{3}$ (the output level corresponding to the unique noncooperative equilibrium of the static market $\langle K_n, \mu_n, y_n, D\rangle$).

By induction on t, $g_{nt} = \frac{1}{2}$ for all n and t. Therefore condition (b) is satisfied with $q_{nt} = \frac{1}{2}$ for all t. Condition (c) is also satisfied, since $D(\int_{K_n} q_{nt}\,d\mu_n) = \frac{1}{2}$ while firms have zero marginal cost. Condition (a) is equivalent to the statement, proved below as Lemma 1, that for all firms to follow the strategy defined by (d), (e), and (f) is a noncooperative equilibrium of M_n. Q.E.D.

LEMMA 1. *For all firms to employ the strategy defined by* (d), (e), *and* (f) *above is a noncooperative equilibrium of the dynamic market* M_n.

Proof. Intuitively, each firm makes a strategic threat to the others. It announces: "Initially, we will participate in a cartel in which all firms produce equal shares of the monopoly output $n/2$. We will do our part to maintain this cartel, as long as there is reciprocity. However, if at any time a lowering of the market price indicates that another firm is exceeding its production limit, we will thereafter protect ourselves by acting as static Cournot oligopolists."

Consider the prospects of a potential deviant firm, if all other firms make this threat. Suppose that it were to adopt a strategy which would lead it, in this environment, to select an output sequence $\langle r_t\rangle_{t\in N}$. It will be shown that $r_t = \frac{1}{2}$ for all t is optimal. Because the strategy (d), (e), (f) produces this sequence if all other firms follow (d), (e), (f), part (a) of the definition of noncooperative equilibrium for a dynamic market is satisfied.

To show this, consider first an output sequence $\langle r_t\rangle_{t\in N}$ with $r_t \neq \frac{1}{2}$ for some t. Let $t_0 = \min\{t \mid r_t \neq \frac{1}{2}\}$. If other firm's outputs are determined by (d), (e), and (f), then the market price at time t is

$$\begin{aligned} D\left(\int_{K_n} q_{nt}\,d\mu_n\right) &= \tfrac{1}{2} && \text{for} \quad t < t_0 \\ &= (n+1-2r_t)/2n && \text{for} \quad t = t_0 \qquad (12) \\ &= (n+2-3r_t)/3n && \text{for} \quad t_0 < t. \end{aligned}$$

Since the outputs of other firms are the same for all sequences $\langle r'_t\rangle_{t\in N}$ which satisfy $t_0 = \min\{t \mid r'_t \neq \frac{1}{2}\}$, r_t must be the output level which maximizes returns at time t for $t_0 \leqslant t$; i.e., by (12),

$$r_{t_0}(n + 1 - 2r_{t_0})/2n - f(r_{t_0}) = \max_{r\in R_+} r(n + 1 - 2r)/2n - f(r), \qquad (13)$$

and

$$r_t(n + 2 - 3r_t)/3n - f(r_t) = \max_{r\in R_+} r(n + 2 - 3r)/3n - f(r) \qquad \text{for} \quad t_0 < t. \qquad (14)$$

Solving Eqs. (13) and (14) yields $r_t = \frac{2}{3}$ for all $t \geqslant t_0$. Thus discounted profits are

$$\begin{aligned}\sum_{t\in N} (0.9)^t \left[r_t D\left(\int_{K_n} q_{nt}\, d\mu_t\right) - f(r_t)\right] &\leqslant \sum_{t<t_0} (0.9)^t\, [(1/2)(1/2) - 0] \\ &\quad + (0.9)^{t_0}\, [(2/3)(1/2) - 0] \\ &\quad + \sum_{t_0<t} (0.9)^t\, [(2/3)(1/3) - 0] \\ &= \left(\sum_{t<t_0} (0.9)^t\right)\Big/4 + (0.9)^{t_0}\,(7/3). \qquad (15)\end{aligned}$$

On the other hand, if $r'_t = \frac{1}{2}$ for all t and if $\langle q'_{nt}\rangle_{t\in N}$ is the sequence of supply vectors which are realized when the firm in question supplies $\langle r'_t\rangle_{t\in N}$, then the firm's discounted profits are

$$\begin{aligned}\sum_{t\in N} (0.9)^t \left[r'_t D\left(\int_{K_n} q'_{nt}\, d\mu_n\right) - f(r'_t)\right] &= \sum_{t<t_0} (0.9)^t\, [(1/2)(1/2) - 0] \\ &\quad + \sum_{t_0<t} (0.9)^t\, (1/2)(1/2) - 0] \\ &= \left(\sum_{t<t_0} (0.9)^t\right)\Big/4 + (0.9)^{t_0}\,(5/2). \qquad (16)\end{aligned}$$

Together, (15) and (16) establish the optimality of the supply sequence $\langle r'_t\rangle_{t\in N}$ for the firm. Thus part (a) of the definition of noncooperative equilibrium is established. To show that part (b) of the definition holds, two cases must be considered at any future time in question. One possibility is that the market price has always been $\frac{1}{2}$ previously. In this case, the firm faces the same decision problem as it did in the initial period, so the earlier part of the proof is still relevant. The other possibility is that the firm has received information that some previous price has been other than $\frac{1}{2}$ (N.B. Although this possibility will not be realized in equilibrium, it must be considered for the reason explained in footnote 4). Then, since other firms will have this information also, they will always produce the Cournot output $\frac{2}{3}$ in the future. Thus it is optimal for the firm under consideration to produce this output in response. Q.E.D.

In a nonatomic market, the output decision of a single firm does not affect the price. Thus a firm cannot threaten its competitors by making its future actions contingent on the present price, because each competitor takes the price to be exogenous to its decision. Another description of this situation is that, if enforcement of a cartel is based only on price information, any firm can be a free rider on the cartel without being discovered. However, firms may be capable of observing the output levels of their competitors directly. Even in a large market, this output information might be supplied by a trade association or by the government. If the information is available, the limit principle may not hold.

THEOREM 4. *There exists a nonatomic dynamic market in which, if firms observe the output levels of their competitors directly, there is a noncooperative equilibrium which is not a price taking equilibrium.*

Proof. Let the market be the non-atomic representation of the sequence in Theorem 3, and let each firm produce $\frac{1}{2}$ unit of output as long as no competitor exceeds that output, but $\frac{2}{3}$ unit if some competitor has produced more than $\frac{1}{2}$ unit at some previous time. The proofs of Theorem 3 and Lemma 1 extend straightforwardly to show that these strategies are in noncooperative equilibrium, but that each firm will produce $\frac{1}{2}$ unit in every period, while price-taking firms would produce $\frac{2}{3}$ unit. Q.E.D.

4. AN INCLUSION THEOREM FOR REPEATED GAMES

The examples presented in the last section show that some justification is needed for applying static inclusion and limit theorems to actual markets. The question of when those results may legitimately be applied is the topic of this section and of the next. In these two sections, conditions will be given which insure that the opportunities for repeated trade in a large dynamic economy will be irrelevant to the decisions taken by agents at any particular time. Under these conditions, all dynamic noncooperative equilibria will be sequences of static equilibria. Therefore, facts about a static market will be true of the corresponding dynamic market as well.

These conditions will be studied in an abstract game theoretic context. This approach has three virtues. First, it makes it easy to deal with issues about agent's information (which have already appeared in Theorem 4). Second, at no extra cost it yields results which extend most of the literature on static price taking behavior to dynamic situations. For instance, the condition under which the dynamic inclusion principle holds is one already imposed by Dubey, Mas-Colell, and Shapley. Third, because the issues raised here are relevant whenever agents repeatedly face situations with "prisoner's dilemma" characteristics, the game theoretic results are of general interest. The theory of repeated games to be used here will now be presented. Then, the interpretation of a dynamic market as a repeated game will be explained.

Intuitively, a *repeated game* is simply a game which is replayed countably many times. A repeated game is defined in terms of three underlying spaces. The *players* are specified by a measure space $\langle K, \mu \rangle$. The measure μ will always be normalized to be a probability measure. For economy of notation, the σ-algebra on which μ is defined will not be referred to explicitly.

The *actions* available to a player are elements of a set A. Although it is assumed here that all players have the same set of feasible actions, this restriction could easily be removed.

The possible *outcomes* of the game at any time form a set X. An element of X specifies the information shared by all players about the results of a play of the game. For instance, in an economy the elements of X might be price vectors. It is important to note that this use of the term "outcome" is slightly different from the usual one, which is that an outcome completely specifies the result of a play (e.g., in an economy, X would be the set of feasible allocations) including aspects which may be unobservable to some of the players (e.g., other players' components of the allocation vector).

Throughout the paper, A and X will be assumed to be complete separable metric spaces, and mappings into these spaces from measurable spaces will be assumed to be Borel measurable.

Each player k is characterized by a *return function* $u_k : A \times X \to R$, and a *discount factor* $\beta_k \in (0, 1)$. A player attempts to maximize the discounted value of his returns from repeated plays of the game.[6]

A *play* q of the game is a measurable assignment of actions to players.[7] A^K will denote the set of plays. The outcome of a play is determined by the *outcome function* F: $A^K \to X$.

A *strategy* s is a rule by which a player determines an action to take at each time, based on his knowledge at that time of the outcomes of past plays, and of his own past actions. If each past action has been generated as a deterministic function of prior outcomes and actions, the player can recover the information about his past actions recursively from his memory of past outcomes (this is easily proved by recursion on t). If the successive plays of the game are indexed by the natural numbers, a player's information at time t about prior history is specified by an element of X^t. Thus, s may be represented by a sequence $\langle s_t \rangle_{t\in N}$, where $s_0 \in A$, and $s_t : X^t \to A$ for $t \in N_+$. The space of strategies is then $A \times \prod_{t\in N+} A^{X^t}$, which will be denoted by S.

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[6] It is assumed that the discounted sum of optimal returns converges. In applications, this assumption typically is satisfied in equilibrium. If necessary, the assumption could be avoided by using a discounted overtaking criterion.

[7] It might be suggested that all (not necessarily measurable) functions from K to A ought to be included in the set of plays and that an outcome of the game should be assigned to every such function. Such a change would not significantly affect the results of this paper. In the case of the limit theorem (Theorem 6), plays of the finite-player games are trivially measurable, and the measurability of the limiting play is proved in the theorem. The inclusion theorem (Theorem 5) would continue to hold as stated, if the definition of anonymity used there were replaced by: a game is anonymous if any two plays differing only in the action of a single player have the same outcome.

A *strategy vector* is an assignment $\langle s^k \rangle_{k\in K}$ of strategies to players such that, if $s^k = \langle s_t^k \rangle_{t\in N}$, then

(a) s_0^k is a measurable function of k, and

(b) for $x_0 \cdots x_{t-1} \in X$, $s_t^k(x_0, \ldots, x_{t-1})$ is a measurable function of k.

These conditions ensure that players' strategies determine recursively a sequence of plays and outcomes. The initial play q_0 is the vector $\langle s_0^k \rangle_{k\in K}$ of initial actions specified by the strategies. The initial outcome is $x_0 = F(q_0)$. The next play is $q_1 = \langle s_1^k(x_0) \rangle_{k\in K}$, and $x_1 = F(q_1)$, and so forth. This process defines an infinite sequence $\langle\langle q_t, x_t \rangle\rangle_{t\in N}$ of plays and outcomes, called the *history* generated by the strategy vector. The sequence $\langle x_t \rangle_{t\in N}$ determined by the history is called the *path*.

The interpretation of dynamic markets as repeated games is straightforward. Outcomes are prices. The return function of firm k specifies its profit as a function of its output level and the price. That is, if f is the total cost function of firm k, $u_k(a, x) = xa - f(a)$. The market discount rate β is β_k for each firm. The outcome function is $F(q) = D(\int_K q \, d\mu)$.

Now it is possible to make precise the informal definition of noncooperative equilibrium which was given in Section 2. Let S denote the space of strategies of the game. That is, $S = A \times \prod_{t\in N+} A^{X^t}$. Also, let $y \colon K \to R^{(A\times X)} \times (0, 1)$ specify the characteristics of players. That is, $y(k) = \langle u_k, \beta_k \rangle$. Three related concepts will now be defined. A static noncooperative equilibrium is a play which is a Nash equilibrium of the simple (nonrepeated) game. A dynamic quasi-equilibrium is a strategy vector which satisfies the defining condition of Nash equilibrium with respect to discounted returns at the initial time. A dynamic noncooperative equilibrium is a perfect (in the sense of footnote 4) dynamic quasi-equilibrium.

DEFINITION. A *static noncooperative equilibrium* of the repeated game $\langle K, A, X, F, \mu, y \rangle$ is a play $q^* \in A^K$ such that, for any other $q \in A^K$, the following implication holds for almost all $k \in K$: If $y(k) = \langle u, \beta \rangle$ and $q(i) = q^*(i)$ for all $i \neq k$, then

$$u(q(k), F(q)) \leqslant u(q^*(k), F(q^*)). \tag{17}$$

A *dynamic quasi-equilibrium* is a strategy vector $v^* \in S^K$ such that, for almost all $k \in K$, the following implication holds for all $v \in S^K$: If $y(k) = \langle u, \beta \rangle$, $v(i) = v^*(i)$ for all $i \neq k$, and $\langle\!\langle q_t, x_t \rangle\!\rangle_{t \in N}$, $\langle\!\langle q_t^*, x_t^* \rangle\!\rangle_{t \in N}$ are the histories generated by v and v^*, respectively, then

$$\sum_{t \in N} \beta^t u(q_t(k), x_t) \leqslant \sum_{t \in N} \beta^t u(q_t^*(k), x_t^*). \tag{18}$$

A *dynamic noncooperative equilibrium* is a strategy vector v^* such that

(a) v^* is a dynamic quasi-equilibrium, and (b) if $t \in N_+$, and $x_0^*, \ldots, x_{t-1}^* \in K$, and for each player k a new strategy is defined by

$$s_0 = s_t^*(x_0^*, \ldots, x_{t-1}^*) \tag{19}$$

and, for $r \in N_+$,

$$s_r(x_0, \ldots, x_{r-1}) = s_{t+r}^*(x_0^*, \ldots, x_{t-1}^*, \ldots, x_0, \ldots, x_{r-1}), \tag{20}$$

then the vector of these new strategies is also a dynamic quasi-equilibrium.

It was asserted at the end of Section 3 that an inclusion theorem holds for dynamic markets in which firms cannot discern changes in the output level of individual competitors. For this condition to be realized in a nonatomic market it is sufficient for the publicly observable outcome of the operation of the market to depend only on the distribution of firm's output level. This property of the outcome function can be generalized to abstract games.

DEFINITION. Denote the set of probability measures on A by $M(A)$. Let $m\colon A^K \to M(A)$ map plays to the distribution of actions which they determine as random variables. I.e., for $q \in A^K$ and $B \subseteq A$, $(m(q))(B) = \mu(\{k \mid q(k) \in B\})$. A game with outcome function F is *anonymous* if $F(q)$ depends on q only through $m(q)$ (i.e., if there is a function $G\colon M(A) \to X$ which makes the diagram

$$\begin{tikzcd} & M(A) \arrow[dr, "G"] & \\ A \arrow[ur, "m"] \arrow[rr, "F"'] & & X \end{tikzcd} \tag{21}$$

commute.

The preceding definitions make it easy to state and prove the result described at the beginning of this section, relating static and dynamic noncooperative equilibria. The inclusion theorem for dynamic markets with price information is an immediate corollary of this result.

THEOREM 5. *If* $v^* \in S^K$ *is a dynamic noncooperative equilibrium of the anonymous repeated game* $\langle K, A, X, F, \mu, y\rangle$, μ *is nonatomic, and* $\langle\langle g_t^* \, x_t^* \rangle\rangle_{t \in N}$ *is the history generated by* v^*, *then every* q_t^* *is a static noncooperative equilibrium of the game.*

Proof. It will be shown that, if some q_r^* is not a static NCE, then v^* is not a dynamic NCE. If q_r^* is not a static NCE, there is a subset $B \subseteq X$ with $\mu(B) > 0$, such that for every $k \in B$ (with $y(k) = \langle u, \beta\rangle$) there exists a play $q \in A^K$ with, $q(i) = q_r^*(i)$ for $i \neq k$, such that (17) fails to hold. That is, $u(q_r^*(k), F(q^*)) < u(q(k), F(q))$. Now, for some arbitrary player $k \in B$, define a new strategy as follows. Let his initial action s be the initial action $q_0^*(k)$ specified by $v^*(k)$, if $r \neq 0$, but let it be $q_0(k)$ if $r = 0$. For every $t > 0$, the function s_t will be a constant function. Everywhere on its domain X^t, the function will take the value $q_t^*(k)$ if $t \neq r$, or $q(k)$ if $t = r$. Define the strategy vector $v \in S^K$ by $v(k) = \langle s_t\rangle_{t \in N}$ and $v(i) = v^*(i)$ for $i \neq k$. It is claimed that (18) fails to hold for v and v^*, and that v^* is therefore not a dynamic NCE because this failure of (18) can be exhibited anywhere on the set B which has positive measure. This claim is substantiated by noting that, if $\langle x_t\rangle_{t \in N}$ is the path generated by v, $x_t = x_t^*$ (this fact is proved below as Lemma 2). By construction of $v(k)$, $q_t(k) = q_t^*(k)$ except for $t = r$. Therefore, for $t \neq r$, the tth summands of the left- and right-hand sides of (18) are identical. Consequently, (18) holds only if $u(q_r(k), x_r) \leq u(q_r^*(k), x_r^*)$. But this last inequality is equivalent to (17), which fails by assumption. Q.E.D.

LEMMA 2. *Let* F *be anonymous,* $v^* \in S^K$, $\langle x_t^* \rangle_{t \in N}$ *be the path generated by* v^*, $r \in N$, *and* $a \in A$. *For some* $k \in K$ *with* $\mu\{k\} = 0$, *define the strategy* $\langle s_t\rangle_{t \in N}$ *as follows: Define* $s_0 = q_0^*(k)$ *if* $0 \neq r$, *but* $s_0 = a$ *if* $0 = r$. *For* $t > 0$, *define* s_t *to be the constant function on* X^t *which is equal everywhere to* $q_t^*(k)$ *if* $t \neq r$, *but to* a *if* $t = r$. *Define* $v \in S^K$ *by* $v(k) = \langle s_t\rangle_{t \in N}$ *and* $v(i) = v^*(i)$ *for* $i \neq k$. *Let* $\langle x_t\rangle$ *be the path generated by* v. *Then for all* $t \in N$, $i \in K - \{k\}$, $q_t(i) = q_t^*(i)$ *and* $x_t = x_t^*$.

Proof. By induction on t. For $t = 0$, $q_t(i)$ is determined directly by $v^*(i)$ for all $i \neq k$. Since $\mu(\{k\}) = 0$, $m(q_0) = m(q^*)$, so $x_0 = s_0^*$ because F is anonymous. Now suppose that the lemma holds for $0, \ldots, t$. Then for $i \neq k$, if $v^*(i) = \langle s_t^* \rangle_{t \in N}$

$$q_{t+1}(i) = s_{t+1}^*(x_0, \ldots, x_t) = s_{t+1}^*(x_0^*, \ldots, x_t^*) = q_{t+1}^*(i).$$

As in the case $t = 0$, $m(q_{t+1}) = m(q_{t+1}^*)$, so $x_{t+1} = x_{t+1}^*$. This completes the induction. Q.E.D.

Theorem 5 is an inclusion theorem, if this term is interpreted broadly to mean a characterization of the equilibria of nonatomic games. It describes the dynamic noncooperative equilibria of a nonatomic anonymous iterated game in terms of the equilibria of the static game which is iterated. If the equilibria of the static game have some property, Theorem 5 entails that the equilibria of the iterated game inherit the property. Thus, the theorem proved in [2] extends immediately to dynamic economies. Theorem 1 of the present paper extends in a similar way.

Suppose that firms in a nonatomic dynamic market receive only price information about the operation of the market. Because price is a function of mean supply, the market is an anonymous game. By Theorem 5, if $\langle\!\langle q_t, x_t \rangle\!\rangle_{t \in N}$ is a dynamic NCE of the market, every q_t is a static NCE. By Theorem 1, then, every q_t is a price-taking equilibrium of the static market. It is evident from the definition of price-taking equilibrium that the sequence $\langle q_t \rangle_{t \in N}$ is therefore a price-taking equilibrium of the dynamic market. This proves the inclusion theorem for dynamic markets with price information only. In fact, the theorem continues to hold if firms receive other information about market aggregates (e.g., the variance of output levels in the market). The significance of this observation will be considered in Section 6.

5. A Limit Theorem for Anonymous Repeated Games with Random Outcomes

Although the inclusion principle is valid for dynamic markets under the hypothesis of anonymity, the limit principle remains invalid. This fact is evident from the counterexample given in the proof of Theorem 3. The reason why the implications of anonymity are different for the two principles is obvious. In a nonatomic market, anonymity guarantees that no firm can have any effect at all on the information which its competitors receive about the market. If inverse demand is strictly downward sloping, though, a unilateral change of output level by a firm in a finite market must have a perceptible effect on the market price. Although the magnitude of this price change becomes small in large replica markets, the change remains perceptible. The limit principle fails because the individual firm's effect on price information does not diminish in large markets, although its effect on the price level shrinks. In order for the limit principle to be valid for a dynamic market, a situation must occur in which price information about supply is unreliable.

One such situation is the existence of random fluctuations, not directly observed by firms, in consumers' demand schedules. Suppose that firms in this situation attempt to maintain a strategic noncooperative equilibrium like that described in Lemma 1. When the market price falls, each firm must decide whether the decline reflects a spontaneous downward shift of the inverse demand function, or whether it was caused by a competitor having exceeded his output quota. In a large market where the scale of an individual firm is insignificant relative to aggregate demand uncertainty, price information cannot provide evidence about the firm's level of output. Just as in the nonatomic market, the firm has incentive to break the cartel agreement.

In this section, a limit theorem will be proved which applies to the situation just discussed. The theorem applies only to equilibria of the type studied in Lemma 1, rather than to all dynamic noncooperative equilibria.

The limit theorem will be stated for anonymous repeated games with random outcomes. A game with random outcomes is one for which, at each play, the outcome is a random vector taking values in X, rather than a determinate value $x \in X$. The random vector which is the outcome depends only on the current play. Players' return functions are now interpreted as von Neumann-Morgenstern utility functions, and players are assumed to maximize expected discounted returns.

The definitions of a repeated game with random outcomes and of an anonymous game with random outcomes are the same as those given in the last section, except that now $F: A^K \to M(X)$ and $G: M(A) \to M(X)$. When plays have random outcomes, a strategy vector will not generate a unique path in X^N. Rather, it will determine a probability distribution over paths. If $B \subseteq X$, $x_0, \ldots, x_{t-1} \in X$, $\langle s^k \rangle_{k \in K}$ is a strategy vector, q is the play defined by $q(k) = s_t^k(x_0, \ldots, x_{t-1})$, and $\nu = F(q)$, then $\nu(B)$ is the probability that $x_t \in B$ conditional on the outcomes of the first t plays having been $x_0, \ldots, x_{t-1}$. In order for this conditional probability to be well defined, $\nu(B)$ must be a measurable function of $x_0, \ldots, x_{t-1}$. A sufficient condition for measurability of this function in the case of an anonymous game is that (a) G is continuous when $M(A)$ and $M(X)$ are endowed with the weak and total variation norm[8] topologies, respectively, and (b) $m(q)$ is measurable as a function of $x_0, \ldots, x_{t-1}$, when $M(A)$ has the weak topology. These conditions will be incorporated in the definitions of anonymity and of strategy vector.

DEFINITION. *A repeated game with random outcomes* $\langle K, A, X, F, \mu, y \rangle$ is defined exactly as was a repeated game in Section 4, except that $F: A^K \to M(X)$. The shorter term *repeated random game* will be used synonymously with this. A random game is *anonymous* if the diagram analogous to that which defines an anonymous deterministic game (but with $M(X)$ instead of X) commutes, and if G is a continuous function from $M(A)$ under the weak topology into $M(X)$ under the total variation norm topology. A *strategy* for a repeated random game is defined as in the deterministic case. An assignment $\langle s^k\rangle_{k\in K}$ of strategies to players is a *strategy vector* for a repeated random game if it satisfies the conditions specified in the deterministic case and if, in addition, when for $t\in N_+$ the function $\theta_t\colon X^t\to M(A)$ is defined by

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[8] If $\mathcal{B}$ is a σ-field and $f: \mathcal{B} \to R$ is a countably additive set function, then the total variation norm of f is defined by $\|f\| = \sup\{\sum_{n \in N} |f(B_n)| \mid \langle B_n \rangle_{n \in N}$ is a sequence of disjoint elements of $\mathcal{B}\}$. If the range of G is any of the usual parametrized families of distribution (e.g., normal, if $X = R^n$) and does not contain any degenerate (e.g., singular covariance normal) distributions, then weak continuity of the parameter as a function of the distribution of actions is sufficient to guarantee continuity of G.

$$\theta_t(x_0,\dots,x_{t-1}) = m(\langle s_t^k(x_0,\dots,x_{t-1})\rangle_{k\in K}), \tag{22}$$

each θ_t is measurable ($M(A)$ having the weak topology).

Let S^K denote the set of strategy vectors. To define the random path generated by a vector $v\in S^K$, let $\Omega = X^N$ and let $\mathscr{B}$ be the Borel σ-field generated by the product topology on Ω. Define the projection functions $\mathbf{x}_t\colon \Omega\to X$ for $t\in N$ by $\mathbf{x}_t(\langle x_0, x_1,\dots, x_t,\dots\rangle) = x_t$. Let $\mathscr{B}_t$ be the smallest σ-field with respect to which $\mathbf{x}_0,\dots,\mathbf{x}_t$ are all measurable (N.B. $\mathscr{B}_t\subseteq\mathscr{B}$, and $\mathscr{B}_t$ is isomorphic as an algebra to the Borel σ-field on X^{t+1}, by [7, p. 6]).

Recall that, for a probability measure π defined on $\mathscr{B}$, a regular conditional probability relative to $\mathscr{B}_t$ is a function $P_t\colon \mathscr{B}x\Omega\to[0,1]$ such that

(a) for fixed $\omega\in\Omega$, $P_t(\cdot\mid\bar{y})$ is a probability measure defined on Ω,

(b) for fixed $B\subseteq\Omega$, $P_t(B\mid\cdot)$ is $\mathscr{B}_t$-measurable, and

(c) for all $B\in\mathscr{B}$ and $C\in\mathscr{B}_t$,

$$\pi(B\cap C) = \int_C P_t(B\mid\omega)\,d\pi(\omega). \tag{23}$$

For every probability measure π defined on $\mathscr{B}$, and every $t\in N$, a regular conditional probability relative to $\mathscr{B}_t$ exists, and any two of these differ only on a set of π measure zero (i.e., if P and P' are regular conditional probabilities relative to $\mathscr{B}_t$, then $\pi(\{\omega\mid\exists B\in\mathscr{B}(P(B\mid\omega)\neq P'(B\mid\omega)\}) = 0)$ [7, p. 147]. Thus, regular conditional probabilities may be used unambiguously to define random paths.

DEFINITION. Let Ω, $\mathscr{B}$, $\mathbf{x}_t$, $\mathscr{B}_t$, and P_t be as above. Let $v\in S^k$, where $v = \langle s^k\rangle_{k\in K}$ and $s^k = \langle s_t^k\rangle_{t\in N}$. The *random path* generated by v is the probability space $\langle\Omega, \mathscr{B}, \pi\rangle$, which satisfies, for every Borel subset B of X,

$$\pi(\mathbf{x}_0(\omega)\in B) = \nu(B), \tag{24}$$

where $\nu = F(\langle s_0^k\rangle_{k\in K})$, and for $t\in N_+$,

$$P_t(\mathbf{x}_t(\omega)\in B\mid\omega) = \nu(B), \tag{25}$$

where $\nu = F(\langle s_t^k(\mathbf{x}_0(\omega),\dots,\mathbf{x}_{t-1}(\omega))\rangle_{k\in K})$, for all ω belonging to some $C_t\in B_{t-1}$ where $\pi(C_t) = 1$.

It is proved in the Appendix that the random path generated by a strategy vector exists and is unique. Definitions of static and dynamic NCE for a random game differ from the definitions in the deterministic case only in that expectations must be taken with respect to distributions of random outcomes and paths.

DEFINITION. *A static noncooperative equilibrium* of the repeated game $\langle K, A, X, F, \mu, \nu \rangle$ with random outcomes is defined as for a deterministic game, except that (17) becomes[9]

$$E_{F(q)}u(q(k), x) \leqslant E_{F(q^*)}u(q^*(k), x). \tag{26}$$

The definition of a *dynamic quasi-equilibrium* is the same as in the deterministic case, except that now the random paths $\langle \Omega, \mathscr{B}, \pi \rangle$ and $\langle \Omega, \mathscr{B}, \pi^* \rangle$ of v and v^*, respectively, are treated rather than histories in the deterministic case. Equation (18) is changed to

$$\begin{aligned} &E_\pi\Big(u(s_0{}^k, \mathbf{x}_0) + \sum_{t \in N_+} \beta^t u(s_t{}^k(\mathbf{x}_0, \dots, \mathbf{x}_{t-1}), \mathbf{x}_t)\Big) \\ &\qquad E_{\pi^*}\Big(u(s_0^{*k}, \mathbf{x}_0) + \sum_{t \in N_+} \beta^t u(s_t^{*k}(\mathbf{x}_0, \dots, \mathbf{x}_{t-1}), \mathbf{x}_t)\Big). \end{aligned} \tag{27}$$

The definition of a *dynamic noncooperative equilibrium* is exactly the same as in the deterministic case.

An explicit definition is now given of the type of dynamic noncooperative equilibrium which was studied in Lemma 1. Recall that in that type of equilibrium, there is a play q_0 to which players are committed at each repetition of the game. This commitment is enforced by mutual threats that, if there is evidence that some player k has not taken action $q_0(k)$, then the other players will thereafter take the actions prescribed by another play q_1. This threat is self-enforcing, because q_1 is a static noncooperative equilibrium. Evidence that a player has departed from q_0 consists of an outcome which lies outside some subset $B \subseteq X$. That is, the players agree to play q_0 as long as outcomes in B have been observed in the past, but always to play q_1 if some outcome outside of B has occurred in the past.

DEFINITION. A *strategic noncooperative equilibrium* of the game $\langle K, A, X, F, \mu, \nu \rangle$ with random outcomes is a dynamic NCE $v \in S^K$ such that. for some triple $\langle q_0, q_1, B \rangle \in A^K \times A^K \times 2^X$, the following conditions are satisfied for every $k \in K$ (with $v(k) = \langle s_t \rangle_{t \in N}$):

(a) q_1 is a static NCE.

(b) $s_0(k) = q_0(k)$.

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[9] The functions $u(q(k), x)$ and $u(s_t{}^k(\mathbf{x}_0, \dots, \mathbf{x}_{t-1}, \mathbf{x}_t)$ are random variables on X and Ω in Eqs. (26) and (27), respectively.

(c) If, for any $t \in N_+$, $x_0, \dots, x_{t-1} \in B$, then $s_t(x_0, \dots, x_{t-1}) = q_0(k)$ and

(d) If, for any $t \in N_+$ and $r < t$, $x_r \notin B$, then $s_t(x_0, \dots, x_r, \dots, x_{t-1}) = q_1(k)$ for all $x_0, \dots, x_{r-1}, x_{r+1}, \dots, x_{t-1} \in X$.

By abuse of notation, the strategy vector defined by (b)–(d) above will also be referred to as $\langle q_0, q_1, B\rangle$.

There is a condition which, if the expectations referred to in its statement are well defined, is necessary and sufficient for a strategy vector $\langle q_0, q_1, B\rangle$ to be a strategic NCE. Some notation is introduced now to facilitate the statement of this condition. Let $k \in K$ be fixed. For $a \in A$, define q_a to be the play which results from q_0 when the action of k is changed to a. That is, $q_a(k) = a$, and $q_a(i) = q_0(i)$ for $i \neq k$. Define $\nu_j = F(q_j)$ for $j = 0, 1$, and $\nu_a = F(q_a)$ for all $a \in A$, and let E_j and E_a be the expectation operators (on functions defined over X) with respect to ν_j and ν_a, respectively.

Lemma 3. *Let $q_0, q_1 \in A^K$, let q_1 be a static NCE, and suppose that for all $k \in K$ (with $y(k) = \langle u, \beta\rangle$), $E_1 u(q_1(k), x)$ is finite and $E_a u(a, x)$ is finite for every $a \in A$. Let $B \subseteq X$, and define*

$$V_k(a) = [E_a u(a, x) + \beta(1 - \beta)^{-1}(1 - \nu_a(B))\, E_1 u(q_1(k), x)]/[1 - \beta \nu_a(B)], \tag{28}$$

for all $a \in A$. Then $\langle q_0, q_1, B\rangle$ is a strategic NCE if and only if, for almost all $k \in K$,

$$V_k(q_0(k)) = \max_{a \in A} V_k(a). \tag{29}$$

Proof. By definition, $\langle q_0, q_1, B\rangle$ is a strategic NCE if and only if it is a dynamic NCE. Furthermore, by the reasoning explained in Lemma 1, the fact that q_1 is a static NCE entails that $\langle q_0, q_1, B\rangle$ is a dynamic NCE if and only if it is a dynamic quasi-equilibrium. Thus, it is sufficient to prove that (27) holds with $v^* = \langle q_0, q_1, B\rangle$ for all other $v \in S^K$ with $v(i) = v^*(i)$ for $i \neq k$, if and only if (29) holds. This proof consists of two parts. First, it is argued that one need only consider (27) with respect to v^* and strategy vectors $v = \langle q_a, q_1, B\rangle$ for $a \in A$. Second, a dynamic programming argument is used to evaluate the expected discounted returns in (27) for paths generated by these strategy vectors.

Both of these arguments are transparent when it is realized that the decision problem of player k is equivalent to a very simple Markov dynamic programming problem. In this equivalent problem there are two states: either all previous outcomes of the game have been in B (state 0), or some outcome has not been in B (state 1). In state 0, taking an action $a \in A$ yields a present return (the expected return $E_a u(a, x)$ from the play q_a) and incurs a risk (the probability $1 - \nu_a(B)$ of producing an outcome not in (B) of causing a change to state 1 in the next period. State 1 is an absorbing state, and in that state the player's best action is $q_1(k)$ which yields return $E_1 u(q_1(k), x)$. A basic theorem of dynamic programming states that, if any optimal control rule exists for this problem, there is an optimal rule which specifies a fixed action $a \in A$ to be taken as long as state 0 persists. If player k follows this rule while other players use the strategies assigned by $\langle q_0, q_1, B\rangle$ the strategy vector $\langle q_a, q_1, B\rangle$ results. Thus, if any strategy of k leads to a violation of (27) the equation fails to hold for some strategy vector of this special form.

It remains to calculate the expected discounted returns to player k from paths of strategy vectors $\langle q_a, q_1, B\rangle$. (N.B. If $a = q_0(k)$, then $\langle q_a, q_1, B\rangle = \langle q_0, q_1, B\rangle$.) Denote his expected discounted return from the path of $\langle q_a, q_1, B\rangle$ by $V_k(a)$. With this interpretation of $V_k(a)$, (29) holds if and only if $\langle q_0, q_1, B\rangle$ is a dynamic quasi-equilibrium. Thus it is only necessary to verify that (28) is true under this interpretation, in order to complete the proof.

To do this, note that $V_k(a)$ must equal the expected discounted return in the equivalent dynamic program under the control rule: perform action a in state 0, and action $q_1(k)$ in state 1. From the first time that state 1 occurs, the expected discounted return is $(1-\beta)^{-1}E_1 u(q_1(k), x)$. Thus, if $W:\{0, 1\} \to R$ is the value function of the control rule, $V_k(a) = W(0)$ and $(1-\beta)^{-1}E_1 u(q_1(k), x) = W(1)$. $W(0)$ is defined by the functional equation

$$\begin{aligned} W(0) &= E_a u(a, x) + \beta[P(\text{the state next period will again be 0}) \cdot \\ &\quad W(0) + P(\text{the state next period will change to 1}) \cdot W(1)]. \end{aligned} \tag{30}$$

Given that the transition probabilities are $\nu_a(B)$ and $(1-\nu_a(B))$, respectively, (30) is equivalent to (28). Q.E.D.

Lemma 3 will be used to prove a limit theorem for anonymous repeated games with random outcomes. To state this theorem, a sequence of replica repeated games must be defined. This definition is a straightforward generalization of that of a sequence of replica dynamic markets.

Definition. Let $H = \langle K, A, X, F, \mu, y\rangle$ be an anonymous repeated game with random outcomes. In particular, let the diagram

$$\begin{tikzcd} & M(A) \arrow[dr, "G"] & \\ A^K \arrow[ur, "m"] \arrow[rr, "F"'] & & M(X) \end{tikzcd} \tag{31}$$

commute. Suppose that K is a finite set. Then, a *sequence of replica repeated games* $\langle H_n\rangle_{n\in N}$ with random outcomes is defined as follows: For every $n \in N$, $H_n = \langle K_n, A, X, F_n, \mu_n, y_n\rangle$, where $K_n = K \times \{0,\ldots, n\}$, μ_n is determined by $\mu_n(\{\langle k, r\rangle\}) = \mu(\{k\})/(n+1)$ for all $\langle k, r\rangle \in K_n$, and $y_n(\langle k, r\rangle) = y(k)$ for all $\langle k, r\rangle \in K_n$. It remains to define the outcome function F_n. Let $m_n\colon A^{K_n} \to M(A)$ map plays of H_n to the distributions of actions which they determine as random variables on $\langle K_n, \mu_n\rangle$. Then define $F_n : A^{K_n} \to M(X)$ by $F_n(q) = G(m_n(q))$ for all $q \in A^{K_n}$. Define the *nonatomic representation* H_∞ of the sequence analogously, where $K_\infty = K \times [0, 1]$, and, for $B \subseteq K_\infty$, $\mu_\infty(B) = \sum_{k \in K} \mu(\{k\}) \cdot \lambda(\{r \in [0, 1] \mid \langle k, r\rangle \in B\})$.

The limit theorem to be proved here states that, under appropriate hypotheses, the limit of strategic noncooperative equilibria is trivial in the sense that the "collusive" play supported in the limit must itself be a static noncooperative equilibrium. Formally, if $\langle H_n\rangle_{n \in N}$ is a sequence of replica repeated games with random outcomes, if $\langle q_0{}^n, q_1{}^n, B_n\rangle$ is a strategic NCE of H_n for each n, and if $\langle\!\langle y_n, q_0{}^n, q_1{}^n\rangle\!\rangle_{n \in N} \to \langle y_\infty, q_0{}^\infty, q_1{}^\infty\rangle$ in distribution, then $q_0{}^\infty$ is a static NCE of H_∞

The statement and proof of the limit theorem are now given.

THEOREM 6 (limit theorem for anonymous repeated games with random outcomes). *Suppose that $\langle H_n\rangle_{n \in N}$ is a sequence of replica repeated games with random outcomes, having H_∞ as nonatomic representation. Let $H = \langle K, A, X, F, \mu, y\rangle$, and let G make* (31) *commute.*[10] *Suppose that $\langle q_0{}^n, q_1{}^n, B_n\rangle$ is a strategic NCE of H_n for every $n \in N$, and that the following hypotheses are satisfied.*

(a) *The countable subset $D \subseteq A$ is dense in A.*

(b) *If $y(k) = \langle u, \beta\rangle$ for some $k \in K$, then $E_{G(\eta)}u(a, x)$ is a finite-valued, continuous function on $A \times M(A)$, where $M(A)$ has the weak topology.*

(c) *The sequence $\langle\!\langle y_n, q_0{}^n, q_1{}^n\rangle\!\rangle_{n \in N}$ converges in distribution.*

Then there is a random vector $\langle y_\infty, q_0{}^\infty, q_1{}^\infty\rangle$ defined on K_∞ such that

$$\langle\!\langle y_n, q_0{}^n, q_1{}^n\rangle\!\rangle_{n \in N} \to \langle y_\infty, q_0{}^\infty, q_1{}^\infty\rangle \tag{32}$$

in distribution, and $q_0{}^\infty$ is a static NCE of H_∞.

---

[10] In particular, it is assumed that G is continuous when $M(X)$ is endowed with the total variation norm topology. This assumption restricts the amount of information that outcomes of the game can carry. For instance, if a deterministic outcome is interpreted as a probability measure concentrated on a single point, then the example constructed to refute the limit principle for repeated deterministic games may be considered as a sequence of replica repeated games with random outcomes. In that example, G is continuous when $M(X)$ is given the weak topology, but not when it is given the total variation norm topology (although the conditional probabilities needed to define the random path are well defined (they are all 1 or 0)). Thus the continuity requirement on G in the theorem cannot be relaxed. An example in which very accurate information is revealed in a large market, so that a strong continuity requirement might be presumed not to hold, is the enforcement by professional associations of prohibitions on advertising by members. At essentially no cost, an association can monitor perfectly whether the prohibition is being observed.

Proof. *The two ideas behind this proof are that the analog (with expected utilities) of Theorem 5 must be true for nonatomic games with random outcomes, and that the functions V_k defined by (28) are continuous in the actions and measures involved in their definition.* If q_0^∞ is not a static NCE of H_∞, the first assertion entails that $\langle q_0^\infty, q_1^\infty, B_\infty\rangle$ cannot be a dynamic quasi-equilibrium for any $B_\infty \subseteq X$. Therefore (29) cannot be satisfied almost everywhere for H_∞ by any $\langle q_0^\infty, q_1^\infty, B_\infty\rangle$. By the second assertion, neither can (29) hold almost everywhere on K_n, for the nearby games H_n and strategy vectors $\langle q_0^n, q_1^n, B_n\rangle$ when n is large.

Formally, the limiting random vector $\langle y_\infty, q_0^\infty, q_1^\infty\rangle$ exists by Skorokhod's theorem [5, p. 50]. Suppose that q_0^∞ is not a static NCE. Then there is a set $J \subseteq K_\infty$, with $\mu_\infty(J) > 0$, on which (26) does not hold for all necessary q when $q^* = q_0^\infty$. Because K is finite, for some $k \in K$ (N.B. k will refer throughout the proof to this fixed element of K), $\mu_\infty(J \cap (\{k\} \times [0, 1])) > 0$. Without loss of generality, it may be assumed that $J \subseteq \{k\} \times [0, 1]$.

Because H_∞ is nonatomic and anonymous, and because D is dense in A, the failure of (26) is equivalent by (b) to the existence of some $d \in D$ such that

$$E_{F_\infty(q_0^\infty)} u(q_0^\infty(\langle k, r\rangle), x) < E_{F_\infty(q_0^\infty)} u(d, x). \tag{33}$$

As above, the countability of D permits (33) to be assumed without loss of generality to hold everywhere on J for some fixed action d of D (N.B. d also to be held fixed throughout the proof). Similarly, by the separability of R_+, it may be assumed that a stronger version

$$E_{F_\infty(q_0^\infty)} u(d, x) - E_{F_\infty(q_0^\infty)} u(q_0^\infty(\langle k, r\rangle), x) > \delta \tag{34}$$

of (33) holds uniformly on J for some $\delta > 0$, and that by (b),

$$| E_{F(q_1^\infty)} u(q_1^\infty(\langle k, r\rangle), x)| < \epsilon. \tag{35}$$

holds uniformly on J for some $\epsilon > 0$.

By (b), (34), and (35), for each r with $\langle k, r\rangle \in J$, there exist open sets $U_{0r}^*, U_{1r}^* \subseteq A$ and $W_{0r}, W_{1r} \subseteq M(A)$ with $q_j^\infty(\langle k, r\rangle) \in U_{jr}^*$ and $m_\infty(q_j^\infty) \in W_{jr}$ for $j = 0, 1$, which satisfy

$$E_{G(\eta)} u(d, x) - E_{G(\theta)} u(a, x) > \delta \tag{36}$$

for $\eta, \theta \in W_{0r}$ and $a \in U_{0r}^*$ and

$$| E_{G(\eta)} u(a, x)| < \epsilon \tag{37}$$

for $\eta \in W_{1r}$ and $a \in U_{1r}^*$.

There are open U_{jr} for $j = 0, 1$ and $\langle k, r\rangle \in J$, such that $q_j^{\infty}(\langle k, r\rangle) \in U_j$. and $\bar{U}_{jr} \subseteq U^*_{jr}$ (N.B. $\bar{U}$ is the closure of U). Let $\Gamma = \bigcup_{\langle k,r\rangle \in J}(U_{0r} \times U_{1r})$. By (a) and Lindelöf's theorem [3, p. 12] there is a countable subset $Y \subseteq [0, 1]$ such that $\{U_{0r} \times U_{1r} \mid r \in Y\}$ is a cover of Γ. Since $\mu_{\alpha}(J) > 0$ and $J \subseteq \{\langle k, r\rangle \mid \langle q_0^{\infty}(\langle k, r\rangle), q_1^{\infty}(\langle k, r\rangle)\rangle \in \Gamma\}$, and since Y is countable, there is some $U = U_{0r} \times U_{1r}$ such that

$$\mu_{\alpha}(\{\langle k, r\rangle \mid \langle q_0^{\infty}(\langle k, r\rangle), q_1^{\infty}(\langle k, r\rangle)\rangle \in U\}) > 0. \tag{38}$$

By [7, p. 4] there is a continuous function $f\colon A \times A \to [0, 1]$ having support in $\bar{U}^*$ (where $U^* = U^*_{0r} \times U^*_{1r}$ for the same r which defines U) juch that $f = 1$ on $\bar{U}$. By (38),

$$\int_{[0,1]} f(\langle q_0^{\infty}(\langle k, r\rangle), q_1^{\infty}(\langle k, r\rangle)\rangle)\, d\lambda(r) > 0. \tag{39}$$

It is evident from (c) and the definition of convergence in distribution that asymptotically (i.e., for all sufficiently large $n \in N$) there exist $r_n \leqslant n$ for which

$$\langle q_0^{n}(\langle k, r_n\rangle), q_1^{n}(\langle k, r_n\rangle)\rangle \in U. \tag{40}$$

For $n \in N$, define $a_j^n = q_j^n(\langle k, r_n\rangle)$ for $j = 0, 1$. Also define (as in Lemma 4) q_d^n to be the play of H_n which results when $\langle k, r_n\rangle$ deviates from q_0^n by taking action d. I.e., $q_d^n(\langle k, r_n\rangle) = d$ and $q_d^n(\langle k', r'\rangle) = q_0^n(\langle k', r'\rangle)$ if $k' \neq k$ or $r' \neq r_n$. Define $\eta_j^n = m_n(q_j^n)$ for $j = 0, 1, d$. By (c),

$$\langle \eta_j^n \rangle_{n \in N} \to m_{\alpha}(q_j^{\infty}) \tag{41}$$

for $j = 0, 1$, and

$$\langle \eta_d^n \rangle_{n \in N} \to m_{\alpha}(q_0^{\infty}). \tag{42}$$

Thus asymptotically $\eta_0^n, \eta_d^n \in W_{0r}$ and $\eta_1^n \in W_{1r}$, so that (36), (37) and (40) imply that for sufficiently large n,

$$E_{G(\eta_d^n)} u(d, x) - E_{G(\eta_0^n)} u(a_0^n, x) > \delta \tag{43}$$

and

$$| E_{G(\eta_1^n)} u(a_1^n, x)| < \epsilon. \tag{44}$$

Equations (43) and (45) will be used now to establish that (29) fails asymptotically for $q_0 = q_0^n$, so that $\langle q_0^n, q_1^n, B_n\rangle$ cannot be a strategic NCE of H_n. Define $\nu_j^n = G(\eta_j^n)$ and let E_j^n be expectation with respect to ν_j^n for $j = 0, 1, d$. It will be shown that asymptotically

$$\begin{aligned}
&[E_0^n u(a_0^n, x) + \beta(1 - \beta)^{-1}(1 - \nu_0^n(B_n))\, E_1^n u(a_1^n, x)]/[1 - \beta\nu_0^n(B_n)] \\
&\quad < [E_d^n u(d, x) + \beta(1 - \beta)^{-1}(1 - \nu_d^n(B_n))\, E_1^n u(a_1^n, x)]/[1 - \beta\nu_d^n(B_n)].
\end{aligned} \tag{45}$$

By (28), (45) is equivalent for H_n to $V_{\langle k, r_n \rangle}(q_0^n(\langle k, r_n \rangle)) < V_{\langle k, r_n \rangle}(d)$, so that (45) contradicts (29) for H_n. Thus the theorem will hold by Lemma 3.

Multiplying both sides of (45) by $[1 - \nu_0^n(B_n)]$ yields

$$
\begin{aligned}
E_0^n u(a_0^n, x) &+ \beta(1-\beta)^{-1}(1 - \nu_0^n(B_n))\, E_1^n u(a_1^n, x) \\
&< [(1 - \beta\nu_0^n(B_n))/(1 - \beta\nu_d^n(B_n))][E_d^n u(d, x) + \beta(1-\beta)^{-1} \\
&\quad \times (1 - \nu_d^n(B_n))\, E_1^n u(a_1^n, x)]. \qquad (46)
\end{aligned}
$$

For each $\xi > 0$ there is an open $W_\xi \subseteq M(A)$, with $m_\infty(q_0^\infty) \in W_\xi$, which satisfies

$$\sup_{B \subseteq X} \left| 1 - [[1 - \beta(G(\eta))(B)]/[1 - \beta(G(\theta))(B)]] \right| < \xi \qquad (47)$$

for $\eta, \theta \in W_\xi$. By (41), (42), and (47), (46) holds asymptotically if

$$E_0^n u(a_0^n, x) - E_d^n u(d, x) + \beta(1-\beta)^{-1}[\nu_d^n(B_n) - \nu_0^n(B_n)]\, E_1^n u(a_1^n, x) < 0 \qquad (48)$$

asymptotically. By (d), (41), and (42), $\lim_{n\to\infty} [\nu_d^n(B_n) - \nu_0^n(B_n)] = 0$, so by (44), (48) holds asymptotically if

$$\overline{\lim_{n \to \infty}}\, [E_0^n u(a_0^n, x) - E_d^n u(d, x)] < 0. \qquad (49)$$

This completes the proof, because (43) implies (49). Q.E.D.

From Theorems 2 and 6, it is evident that the paths of strategic noncooperative equilibria of a sequence of replica dynamic markets with unobserved random demand fluctuations converge to a price-taking equilibrium of the nonatomic representation of the sequence. One difficulty arises in the application of Theorem 6 here. That is, that hypothesis (d) fails if firms have unbounded production sets. However, if marginal costs become sufficiently high as output levels rise, this difficulty may be handled by imposing prior bounds, uniform in n, on the output levels which firms would consider. Hypothesis (d) is satisfied in the resulting sequence of replica markets because inverse demand is continuous and production sets are compact.

6. CONCLUDING REMARKS

The results which have been presented here concern the extent to which price-taking behavior characterizes the noncooperative equilibria of large economies. Theorems 1 and 2 of the present paper exemplify the positive results for this question which hold under the simplifying assumption that trade in each market occurs only once, during a single, initial trading period in which all markets clear simultaneously. Theorems 3 and 4 show that the validity of these results for actual markets, which continue to be active over time cannot be taken for granted. Finally, Theorems 5 and 6 describe the extent to which results for static markets can legitimately be applied to temporal trade. In this section, the practical significance of these results will be examined.

Most economists are in rough agreement about which actual markets are paradigm cases which corroborate the static theory of competitive behavior in large markets, and which markets are accounted for less successfully by the theory. Theorems 5 and 6 justify the application of the static theory to the paradigm cases, and they also clarify the status of some of the marginal cases. Theorem 5 asserts (in conjunction with Theorem 3) that, if firms in a market literally have no effect on the price, then the market must be competitive unless firms have access to extremely disaggregated information about their competitors. Theorem 6 states that, in the presence of imperfections which may reasonably be thought to exist in the price system, the assumption in Theorem 5 that individual firms do not influence the price may be taken as a good approximation of the situation in large finite markets. Thus, when a market contains many uncoordinated sellers, the static theory may legitimately be used to predict that the market will be competitive.

Problems typically arise in deciding what the words "many" and "uncoordinated" mean, when the applicability of the theory of large markets to a particular market is in question. Thus one class of marginal cases for the static theory consists of moderately concentrated industries (typically having fewer than a dozen major suppliers), and another class consists of markets in which the actions of a trade or professional association might affect the behavior of sellers. One school of thought in the economics profession holds that collusive agreements of even the smallest size are very unstable and that noncooperative equilibria converge rapidly to price-taking as the number of sellers is increased, so that there is a strong presumption that markets in both of these classes are competitive. An apparently conflicting view is held by a number of economists who have been persuaded of the monopolistic effects of professional associations in medicine, law, and so forth, on the basis of evidence which for the most part has been rather impressionistic.

Each of these views presupposes a set of beliefs about what determines whether firms in an industry would be able to detect and punish a competitor which attempted to violate a cartel agreement. It has long been recognized in the informal literature on industrial organization that this question, more than the question of how concentrated the industry is according to the usual quantitative measures, is important in assessing the competitiveness of an industry with restricted entry. By stating precise, formal criteria for cartel organization in an industry to be enforceable, Theorems 5 and 6 point the way toward an embodiment of this informal tradition in a rigorous, empirically satisfactory theory.[11] Industry studies based on such a theory would help to resolve long-standing empirical disputes about the extent of cartel organization, and might contribute to the determination of antitrust policy on a more rational basis.

APPENDIX: EXISTENCE AND UNIQUENESS OF A RANDOM PATH

Let $v = \langle s^k\rangle_{k\in K} \in S^K$, where $s^k = \langle s_t^k\rangle_{t\in N}$ for each t. It will be proved that there is a unique Borel measure on $\Omega = X^N$ which satisfies definition of a random path of v of an anonymous repeated random game.

LEMMA 4. *To prove the existence and uniqueness of a random path, it is sufficient to prove that there is a unique sequence* $\langle \pi_t\rangle_{t\in N}$ *such that*

(a) π_t *is a measure defined on* $\mathscr{B}_t$ *for each* t,

(b) *if* $t < t'$, *then* π_t *is the restriction of* $\pi_{t'}$ *to* $\mathscr{B}_t$,

(c) *the equation*

$$\pi_0 = F(\langle s_0^k\rangle_{k\in K}) \tag{50}$$

holds, and

(d) *if* $t \in N$, *and* P_t^* *is regular conditional probability for* π_t *relative to* $\mathscr{B}_{t-1}$, *then for every Borel subset* B *of* X,

$$P_t^*(\mathbf{x}_t(\omega) \in B \mid \omega) = \nu(B), \tag{51}$$

where $\nu = F(\langle s_t^k(\mathbf{x}_0(\omega), \ldots, \mathbf{x}_{t-1}(\omega))\rangle_{k\in K})$, *for all* ω *belonging to some* $C_t \in \mathscr{B}_{t-1}$ *where* $\pi_t(C_t) = 1$.

Proof. By the Kolmogorov consistency theorem [7, p. 139], for any sequence satisfying (a)–(d) there is a unique Borel measure π on Ω such that π_t is the restriction of π to $\mathscr{B}_t$, for every t. Thus (50) and (51) are equivalent to (24) and (25), so $\langle \Omega, \mathscr{B}, \pi\rangle$ is a random path generated by v. If $\langle \Omega, \mathscr{B}, \pi'\rangle$ is any random path generated by v, and if π_t' is the restriction of π' to $\mathscr{B}_t$, then $\langle \pi_t'\rangle_{t\in N}$ satisfies (a)–(d). By the uniqueness of the Kolmogorov extension, $\langle \pi_t\rangle_{t\in N} \neq \langle \pi_t'\rangle_{t\in N}$ if $\pi \neq \pi'$. Thus, uniqueness of the sequence satisfying (a)–(d) is sufficient for the uniqueness of the random path. Q.E.D.

LEMMA 5. *A unique sequence* $\langle \pi_t\rangle_{t\in N}$ *exists which satisfies* (a)–(d) *of Lemma* 4.

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[11] In particular, Eq. (29) suggests how the hypothesis that there is collusion in a particular industry might be tested on the basis of a time series of prices and market shares for that industry. Evidence from such a test would be more persuasive than that from cross-industry comparisons on which current attempts to measure the extent of collusion are based.

Proof. A sequence satisfying (a)–(d) will be constructed recursively. At each stage the measure chosen will be determined uniquely. To begin with, (50) defines π_0. Supposing that π_t has been chosen, condition (d) will be used to uniquely define π_{t+1} on $\mathscr{B}_{t+1}$. Define a *block* in X^{t+2} to be a product set $C \times B$, where $C \in \mathscr{B}_t$[12] and B is a Borel subset of X. Define $\pi^*\colon \mathscr{B}_t \times \mathscr{B}_0 \to [0, 1]$ by

$$\pi^*(C \times B) = \int_C P^*_{t+1}(\mathbf{x}_{t+1}(\omega) \in B \mid \omega)\, d\pi_t(\omega), \tag{52}$$

where Eq. (51) is taken as the *definition* of P^*_t. (N.B. The measurability of $P^*_{t+1}(\mathbf{x}_{t+1} \in B \mid \cdot)$ with respect to $\mathscr{B}_t$ is guaranteed by the definitions of strategy vector and anonymous game.) Now define $\mathscr{B}^* \subseteq \mathscr{B}_{t+1}$ to be the class of finite unions of disjoint blocks. $\mathscr{B}^*$ is a Boolean algebra [3, p. 185], and π^* determines a finitely additive measure on $\mathscr{B}^*$. In fact, it is evident from Lebesque's monotone convergence theorem that π^* is countably additive on $\mathscr{B}^*$. By the Caratheodory–Hahn extension theorem [3, p. 136], π^* extends uniquely to a measure on the smallest σ-field containing $\mathscr{B}^*$. By [7, p. 6], this σ-field is $\mathscr{B}_{t+1}$. Let the extension measure be π_{t+1}. Conditions (a) and (b) are trivially verified, and it follows from (23) that p^*_{t+1} defined by (51) are trivially verified, and it follows from (23) that p^*_{t+1} defined by (51) agrees almost surely with regular conditional probability for $B \in \mathscr{B}^*$, so that π_{t+1} satisfies (d) and is the unique extension of π_t which does so. This completes the induction. Q.E.D.

The existence and uniqueness of the random path of v are an immediate consequence of Lemmas 4 and 5:

THEOREM 7. *If H is an anonymous repeated game with random oucomes and v is a strategy vector for H, then v generates a unique random path* $\langle \Omega, \mathscr{B}, \pi \rangle$.

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[12] In this proof, $\mathscr{B}_t$ will indicate both the Borel subsets of X^{t+1} and the smallest subfield of X^N with respect to which $\mathbf{x}_0, \ldots, \mathbf{x}_t$ are Borel measurable. The intended reference will be clear from context.

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JOURNAL OF ECONOMIC THEORY 22, 183–207 (1980)

The No-Surplus Condition as a Characterization of Perfectly Competitive Equilibrium*

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Received January 5, 1979

In the Walrasian definition of competitive equilibrium, attention is confined to the consistency of individuals' plans on the presumption that each agent regards itself as a price-taker. Thus, Walrasian equilibrium may exist where there is no supporting evidence for that presumption—e.g., in an economy with a small number of agents. The self-imposed limitations of the definition imply that it describes necessary but not sufficient conditions for perfectly competitive equilibrium—i.e., an equilibrium for which the presumption of price-taking is justified. In this paper we propose an alternative definition that brings out the perfectly competitive character of the equilibrium that is implicitly behind the usual interpretation.

The approach adopted here has as a basic premise that the description of perfect competition should be invariant to the number of agents. Contrary to what appears to be the typical view that it is locigally tied to the study of economies with large numbers of agents, we find that an analysis of the conditions for perfectly competitive equilibrium with small numbers yields a characterization that is essentially the same as it is for large. What differs between economies with small and large numbers is simply the likelihood of its occurrence.

Consider a group of agents and an allocation that maximizes the utility of one of them, say j, subject to requirement that the others are made no worse off than they would be by reallocating resources among themselves, without j. At such an allocation, j is extracting all the gains from trade he might reasonably expect by, as it were, joining the economy; and, the others may be receiving absolutely no surplus from the presence of j. Suppose an economy exhibits an allocation that simultaneously fulfills this condition for all agents taken one at a time. Such a situation is a special case of what will be called a *no-surplus allocation* that we propose as an alternative to the Walrasian definition of competitive equilibrium.

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* This work was supported by NSF Grant SOC77-18086.

0022-0531/80/020183-25$02.00/0

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Some justifications for the proposed definition are the following:

(1) Whether the number of agents is large or small, a no-surplus allocation is a Walrasian equilibrium.

(2) A no-surplus allocation is equivalent to the condition that each agent faces perfectly elastic demand schedules, at Walrasian prices, for the goods it sells.

(3) Although almost all Walrasian allocations in economies with small numbers are not no-surplus, in economies with large numbers and a finite number of commodities almost all Walrasian allocations are.

In this paper we shall demonstrate (1)–(3) for the most elementary but nevertheless representative case of finite exchange economies and for sequences of exchange economies consisting of replicas of a fixed finite economy. A novelty of the demonstration is that for large economies we shall permit the presence of external effects. More general sequences of economies exhibit similar conclusions (see Ostroy [14]). Related results have been established by Makowski [10] for economies with production by firms.

The concept of no-surplus is reminiscent of the marginal productivity theory of distribution. With such an allocation each agent may be said to be obtaining its marginal product and since this is true for all agents simultaneously, the total product (allocation) may be said to be entirely exhausted by the sum of the marginal products (see Theorem 2, below). The analogy is further exploited by Makowski and it is shown in Ostroy [15] that the marginal-productivity-product-exhaustion and no-surplus descriptions of perfect competition are essentially equivalent for economies with a continuum of traders.

In the final section, the no-surplus and core characterizations of perfectly competitive equilibrium are compared. Elsewhere, we shall exhibit the connections with the Shapley value (Shapley [18]), and the concept of incentive-compatibility of Walrasian equilibria (Roberts and Postlewaite [16]). Mas-Colell [12] has shown that the no-surplus condition can be used to abstract some of the common elements of the Core and Value Convergence Theorems. In the context of economies regarded as games with transferable utility, Geanakoplos [7] has independently demonstrated that the marginal-productivity-product-exhaustion condition can be used to derive the equivalence of the core, Shapley value, bargaining set, and Walrasian equilibria.

I. NOTATION AND PRELIMINARY DEFINITIONS

The concepts and results below are formulated for an exchange economy $\mathscr{E} = \{(S_i, w_i)\}_{i=1}^{i=n}$, where S_i describes the preferences and w_i the initial endowments of a typical agent i. It is assumed throughout that the relevant commodity space is R_+^ℓ so that $w_i \in R_+^\ell$. An *allocation* is denoted by $X = (x_i)$, $x_i \in R_+^\ell$; and, unless otherwise noted, X will also be assumed to satisfy the feasibility condition, $\sum(x_i - w_i) = 0$.

For an allocation X, $S_i(X)$ is a subset of R_+^ℓ. If $y \in S_i(X)$, then i would prefer y to x_i given the allocations x_j in X for $j \neq i$. Let $\partial S_i(X)$ be the R_+^ℓ-boundary of $S_i(X)$, the set of points in the closure of $S_i(X)$ that does not belong to its R_+^ℓ-interior. For each $i = 1,\ldots, n$ and $X = (x_i)$, assumptions on preferences will include (some of) the following:

A_1: $S_i(X)$ is open in R_+^ℓ;

A_2: $x_i \in \partial S_i(X)$;

A_3: $p \in R^\ell$, $p \neq 0$, $\inf p[S_i(X) - w_i] = 0$, and $y \in S_i(X)$ implies $p(y - w_i) > 0$;

A_4: $S_i(X)$ is convex;

A_5: $X' = (x_i')$ and $x_j' = x_j$ implies $S_j(X') = S_j(X)$;

A_6: $\{(y, X): y \in S_i(X)\}$ is open in $R_+^\ell \times (\underbrace{R_+^\ell \times \cdots \times R_+^\ell}_{n})$

Consistent with the interpretation of $S_i(X)$ as indicating strictly preferred elements, A_1 and A_2 imply that the underlying preference ordering is irreflexive since $S_i(X)$ does not contain its boundary and $x_i \in \partial S_i(X)$. A_2 is a version of local nonsatiation that also implies $S_i(X)$ is not empty. A_3 rules out the possible difficulties that arise from the fact that $S_i(X)$ is merely R_+^ℓ-open. If $\inf p[S_i(X) - w_i] = 0$ and $pw_i \neq 0$, it is well known that A_3 is superfluous. However, since the conditions that would guarantee $pw_i \neq 0$ are not otherwise needed for the results below, we simply impose A_3. Note that A_3 obviates the need for any assumptions on w_i or $\sum w_i$ being in the interior of R_+^ℓ. A_4 is essential for a linear price description of perfectly competitive, Walrasian equilibrium. A_5 says that an agent's preferences are unaffected by any part of the allocation that is not assigned to the agent—i.e., there are no external effects. The continuity condition, A_6, will not be used in any of the formal results but it will be called upon, informally, to extend some conclusions.

In the following we shall be concerned with the influence of any single agent j on the rest of the economy and it will be convenient to adopt the convention in which $(\cdot)^j$ indicates the corresponding sum excluding j. Thus, $S^j(X) = \sum_{i\neq j} S_i(X)$, $w^j = \sum_{i\neq j} w_i$, and $x^j = \sum_{i\neq j} x_i$. Similarly, we shall use $(\cdot)^0$ to indicate that the sum excludes no agent. Thus, $S^0(X) = \sum S_i(X)$, $w^0 = \sum w_i$, and $x^0 = \sum x_i$.

Remark 1. Since $\sum x_i = w^0$, A_1 and A_2 imply that the condition $w^0 \notin S^0(X)$ is equivalent to $w^0 \in \partial S^0(X)$. With A_5, $w^0 \in \partial S^0(X)$ has the usual interpretation that X is Pareto-optimal—there is no other feasible allocation that would be preferred by all agents. However, without A_5, if $w^0 \in \partial S^0(X)$ no such conclusion is warranted. We can only maintain the rather trivial implication that if any agent is to reach a more preferred allocation *without relying on trade with others*, total resources must be other than w^0.

II. THE NO-SURPLUS CONDITION AND ITS DUAL IN $\mathscr{E}$

For the following definitions and throughout the remainder of the paper A_1 and A_2 are assumed.

Trade is productive for all the members of $\mathscr{E}$ whenever there exists an allocation $X = (x_i)$ such that each agent can achieve a surplus through trade —i.e., $x_i \in S_i(W)$, where $W = (w_i)$ is the initial allocation. We may say that the surplus produced through trade is maximal when X is *Pareto-optimal* (PO).

$$\text{PO:} \qquad w^0 \in \partial S^0(X).$$

Following the logic of marginal productivity theory, can an allocation be found that uniquely imputes the total surplus produced through trade to the separate contributions of the participating agents? To formulate this condition we require that each agent receive neither more nor less than its marginal product.

For an exchange economy, we shall say that at the allocation X no agent is receiving more than it is worth if it satisfies the *nonnegative surplus* (NNS) condition.

$$\text{NNS:} \qquad w^j \notin S^j(X), \qquad j = 1,\ldots, n.$$

NNS places an upper bound on the extent to which an agent can exploit its monopoly power by saying, in effect, that no seller j can enforce an outcome in which its customers would do better by refusing to deal with j and going elsewhere.

At an allocation X no agent is receiving less than it is worth if it satisfies the *nonpositive surplus* (NPS) condition.

$$\text{NPS:} \qquad w^j \in \operatorname{cl} S^j(X), \qquad j = 1,\ldots, n \qquad (\text{cl} = \text{closure}).$$

Otherwise, if $w^j \notin \operatorname{cl} S^j(X)$, j is contributing a positive surplus and can therefore claim to be receiving less than it is worth to the rest of the economy. (Unless there are no external effects, A_5, the NNS and NPS conditions may not reflect their marginal productivity interpretations in finite economies. See Remark 2, below.)

An allocation X satisfies NNS and NPS if and only if $w^j \in \partial S^j(X)$, $j = 1,\ldots, n$, where $\partial S^j(X)$ the R_+^l-boundary of $S^j(X)$. To continue the analogy with marginal productivity theory, if each agent is paid its marginal product, will the sum of the payments just exhaust the total product? For an exchange economy, the question is whether there exists a *no-surplus* (NS) allocation – i.e., an X such that

$$\text{NS:} \qquad w^m \in \partial S^m(X), \qquad m = 0, 1,\ldots, n.$$

The geometrical properties that flow from the convexity hypothesis, A_4, can be exploited to give additional perspective on the concept of NS, especially its relation to Walrasian equilibrium.

Let $P = \{p \in R^l \colon \|p\| = 1\}$, where $\|\cdot\|$ denotes the Euclidean norm, be the set of vectors on the surface of the unit ball of R^l. Define

$$Q^m(X) = \{p \in P \colon \inf p[S^m(X) - w^m] \geqslant 0\}, \qquad m = 0, 1,\ldots, n.$$

If $w^m \notin S^m(X)$, then $Q^m(X) \neq \varnothing$ and conversely, if A_3 is assumed, $Q^m(X) \neq \varnothing$ will be shown to imply $w^m \notin S^m(X)$. This will be used to give a description of X as an NS allocation in terms of the sets $Q^m(X)$.

THEOREM 1. *Let* $\mathscr{E}$ *satisfy* A_1–A_4. *Then* X *is an* NS *allocation if and only if* $Q^m(X) \neq \varnothing$, $m = 0, 1,\ldots, n$, *and*

(i) $Q^0(X) \subset Q^j(X), j = 1,\ldots, n,$

(ii) $p \in Q^m(X)$ *implies* $\inf p[S^m(X) - w^m] = 0$, $m = 0, 1,\ldots, n$.

Proof. If $Q^m(X)$, $m = 0, 1,\ldots, n$, satisfy (i) and (ii) and $p \in Q^0(X)$, then $\inf p[S^0(X) - w^0] = \inf p[S^j(X) - w^j]$, $j = 1,\ldots, n$. By construction,

$$\inf p[S^0(X) - w^0] = \inf p[S^j(X) - w^j] + \inf p[S_j(X) - w_j]. \qquad (1)$$

Therefore, $\inf p[S_j(X) - w_j] = 0$ and by A_3 if $y \in S_j(X)$, $p(y - w_j) > 0$. Thus, for any $p \in Q^0(X)$, if $y \in S^m(X)$, $p(y - w^m) > 0$, $m = 0, 1,\ldots, n$. This and (ii) imply that $w^m \notin S^m(X)$, all m.

To show that $w^m \in \partial S^m(X)$ it suffices to show that $w^m \in \operatorname{cl} S^m(X)$. Suppose the contrary and let $0 \neq \|z^m\| = \inf \|\operatorname{cl} S^m(X) - w^m\|$. By A_4 and a standard argument used in the proof of the Separation Theorem, we may set $p = (\|z^m\|)^{-1} z^m \in P$ and obtain the conclusion that $p \in Q^m(X)$. But $0 \neq pz^m = \|z^m\| \leqslant \inf p[S^m(X) - w^m]$, contradicting (ii).

If X is NS, the Separation Theorem implies that $Q^m(X) \neq \varnothing$, $m = 0, 1,\ldots, n$. Let $p \in Q^0(X)$. Since $w^m \in \operatorname{cl} S^m(X)$, $\inf p[S^m(X) - w^m] \leqslant 0$. Again by A_4 and (1), above, we may conclude that $\inf p[S^j(X) - w^j] = 0$, $j = 1,\ldots, m$. This establishes (i).

If $p \in Q^m(X)$ and $\inf p[S^m(X) - w^m] > 0$, then $S^m(X)$ can be strictly separated from w^m, which contradicts the fact that $w^m \in \partial S^m(X)$. This establishes (ii).

The following gives a sufficient condition for X to be an NS allocation.

COROLLARY 1.1. *Let $\mathscr{E}$ satisfy* A_1–A_4. *If X is such that $Q^0(X) = Q^j(X) \neq \varnothing$, $j = 1,\ldots, n$, then X is* NS.

Proof. It suffices to show condition (ii) of Theorem 1. Since $\bigcap_j Q^j(X) = Q^j(X) = Q^0(X)$ and $p \in \bigcap Q^j(X)$, we have $p(x^j - w^j) \geqslant \inf p[S^j(X) - w^j] \geqslant 0$. This implies $p(x^m - w^m) = 0$, $m = 0, 1,\ldots, n$ and therefore, $\inf p[S^m(X) - w^m] = 0$, $m = 0, 1,\ldots, n$.

To show that the above sufficient condition is practically necessary for NS, we demonstrate

COROLLARY 1.2. *Let $\mathscr{E}$ satisfy* A_1–A_4 *and let X be an allocation such that the set of net trade vectors $\{(x_i - w_i)\}$ is not contained in a subspace of dimension less than $(l - 1)$. Then X is* NS *if and only if there is a $p \in P$ such that $Q^0(X) = Q^j(X) = \{p\}$, $j = 1,\ldots, n$.*

Proof. By Corollary 1.1, if $Q^0(X) = Q^j(X)$, X is NS.

If X is NS, by the argument in Corollary 1.1, there is at least one $p \in P$ such that $p(x_i - w_i) = 0$, $i = 1,\ldots, n$. Since the dimension of the span of $\{(x_i - w_i)\}$ is not less than $(l - 1)$ and $p \in P$ is in its orthogonal complement the dimension is not more than $(l - 1)$ and the dimension of the orthogonal complement is therefore 1.

It has been demonstrated by Mas-Colell [11] and Shafer and Sonnenschein [17] that only those properties of preferences embodied in A_1–A_4 and A_6 are required for the existence of Walrasian equilibrium. Following this lead, as well as a similar framework used by Vind [22], the concept of NS is also formulated so as to depend only on the geometrical character of the preference mapping $S_i(X)$. Since these restrictions do not necessarily include completeness or transitivity, preferences need not be numerically representable. However, additional confirmation of the links between the NS concept and the marginal productivity theory of distribution is gained by assuming that $S_i(X)$ may be derived from a function $u_i : R_+^l \to R$ as $\{y : u_i(y) > u_i(x_i)\}$.

Let $\mathscr{E}_u = \{(u_i, w_i)\}$ be a numerical representation of $\mathscr{E} = \{(S_i, w_i)\}$. An allocation $X = (x_i)$ produces the utility vector $u(X) = (u_1(x_1),\ldots, u_n(x_n))$. Assuming each u_i is continuous, define

$$v_u{}^0 = \max_{\substack{y_i \in R_+^{\ell} \\ \Sigma y_i = w^0}} \sum u_i(y_i),$$

$$v_u{}^j = \max_{\substack{y_i \in R_+^{\ell} \\ \Sigma_{i \neq j} y_i = w^j}} \sum_{i \neq j} u_i(y_i).$$

Regarding v_u^0 as the maximum total product from $\mathscr{E}_u$ and v_u^j as the maximum when j does not participate, the marginal product of j in $\mathscr{E}_u$ is $(v_u^0 - v_u^j)$. We shall say that $\mathscr{E}_u$ exhibits *product-exhaustion* (PE) if there is an allocation $X = (x_i)$ such that all agents receive their marginal products, i.e., for each $j = 1, \ldots, n$,

$$\text{PE:} \qquad u_j(x_j) = v_u^0 - v_u^j.$$

While the NS concept of marginal productivity is entirely ordinal, the definition of PE depends upon the particular numerical scaling used to represent preferences. To remove some spurious differences let $U(\bar{u}_i) = \{u_i : u_i = \alpha_i \bar{u}_i, \alpha_i > 0\}$ be the set of positive scalar multiples of the function $\bar{u}_i$ and let $\mathscr{E}_U = \{(U(\bar{u}_i), w_i)\}$ describe an equivalence class of numerical representations for $\mathscr{E} = \{(S_i, w_i)\}$. $\mathscr{E}_U$ will be said to satisfy PE if there is an $\mathscr{E}_u \in \mathscr{E}_U$ exhibiting PE.

THEOREM 2. *Let $\mathscr{E} = \{(S_i, w_i)\}$ be representable by $\mathscr{E}_U = \{(\bar{u}_i), w_i)\}$, where each $\bar{u}_i$ is continuous, concave and increasing on R_+^ℓ. Then $X = (x_i)$ is an* NS *allocation for $\mathscr{E}$ if and only if $\mathscr{E}_U$ satisfies* PE.

The proof depends on the following property of exchange economies with concave utilities. It is one of the derivable implications of the fact that such an economy can be regarded as a balanced game. (See Shapley and Shubik [19] and Billera and Bixby [3].)

LEMMA 1. *If each u_i is continuous and concave on R_+^ℓ, then $\mathscr{E}_u = \{(u_i, w_i)\}_{i=1}^{i=n}$ satisfies*

$$v_u^0 \geq (n-1)^{-1} \sum_{j=1}^{j=n} v_u^j.$$

Proof. Let $\sum_{i \neq j} u_i(x_i^j) = v_u^j$, $j = 1, \ldots, n$. Such an allocation exists by virtue of the continuity of each u_i and the compactness of the domain over which the maximum is to be achieved. For each $i = 1, \ldots, n$, put

$$x_i = (n-1)^{-1} \sum_{j \neq i} x_i^j.$$

Then,

$$(n-1)^{-1} \sum_{j=1}^{j=n} v_u^j = (n-1)^{-1} \sum_{j=1}^{j=n} \sum_{i \neq j} u_i(x_i^j) = \sum_{i=1}^{i=n} (n-1)^{-1} \sum_{j \neq i} u_i(x_i^j)$$

$$\leq \sum u_i \left((n-1)^{-1} \sum_{j \neq i} x_i^j \right) = \sum u_i(x_i),$$

where the inequality follows from concavity.

Since $\sum_{i \neq j} x_i^j = w^j$ and $(n-1)^{-1} \sum w^j = w^0$,

$$\sum x_i = (n-1)^{-1} \sum_{i} \sum_{j \neq i} x_i^j = (n-1)^{-1} \sum_{j} \sum_{i \neq j} x_i^j$$

$$= (n-1)^{-1} \sum w^j = w^0,$$

and therefore $X = (x_i)$ is an allocation for $\mathscr{E}_u$. Thus, $v_u^0 \geqslant \sum u_i(x_i)$ which, with the above inequality, establishes the desired conclusion.

Proof of Theorem 2. If $X = (x_i)$ satisfies PE for $\mathscr{E}_u$ then $u_j(x_j) = v_u^0 - v_u^j$ and

$$\sum u_j(x_j) = \sum (v_u^0 - v_u^j) = n v_u^0 - \sum v_u^j \geqslant v_u^0,$$

where the inequality follows from Lemma 1. Since $X = (x_i)$ is an allocation for $\mathscr{E}_u$, $\sum u_j(x_j) = v_u^0$ and therefore $w^0 \in \partial S^0(X)$. From PE,

$$\sum_{i \neq j} u_i(x_i) = \sum_{i=1}^{i=n} u_i(x_i) - u_j(x_j) = v_u^0 - (v_u^0 - v_u^j) = v_u^j,$$

and therefore $w^j \in \partial S^j(X)$, $j = 1, \ldots, n$.

For the converse, let

$$V^0 = \left\{ v = (v_1, \ldots, v_n) : v_i \leq \bar{u}_i(y_i), \sum y_i = w^0, y_i \in R_+^\ell \right\}.$$

By the concavity and continuity of each $\bar{u}_i$, V^0 is known to be closed and convex. When X is NS, $w^0 \in \partial S^0(X)$, and therefore $\bar{u}(X) \in \partial V^0$. The fact that each $\bar{u}_i$ is increasing is well known to imply that if $v \in V^0$, then $[v - \bar{u}(X)] \notin R_+^n \backslash \{0\}$. Thus, by the Support Theorem for convex sets, there is an $\alpha = (\alpha_1, \ldots, \alpha_n)$, $\alpha_i \geq 0$, $\sum \alpha_i = 1$ such that $\sup_{v \in V^0} \alpha v = \alpha \bar{u}(X)$.

Further, the hypothesis that X is NS and each $\bar{u}_i$ is increasing implies that $\alpha_j > 0$ whenever $w_j \neq 0$. Otherwise, if $\alpha_j = 0$ and $w_j \neq 0$

$$\sum_{i \neq j} \alpha_i \bar{u}_i(x_i) = \sum_{i=1}^{i=n} \alpha_i \bar{u}(x_i) = \max_{\substack{y_i \in R_+^\ell \\ \sum_{i \neq j} y_i = w^0}} \sum_{i \neq j} \alpha_i \bar{u}_i(y_i) > \max_{\substack{y_i \in R_+^\ell \\ \sum_{i \neq j} y_i = w^j}} \sum_{i \neq j} \alpha_i \bar{u}_i(y_i).$$

But the inequality contradicts the assumption that $w^j \in \partial S^j(X)$.

Without loss of generality assume each $w_i \neq 0$, and therefore each $\alpha_i > 0$. Putting $u_i = \alpha_i \bar{u}_i$, $i = 1, \ldots, n$, we have for the economy $\mathscr{E}_u$,

$$\sum u_i(x_i) = v_u^0 = \max_{\substack{y_i \in R_+^\ell \\ \sum y_i = w^0}} \sum \alpha_i \bar{u}_i(y_i).$$

Since $w^j \in \partial S^j(X)$ and u_i is increasing, $v_u^j \geqslant \sum_{i \neq j} u_i(x_i)$. If $v_u^j > \sum_{i \neq j} u_i(x_i)$ for some j, then

$$\sum_j v_u^j > \sum_j \sum_{i \neq j} u_i(x_i) = (n-1)\, v_u^0,$$

contradicting Lemma 1. Therefore, for each j, $\sum_{i \neq j} u_i(x_i) = v_u^j$, and we have the desired conclusion that $u_j(x_j) = v_u^0 - v_u^j$.

III. WALRASIAN AND PERFECTLY COMPETITIVE EQUILIBRIUM

A Walrasian equilibrium (WE) for $\mathscr{E}$ is a pair (X, p), where X is an allocation and $p \in P$, such that for each i,

$$\text{WE:} \qquad p(x_i - w_i) = 0 \qquad \text{and} \qquad y \in S_i(X) \ \text{implies} \ p(y - w_i) > 0.$$

It is useful to give an alternative description.

PROPOSITION 1. *Let* $\mathscr{E}$ *satisfy* A_1–A_3. *Then* (X, p) *is a* WE *if and only if* $p \in \bigcap_{j=1}^{j=n} Q^j(X)$.

Proof. If (X, p) is a WE, $\inf p[S_i(X) - w_i] \geqslant 0$. This yields by summation $\inf p[S^j(X) - w^j] \geqslant 0$ and therefore $p \in Q^j(X)$.

If $p \in \bigcap Q^j(X)$, $\inf p[S^j(X) - w^j] \geqslant 0$. By A_2, $px_i \geqslant \inf p\, S_i(X)$ and therefore $px^j \geqslant \inf p\, S^j(X)$. Thus, $p(x^j - w^j) = p(w_j - x_j) \geqslant \inf p[S^j(X) - w^j] \geqslant 0$, for all $j = 1, \ldots, n$. This implies $p(x_j - w_j) = \inf p[S_j(X) - w_j] = 0$, all j. The remaining part of the definition of WE follows from A_3.

Inspection of Theorem 1 and Proposition 1 yields.

PROPOSITION 2. *Let* $\mathscr{E}$ *satisfy* A_1–A_4. *If X is an* NS *allocation, there is a* $p \in P$ *such that* (X, p) *is a* WE.

Comparing the definition of X as a WE allocation to the definition of an NS allocation the former simply requires that $\bigcap_j Q^j(X) \neq \varnothing$ while the latter demands that, in "most" cases, $Q^j(X) = Q^0(X)$. (Compare Corollaries 1.1 and 1.2.) Assuming that $Q^0(X) = \{p\}$, if $Q^j(X) \neq \{p\}$, there are valuations of resources for the "one-fewer" sets of agents that are not common to the valuation of resources by the economy as a whole. The definition of WE resolves the issue of resource valuation by imposing the condition that only those elements of each $Q^j(X)$ that are common to $Q^0(X)$ are acceptable. This is nothing other than the well-known restriction that individual agents must take prices as given. However, with an NS allocation, we shall show that such a restriction is superfluous. This will be accomplished by constructing a definition of equilibrium in which competitive self-interest dictates price-taking behavior and that the definition is equivalent to the NS condition. Informally, demands are perfectly elastic at the price vector p when the attempt by any agent to set prices for the goods it supplies at levels higher than p results in the loss of all sales *and* markets for the goods supplied by others clear without any adjustment in their prices. To make this experiment possible, each agent must have control over the prices of the goods it supplies.

Normally the commodity space, R_+^ℓ, would be chosen so that the number of goods, l, is as small as possible. It aggregates into one all those goods for which individuals have identical, constant marginal rates of substitution. The resulting interpretation is that each commodity represents a market with the presumption that on any market one and only one price will be charged. Upon such a foundation the concept of WE is applied to determine market-clearing prices. Here we do not take the market with its single price for granted. Rather, the purpose is to open up the model to price-setting by individual agents so as to allow "markets" to emerge as a conclusion.

We shall assume that commodities are disaggregated to bring the specification of initial endowments into the appropriate form of a *personalized commodity space* in which each agent is the unique supplier of its own goods. Formally, commodities are in personalized form if

$$w_i, w_j \in R_+^\ell, \qquad i \neq j, \text{ implies } w_i w_j = 0.$$

For any agent j, we may divide the l (disaggregated) commodities into those that could be supplied by j and those that could not. For $p \in R^\ell$, let p_j be the vector of prices of the former and p^j the prices of the latter. For any p and j, $p = (p^j, p_j)$, let $q(p; j)$ denote any vector $q = (q^j, q_j) \in R^\ell$ such that $q^j = p^j$ and $q_j \geq p_j$ and $q_j \neq p_j$. At $q(p; j)$ the price of at least one of j's goods is higher than at p while the prices of the remaining goods, that j cannot control, are the same as in p.

The pair (X, p) is a *perfectly determinate price equilibrium* (PD) for $\mathscr{E}$ if for each $j = 1,\dots, n$ and any $i \neq j$, if y satisfies

$$\mathrm{PD}_1: \quad q(p; j)(y - w_i) = 0, \ (q(p; j) - p)y \neq 0, \quad \text{then} \quad y \notin \mathrm{cl}\, S_i(X)$$

and there exists y_i^j, $i \neq j$, such that

$$\mathrm{PD}_2: \quad q(p; j)(y_i^j - w_i) = 0, \quad y_i^j \in \mathrm{cl}\, S_i(X) \quad \text{and} \quad \sum_{i \neq j} y_i^j = w^j.$$

To interpret, the opportunities available to i are strictly smaller when they are defined by the price vector $q(p; j)$ rather than p since the price of all goods other than j's are the same but j's prices are higher. However, PD states that if i is a purchaser from j, $(q(p; j) - p)y \neq 0$, i has not made the best use of his smaller opportunity set since y is not in cl $S_i(X)$, while if i had refused to deal with j, then trading at the same prices, he could have found buyers and sellers willing to make exchange leading to y_i^j in cl $S_i(X)$. If preferences were transitive and continuous, y_i^j would be strictly preferred to y_i. We shall regard the existence of (X, p) satisfying PD as synonymous with a price equilibrium exhibiting perfectly elastic demands.

THEOREM 3. *Let* $\mathscr{E}$ *satisfy* A_1–A_5 *and assume initial endowments are personalized. Then* X *is an NS allocation if and only if there is a* $p \in P$ *such that* (X, p) *is* PD.

Proof. From Corollary 1.1, if X is NS, there is a $p \in P$ such that (X, p) is a WE. Thus, $pw_i = px_i = \inf pS_i(X)$. If $q(p; j)(y - w_i) = 0$ and $(q(p; j) - p)y \neq 0$, then $py < px_i$ and therefore $y \notin \text{cl}\, S_i(X)$ and PD_1 is satisfied. Since $p \in Q^0(X)$ may be chosen so that $p \in Q^j(X)$ and because X is NS, $pw^j = \inf pS^j(X) = px^j$. Also by NS there exists $y_i^j \in \text{cl}\, S_i(X)$ such that $\sum_{i \neq j} y_i^j = w^j$ and therefore for all i and j, $i \neq j$, $py_i^j \geqslant \inf pS_i(X)$, which implies $p(y_i^j - w_i) = 0$ or $q(p; j)(y_i^j - w_i) = 0$, and PD_2 is established.

For the converse it suffices to show that if (X, p) is PD, it is a WE, and therefore satisfies $p \in \bigcap_j Q^j(X)$. If (X, p) is a WE, PD_2 clearly implies NS.

From PD_2 we have $p(y_i^j - w_i) = 0$, and since $y_i^j \in \text{cl}\, S_i(X)$, $\inf pS_i(X) \leqslant py_i^j$. If $\inf pS(X) < py_i^j = pw_i$ for some i, then we may choose $q(p; j)$ close to p and y close to x_i such that $q(p; j)(y - w_i) = 0$ and $(q(p; j) - p)y \neq 0$ such that $y \in \text{cl}\, S_i(X)$, contradicting PD_1. Therefore, $pw_i = py_i^j = \inf pS_i(X) = px_i$ for all i and therefore $pw_i = px_i$. The remainder of the definition of WE follows from A_3.

An implication of Theorem 3 is that PD or NS may be regarded as the definition of perfectly competitive equilibrium. With it we have the conclusion that agents will rationally choose to set as prices for the commodities they supply the ones dictated by WE.

Remark 2. Without A_5, the absence of external effects, the definition of perfectly competitive equilibrium for finite exchange economies would be deficient. If j set a higher price and sells nothing because his customers go elsewhere, the resulting allocation could be substantially different and, with external effects, preferences could be substantially changed. Thus, even though WE satisfies NS, an agent can possess monopoly power by recognizing the possible adverse effects to customers who might otherwise go elsewhere. This is an added complication due to small numbers. With large numbers external effects do not create a separate problem because the possible disruption to others caused by refusing to deal with any one agent vanishes as the relative size of the agent vanishes.

A trivial illustration of an NS economy occurs whenever the initial endowments for $\mathscr{E}$ are PO. A nontrivial illustration is provided by the following:

EXAMPLE 1. Let the tastes of agent $i = 1, 2, 3$ be represented by the utility function $u_i(x_i) = u_i(x_{i1}, x_{i2}, x_{i3})$, initial endowments by the matrix W and final allocations by the matrices X_α, where

$$\begin{aligned} u_1(x_1) &= [x_{11}(x_{12} + x_{13})]^{1/2}, \\ u_2(x_2) &= [x_{22}(x_{21} + x_{23})]^{1/2}, \\ u_3(x_3) &= [x_{33}(x_{31} + x_{32})]^{1/2}; \end{aligned}$$

$$W = \begin{array}{c|ccc} {}_i\diagdown^c & & & \\ \hline & 2 & 0 & 0 \\ & 0 & 2 & 0 \\ & 0 & 0 & 2 \end{array}, \quad X_\alpha = \begin{array}{c|ccc} {}_i\diagdown^c & & & \\ \hline & 1 & \alpha & 1-\alpha \\ & 1-\alpha & 1 & \alpha \\ & \alpha & 1-\alpha & 1 \end{array}$$

To show that $\mathscr{E}$ is an NS economy, it must be demonstrated that for any X such that $w^j \notin S^j(X)$, $j = 1, 2, 3$, $w^m \in \partial S^m(X)$, $m = 0, 1, 2, 3$. Taking $X = X_\alpha$, $0 < \alpha < 1$, this result is easily verified. Note that these allocations are at the opposite extreme from no-trade since their attainment requires that all agents participate in trade. Of course, because each X_α is an NS allocation and there are only three traders, any pair could also do as well by itself. Each agent's participation in an NS allocation is essential only to itself.

It may also be verified that each X_α is a WE (Proposition 2) at prices $p = (r, r, r)$, $r > 0$; and, if any agent j were to raise the price of its good above its value in p, j would sell nothing and excess demands among the other agents be zero without disturbing the other prices (Theorem 3). Further, for any X_α, each agent obtains its marginal product—i.e., $u_j(x_j) - 1 = v_u^0 - v_u^j = 3 - 2$ (Theorem 2).

Remark 3. Note that in Example 1 individual preferences are not strictly convex. (Strict convexity obtains if every line segment connecting two distinct points on $\partial S_i(X)$ lies in $S_i(X)$.) When the commodity space is put in personalized form strict convexity and NS imply that there can be no trade.

At an NS allocation each agent j must be faced with an $S^j(X)$, a portion of whose boundary coincides with the hyperplane $\{y: p(y - w^j) = 0\}$, where p is a WE price vector. If the hyperplane supported $S^j(X)$ but $S^j(X)$ were strictly convex, the only point of intersection would be at $x^j \in \partial S^j(X)$ and therefore $w^j \in \partial S^j(X)$ if and only if $x^j = w^j$.

Since strict convexity is the rule even after commodities are put in personalized form, this observation shows that most finite economies will fail to admit an NS allocation. It also exhibits a geometric parallel with the apparently quite different conclusions for large economies. When the scale of each agent becomes very small, the quantities x^j and w^j will differ very little from w^0 and therefore from each other. In this case, $x^j \in \partial S^j(X)$ will imply $w^j \in \partial S^j(X)$ provided that the boundary of $S^j(X)$ in the neighborhood of x^j is, at least to a first approximation, linear—i.e., differentiable. Since differentiability of aggregate preferences will be the rule when preferences are convex, this explains why most large economies will exhibit NS.

IV. The No-Surplus Condition and Its Dual in $\mathscr{E}_\infty$

In this section the characterization of perfectly competitive equilibrium established for finite exchange economies is extended to a simple sequence of economies in which the number of agents is indefinitely increasing. The limit of this process of replication will describe an economy $\mathscr{E}_\infty$. Although not all will be given here, each result in Sections II and III for the economy $\mathscr{E}$ has an extension to $\mathscr{E}_\infty$. This will be emphasized by the notation Theorem 1_∞ or Proposition 2_∞, below, to denote the analogs of Theorem 1 and Proposition 2 for $\mathscr{E}$. Whereas for $\mathscr{E}$ an NS allocation is a WE but not conversely, for $\mathscr{E}_\infty$ the concepts of WE and NS are practically equivalent.

By dividing up the economy $\mathscr{E} = \{(S_i, w_i)\}_{i=1}^{i=n}$, another economy may be constructed with a larger number of agents each of whom operates on a smaller scale. Let $\mathscr{E}_k = \{(S_{i_h}, w_{i_h})\}$, $i = 1,\dots, n$ and $h = 1,\dots, k$, be such that for each i and h,

$$k w_{i_h} = w_i .$$

For an allocation $X = (x_i)$ for $\mathscr{E}$, define $[X]^k = (x_{i_h})$, $i = 1,\dots, n$ and $h = 1,\dots, k$ such that for each i and h,

$$k x_{i_h} = x_i .$$

Since X is a feasible allocation for $\mathscr{E}$ ($\sum x_i = \sum w_i$), $[X]^k$ is a feasible allocation for $\mathscr{E}_k$—i.e.,

$$\sum_i \sum_h x_{i_h} = k \sum_i x_{i_h} = \sum x_i = \sum w_i = k \sum_i w_{i_h} = \sum_i \sum_h w_{i_h} .$$

Attention will be confined to allocations for $\mathscr{E}_k$ that can be written as $[X]^k$—all agents of the same type i receive the same allocation. Further, the sequences of allocations considered for the sequence of economies $\{\mathscr{E}_k\}$, $k = 1, 2,\dots$, will be restricted to $\{[X]^k\}$—variations in the allocation as k varies are based entirely on scalar multiplication of the fixed allocation X for $\mathscr{E}$.

To complete the description of $\mathscr{E}_k$, define $S_{i_h}([X]^k)$ such that for each i and h,

$$k S_{i_h}([X]^k) = S_i(X).$$

With A_5, $\{\mathscr{E}_k\}$ is a sequence of Edgeworth replica economies that approaches a special case of a continuum of traders economy (Aumann [1]). Without A_5, the above assumption on preferences preserves the replication hypothesis by requiring that the consequences of external effects vary in proportion to the size of the agents. Twice as many individuals each consuming half as much creates no change in aggregate external effects.

For $k = 1, 2, \ldots$, let $\{j(k)\}$ be a sequence that selects one of the agents of one of the types $j = 1, \ldots, n$ from each $\mathscr{E}_k$. With the above restrictions, $j(k)$ could be set equal to j_k, the last agent of type j in $\mathscr{E}_k$. The total resources of all agents in $\mathscr{E}_k$ except $j(k)$ is

$$w^{j(k)} = \sum_{i \neq j} \sum_{h} w_{i_h} + \sum_{h \neq k} w_{j_h} = k \sum_{i \neq j} w_{i_h} + (k - 1)\, w_{j_h} = w^j + \frac{k-1}{k}\, w_j .$$

Analogously, define

$$\begin{aligned} S^{j(k)}([X]^k) &= \sum_{i \neq j} \sum_{h} S_{i_h}([X]^k) + \sum_{h \neq k} S_{j_h}([X]^k) \\ &= k \sum_{i \neq j} S_{i_h}([X]^k) + (k - 1)\, S_{j_h}([X]^k) = S^j(X) + \frac{k-1}{k}\, S_j(X). \end{aligned}$$

The last equality allows us to write $S^{j(k)}([X]^k)$ in the more compressed form $S^{j(k)}(X)$ without any ambiguity.

To extend the NS concept to $\mathscr{E}_\infty$, define

$$d(\partial S^{j(k)}(X), w^{j(k)}) = \inf \| \partial S^{j(k)}(X) - w^{j(k)} \|.$$

The distance is nonzero if and only if either (1) $w^{j(k)} \notin \operatorname{cl} S^{j(k)}(X)$, in which case $j(k)$ is contributing a positive surplus to $[X]^k$, or (2) $w^{j(k)} \in S^{j(k)}(X)$ and $j(k)$ is contributing a negative surplus. We shall say that $\{[X]^k\}$ satisfies the NS condition for $\mathscr{E}_\infty$ (NS_∞) if for any $\{j(k)\}$, $k = 1, 2, \ldots$,

$$\mathrm{NS}_\infty\colon \qquad \lim k\, d(\partial S^{j(k)}(X), w^{j(k)}) = 0.$$

NS_∞ requires not only that the surplus contributed by any one agent, positive or negative, go to zero as k increases, but it must go to zero sufficiently rapidly so that the surplus contributed by any agent to all other agents goes to zero faster than $(k)^{-1}$, a measure of the size of $j(k)$.

For the allocation $[X]^k$ for $\mathscr{E}_k$, define

$$Q^{j(k)}([X]^k) = \{p \in P\colon \inf p[S^{j(k)}(X) - w^{j(k)}] \geqslant 0\}.$$

There is no ambiguity in writing $Q^{j(k)}([X]^k)$ as $Q^{j(k)}(X)$. For a sequence $\{j(k)\}$ in which $j(k) = j$, let

$$Q^{j(\infty)}(X) = \bigcap_{k=1}^{k=\infty} Q^{j(k)}(X).$$

For $\mathscr{E}$, Corollaries 1.1 and 1.2 showed that $Q^j(X) = Q^0(X)$, $j = 1,\ldots, n$, is sufficient and practically necessary for X to be NS. In the following extension of Theorem 1 describing NS_∞ in terms of its dual, it is shown that for $\mathscr{E}_\infty$ there is complete equivalence.

THEOREM 1_α. *Let $\mathscr{E}$ satisfy* A_1–A_4. *Then* $\{[X]^k\}$ *is* NS_∞ *for* $\{\mathscr{E}_k\}$ *if and only if* $Q^{j(\infty)}(X) = Q^0(X)$, $j = 1,\ldots, n$.

The Theorem is proved with the aid of the following.

LEMMA 2. *Let $\mathscr{E}$ satisfy* A_1, A_2, *and* A_4 *and let* $\{[X]^k\}$ *be a sequence of allocations for* $\{\mathscr{E}_k\}$ *such that* $Q^{j(k)}(X) \neq \varnothing$. *Then if* $j(k) = j$, $k = 1, 2,\ldots,$

(i) $\{Q^{j(k)}(X)\}$ *is a decreasing sequence of closed subsets of* P.

(ii) $Q^{j(\infty)}(X) \subset Q^0(X)$.

Proof. To show that $Q^{j(k)}(X)$ is closed, let $p^r \in Q^{j(k)}(X)$ and $\lim p^r = p$. Thus, $\inf p^r[S^{j(k)}(X) - w^{j(k)}] \geqslant 0$, or $p \in Q^{j(k)}(X)$.

To establish that $\{Q^{j(k)}(X)\}$ is decreasing, it suffices to show that for any $k > 1$ and $j(k) = j(k-1)$, if $p \in Q^{j(k)}(X)$ then $p \in Q^{j(k-1)}(X)$. Letting $p \in Q^{j(k)}(X)$,

$$\begin{aligned}
&\inf p[S^{j(k)}(X) - w^{j(k)}] \\
&\quad = \inf p\left[S^j(X) + \frac{k-1}{k} S_j(X) - \left(w^j + \frac{k-1}{k} w_j\right)\right] \\
&\quad = \inf p[S^j(X) - w^j] + \frac{k-1}{k} \inf p[S_j(X) - w_j] \geqslant 0. \qquad (1)
\end{aligned}$$

By A_2, $\inf p[S^0(X) - w^0] \leqslant 0$ and since

$$\inf p[S^0(X) - w^0] = \inf p[S^{j(k)}(X) - w^{j(k)}] + (k)^{-1} \inf p[S_j(X) - w_j],$$

we must have

$$\inf p[S_j(X) - w_j] \leqslant 0. \qquad (2)$$

When $0 \geqslant \alpha = \inf p[S_j(X) - w_j]$, $((k-2)/(k-1))\alpha \geqslant ((k-1)/k)\alpha$. Thus, (1) and (2) yield

$$\inf p[S^j(X) - w^j] + \frac{k-2}{k-1} \inf p[S_j(x) - w_j] \geqslant 0. \qquad (3)$$

But (3) means that $p \in Q^{j(k-1)}(X)$, which demonstrates (i).

To establish (ii), first note that since $Q^{j(\infty)}(X)$ is the intersection of a decreasing sequence of nonempty compact sets, it is nonempty. If $p \in Q^{j(\infty)}(X)$, then the inequality in (1) is satisfied for all k which obviously implies that $\inf p[S^j(X) - w^j] + \inf p[S_j(X) - w_j] = \inf p[S^0(X) - w^0] \geqslant 0$, or $p \in Q^0(X)$.

Proof of Theorem 1_∞. If $\bigcap_k Q^{j(k)}(X) = Q^0(X)$, then since part (i) of Lemma 2 shows that $\{Q^{j(k)}(X)\}$ is decreasing we must have $Q^0(X) \subset Q^j(X)$, $j = 1,\ldots, n$. Therefore, Proposition 1 allows us to conclude that for any $p \in Q^0(X)$, $\inf p[S_j(X) - w_j] = p(x_j - w_j) = 0$, $j = 1,\ldots, n$.

Let $\| z^{j(k)} \| = \inf \| \mathrm{cl}\, S^{j(k)}(X) - w^{j(k)} \|$. If for all $j(k)$, $\| z^{j(k)} \| = 0$, there is nothing more to prove—i.e., $\| z^{j(k)} \| = 0$, $Q^{j(k)}(X) \neq \varnothing$ and A_3 imply that $w^{j(k)} \in \partial S^{j(k)}(X)$. Therefore, assume $\| z^{j(k)} \| \neq 0$. Letting $p^{j(k)} = (\| z^{j(k)} \|)^{-1} z^{j(k)}$, we obtain as in the proof of Theorem 1 that $p^{j(k)} \in Q^{j(k)}(X)$. But

$$\begin{aligned} 0 < k \| z^{j(k)} \| = k p^{j(k)} z^{j(k)} &\leqslant k \inf p[S^{j(k)}(X) - w^{j(k)}] \\ &< k p^{j(k)}(x^{j(k)} - w^{j(k)}) = p^{j(k)}(w_j - x_j). \end{aligned}$$

The last equality follows from the identity

$$\begin{aligned} k(x^{j(k)} - w^{j(k)}) &= k\left[x^j + \frac{k-1}{k} x_j - \left(w^j + \frac{k-1}{k} w_j\right)\right] \\ &= k(x^j - w^j) + (k-1)(x_j - w_j) = (w_j - x_j). \qquad (1) \end{aligned}$$

Since $p \in Q^0(X)$ implies $p(w_j - x_j) = 0$ and by hypothesis $\inf \| Q^0(X) - p^{j(k)} \| \to 0$, we have the desired conclusion that $k \| z^{j(k)} \| \to 0$.

For the converse, we first show that NS_∞ implies $w^0 \notin S^0(X)$ and therefore $Q^0(X) \neq \varnothing$. Suppose the contrary that $w^0 \in S^0(X)$. Then since $w^{j(k)} \to w^0$, by A_1 we have for sufficiently large k that $w^{j(k)} \in S^0(X)$. Now, $S^0(X) = S^{j(k)}(X) + S_{j(k)}(X)$ and for k large, $S_{j(k)}(X)$ intersects any ϵ-neighborhood of the origin. Therefore, for all sufficiently large k, $w^{j(k)} \in S^{j(k)}(X)$, which contradicts the hypothesis that $\{[X^k]\}$ is NS_∞.

Let $p \in Q^0(X)$. Then A_2 implies

$$0 = \inf p[S^0(X) - w^0] = \inf p[S^{j(k)}(X) - w^{j(k)}] + (k)^{-1} \inf p[S_j(X) - w_j].$$

Multiplying by k, we obtain

$$0 = k \inf p[S^{j(k)}(X) - w^{j(k)}] + \inf p[S_j(X) - w_j].$$

Since $\{[X]^k\}$ is NS_∞,

$$\lim\sup\{k \inf p[S^{j(k)}(X) - w^{j(k)}]\} \leqslant 0.$$

Thus, $\inf p[S_j(X) - w_j] \geqslant 0$, $j = 1,\ldots, n$, which implies $\inf p[S_j(X) - w_j] = 0$, all j. Therefore,

$$\inf p[S^j(X) - w^j] + \frac{k-1}{k} \inf p[S_j(X) - w_j] \geqslant 0,$$

which means that for all k, $p \in Q^{j(k)}(X)$. This establishes $Q^0(X) \subseteq Q^{j(\infty)}(X)$. Part (ii) of Lemma 2 gives the opposite inclusion.

To exhibit the connections with WE, simply extend Proposition 1. Clearly, if (X, p) is a WE for $\mathscr{E}$, $([X]^k, p)$ is a WE for $\mathscr{E}_k$, $k = 1, 2, \ldots$; and, with A_4, the converse holds.

Proposition 1_∞. *Let* $\mathscr{E}$ *satisfy* A_1–A_4. *Then* (X, p) *is a* WE *for* $\mathscr{E}$ *if and only if* $p \in \bigcap_j Q^{j(\infty)}(X)$.

The following extension of Proposition 2 is contained in the proof of Theorem 1_∞.

Proposition 2_∞. *Let* $\mathscr{E}$ *satisfy* A_1–A_4. *If* $\{[X]^k\}$ *is* NS_∞, *there is a* $p \in P$ *such that* $([X]^k, p)$ *is a* WE *for* $\mathscr{E}_k$.

The relation between WE and NS_∞ is summarized by

Corollary $1_\infty \cdot 1$. *Let* $\mathscr{E}$ *satisfy* A_1–A_4 *and let* X *be an allocation for* $\mathscr{E}$ *such that* $Q^0(X) = \{p\}$. *Then* (X, p) *is a* WE *if and only if* $\{[X]^k\}$ *is* NS_∞.

Proof. If (X, p) is a WE for $\mathscr{E}$, then $Q^{j(k)}(X) \neq \varnothing$ and therefore by Lemma 2, $Q^{j(\infty)}(X) \subset Q^0(X)$. Since $Q^0(X) = \{p\}$, $Q^{j(\infty)}(X) = Q^0(X)$ and by Theorem 1_∞, $\{[X]^k\}$ is NS_∞.

The converse implication is Proposition 2_∞.

As is the case for $\mathscr{E}$ (see Corollary 1.2), the hypothesis in Corollary $1_\infty \cdot 1$ can, with a mild qualification, be derived as a conclusion.

Corollary $1_\infty \cdot 2$. *Let* $\mathscr{E}$ *satisfy* A_1–A_4 *and assume that for any* X *for which* $Q^{j(k)}(X) \neq \varnothing$, $\{(x_i - w_i)\}$ *does not lie in a subspace of dimension less than* $(l - 1)$. *Then* $\{[X]^k\}$ *is* NS_∞ *if and only if* $Q^0(X) = \{p\}$.

Proof. If $Q^0(X)$ is not a singleton, the dimensionality assumption on $\{(x_i - w_i)\}$ implies that there is at least one $p \in Q^0(X)$ and one j such that $p(w_j - x_j) < 0$. The identity (1) in the proof of Theorem 1_∞ means that $kp[x^{j(k)} - w^{j(k)}] < 0$. By A_2, $k \inf p[S^{j(k)}(X) - w^{j(k)}] \leqslant kp(x^{j(k)} - w^{j(L)})$. Thus, there is a $p \in Q^0(X)$ that does not belong to any $Q^{j(k)}(X)$. But this contradicts the condition $Q^0(X) = Q^{j(\infty)}(X)$ that is established in Theorem 1_∞ as necessary for NS_∞.

The converse implication follows from Lemma 2, part (ii), and Theorem 1_∞.

Remark 4. The principal mathematical distinction between $\mathscr{E}$ and $\mathscr{E}_\infty$ is the conclusion in part (ii) of Lemma 2. For $\mathscr{E}$, the analog of (ii)—$Q^j(X) \subset Q^0(X)$—does not hold even for a WE. This discrepancy accounts for the differences between Theorems 1 and 1_∞ and Corollaries $1 \cdot 1$ and $1_\infty \cdot 1$.

To complete the extension from $\mathscr{E}$ to $\mathscr{E}_\infty$, analogs of Theorems 2 and 3 should be given demonstrating that NS_∞ is equivalent, respectively, to product-exhaustion and to perfectly determinate prices for $\mathscr{E}_\infty$. Results similar to what would be called here Theorem 2_σ are found in Mas-Colell [12] and Geanakoplos [7].

Because commodities are personalized in the statement of Theorem 3, a literal extension would require an infinite-dimensional commodity space. (This is the subject of Ostroy [15].) Here, however, such a construction is unnecessary. We need only consider a space of dimension $2l$ so as to distinguish the goods supplied by any one $j(k)$ and those supplied by others. Rather than proceed with the formal details, we shall outline an independent proof that suggests why NS$_\infty$ would imply the perfect determinacy of prices. It also shows, by invoking the continuity assumption on preferences, A$_6$, that we shall be able to obtain a perfectly competitive equilibrium even in the presence of external effects. (Compare Remark 2.)

If X is an allocation for $\mathscr{E}$, let $Y^k = (y^k_{i_h})$ be an allocation for $\mathscr{E}_k$ defined by

$$\begin{aligned} y^k_{i_h} &= w_{i_h}, && \text{if} \quad i_h = j_k, \\ &= x_{i_h}, && \text{if} \quad i = j \quad \text{and} \quad i_h \neq j_k, \\ &= x_{i_h} - (k)^{-1}(x_{i_h} - w_{i_h}), && \text{if} \quad i \neq j. \end{aligned}$$

Y^k describes a reallocation away from $[X]^k$ in which everyone refuses to trade with j_k. Those agents of the same type as $j(k)$ obtain the same outcome as in $[X]^k$ while the members of the other types cut back their excess demands by $(k)^{-1}$. Note that $\sum_i \sum_h y_{i_h} = w^0$, so Y^k is in fact a feasible allocation for $\mathscr{E}_k$.

The total disruption caused by refusing to trade with j_k may be measured by

$$\| Y^k - [X]^k \| = \sum_i \sum_h \| y^k_{i_h} - x_{i_h} \|.$$

Substituting the definition of Y^k, this equals

$$\sum_{i \neq j} \sum_h \|(x_{i_h} - (k)^{-1}(x_{i_h} - w_{i_h}) - x_{i_h})\| + \|(w_{j_k} - x_{j_k})\|$$

$$\sum_{i \neq j} \|(w_{i_h} - x_{i_h})\| + \|(w_{j_k} - x_{j_k})\| = (k)^{-1} \sum_i \|(w_i - x_i)\|.$$

Therefore, $\lim \| Y^k - [X]^k \| = 0$.

If (X, p) is a WE for $\mathscr{E}$ and $Q^0(X) = \{p\}$, then even if there are external effects, $[X]^k$ is a stable, noncooperative equilibrium for $\mathscr{E}_k$ when k is large because if any agent were to raise the prices of its commodities, the others could, with impunity, go elsewhere.

More precisely, when $[X]^k$ is a WE allocation for $\mathscr{E}_k$, $w^{j(k)} \notin S^{j(k)}(X)$. By invoking A$_6$ and $\lim \| Y^k - [X]^k \| = 0$, we have for all k sufficiently large, $w^{j(k)} \notin S^{j(k)}(Y^k)$, and therefore $Q^{j(k)}(Y^k) \neq \varnothing$. Lemma 2 may be extended, using A_6 and $\lim \| Y^k - [X]^k \|$, to show that for some k', $\bigcap_{k>k'} Q^{j(k)}(Y^k) \subset Q^0(X)$. Similarly, if $Q^0(X) = \{p\}$ then $\bigcap_{k>k'} Q^{j(k)}(Y_k) = Q^0(X)$ for every $\{j(k)\}$ and the arguments used in the proof of Theorem 1_α may be generalized to yield

$$\lim\{k \inf \| \partial S^{j(k)}(Y^k) - w^{j(k)} \|\} = 0.$$

We conclude this section with an example illustrating that the condition $Q^{j(\alpha)}(X) = Q^0(X)$ in Theorem 1_∞ is necessary for ANS by showing that without it there is nonvanishing monopoly power to small-scale agents. To avoid complications, A_5 is assumed.

EXAMPLE 2. Consider the economy of equal numbers of type 1 and type 2 traders illustrated by the Edgeworth–Bowley box of Fig. 1.

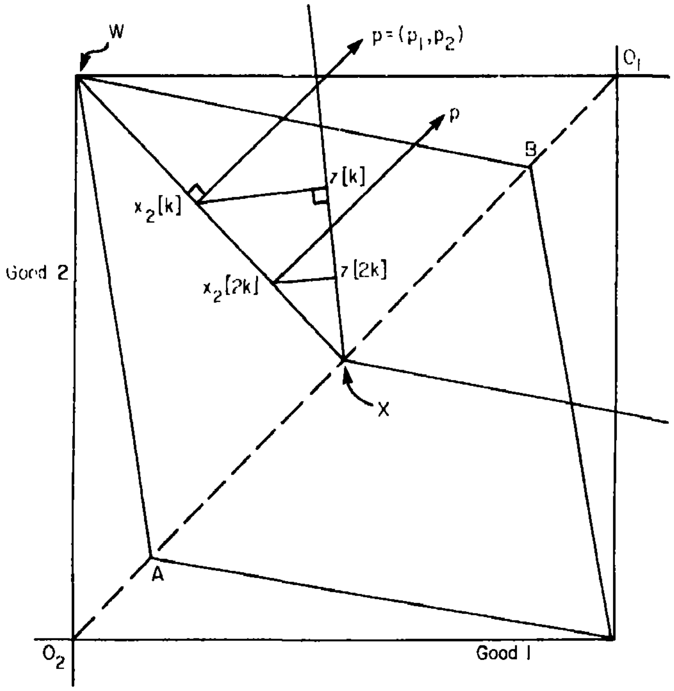

FIGURE 1

Let $\mathscr{E}_k^{j(k)}$ denote the economy consisting of all agents in $\mathscr{E}_k$ except $j(k)$, $j = 1, 2$. No matter what the value of k, the WE allocations for $\mathscr{E}_k$ include all points along the line AB, while the WE allocations for $\mathscr{E}_k^{1(k)}$ or $\mathscr{E}_k^{2(k)}$ include only the points A or B, respectively.

The example is of some historical interest because of its similarity to Edgeworth's master–servant example in which each master has need of only one servant and one servant cannot serve two masters. If the minimum wage at which servants would offer themselves is α and the maximum wage at which masters would accept is β, Edgeworth noted that as long as there is an equal number of masters and servants, then no matter how many there are, the equilibrium wage is indeterminate, lying somewhere between α and β ($\alpha < \beta$). If there is one more master than servant (or one more servant than master) indeterminacy disappears and the wage rate becomes β (or α). Edgeworth attributed the indeterminacy to the indivisibility of the good "domestic service" and on the basis of his example modified his proposition that large numbers of traders in a market lead to a determinate outcome only when goods are divisible. The self-evident similarity of the master–servant example to Example 2, in which goods are divisible, suggests that some other explanation is called for. What is common to Edgeworth's example and Example 2 is the failure of NS_∞.

To see this, consider the WE allocation $X = (x_1, x_2)$ in Fig. 1 replicated k times for $\mathscr{E}_k$ and consider how close the members of $\mathscr{E}_k^{1(k)}$ can come to doing as well. From the construction of Y^k, above, all of the type 1 traders can share x_1 and all of the type 2's may share $x_2[k] = x_2 - (1/k)(x_2 - w_2)$. It would then require $z[k] = (z_1[k], z_2[k])$ additional resources to minimize, with respect to Euclidean distance, the amounts of additional resources necessary for all the type 2 agents to be as satisfied as they are in $[X]^k$. If we double k, we may decrease the additional resources to make *each* type 2 agent as satisfied, but we cannot decrease the amounts by more than one-half, i.e., $2\|z[2k]\| = \|z[k]\|$. Thus, the influence of any type 1 agent, measured in this way, does not decrease as k increases, and $\{[X]^k\}$ does not satisfy NS_∞.

To exhibit the consequences for the elasticity of demand, assume that other traders are transacting at the WE prices p. Then, the maximum exchange rate a seller of type 1 can set, without losing all his business, does not go to p as k increases. After the type 2 traders have purchased all that the $(k - 1)$ type 1 traders are willing to supply at p, the remaining type 1 trader can offer any exchange rate ρ such that $p_1/p_2 < \rho < z_1[k]/z_2[k]$ and sell, when k is large, as much of commodity 1 as it likes.

If the preferences of the type 2 traders were differentiable at the WE allocation X, and therefore there were a unique supporting hyperplane to $S^0(X) = S_1(X) + S_2(X)$ at w^0, $\{[X^k]\}$ would, by Corollary $1_\infty \cdot 1$, satisfy NS_∞. In this case (not illustrated), the angle between p and $z[k]$ would tend to zero as k increased and the sequence $\{[X^k]\}$ would be approaching one in which WE prices p became PD.

Remark 5. Let $\delta(A, B)$ denote the Hausdorff distance between two nonempty closed sets in R^l. The condition (a) $\bigcap_{k=1}^{k=\infty} Q^{j(k)}(X) = Q^0(X)$ used to characterize NS_∞ in Theorem 1_∞ is equivalent to (b) $\lim \delta(Q^{j(k)}(X), Q^0(X)) = 0$. Condition (b) is shown in [15] to replace (a) in economies that do not have a finite number of types of agents.

Let $W(\mathscr{E}_k)$ and $W(\mathscr{E}_k^{j(k)})$ be subsets of P denoting the (closed) sets of price vectors corresponding to WE allocations in $\mathscr{E}_k$ and $\mathscr{E}_k^{j(k)}$. Since $\lim \delta(W(\mathscr{E}_k^{j(k)}), W(\mathscr{E}_k))$ does not exist in Example 2, it might be conjectured that this is the source of nonvanishing monopoly power of individual agents and that (c) $\lim \delta(W(\mathscr{E}_k^{j(k)}), W(\mathscr{E}_k)) = 0$ is an alternative to $\lim \delta(Q^{j(k)}(X), Q^0(X)) = 0$ as a characterization of NS_∞. This conjecture is false in both directions—(b) neither implies nor is implied by (c).

To show that (c) does not suffice for (b), simply perturb the initial endowments in Example 2 so that $W(\mathscr{E}_k) = \{p\}$ for all k but the WE allocation remains at the point where preferences are kinked. It may then be demonstrated that (c) holds. However, as long as preferences are not differentiable at the WE allocation, the same argument exhibiting the failure of NS_∞ in Example 2 leads to the same conclusion here.

To show that (c) is not necessary, consider an Edgeworth–Bowley box example where preferences are differentiable in the interior and there are a continuum of WE prices and allocations for each $\mathscr{E}_k$. Examples may be constructed to show that $W(\mathscr{E}_k^{j(k)})$ does not approach $W(\mathscr{E}_k)$—i.e., (c) fails. (For a graphical illustration see Bewley [2], Example 2].) Nevertheless, the smoothness of preferences implies that as k increases the total surplus attributable to any agent at any Walrasian allocation goes to zero. Even though the removal of any one would cause an abrupt change in WE prices, no agent can usefully exploit this. The failure of markets to clear when an agent threatens to withdraw his supplies if he does not receive a higher price may be essentially confined to the disappointed demands of the very trader who is asking for more.

V. NO-SURPLUS AND CORE EQUIVALENCE AS ALTERNATIVE CHARACTERIZATIONS OF PERFECTLY COMPETITIVE EQUILIBRIUM

The presence of external effects vitiates any comparisons between the core, which is limited to allocations that are truly Pareto-optimal, and NS. (See Remark 1.) Thus, in the following discussion A_5 is assumed. It will be useful to separate the comparisons for $\mathscr{E}$ from those for $\mathscr{E}_\infty$.

$\mathscr{E}$: Let T be any nonempty subset of $I = \{1, \ldots, n\}$, the index set of agents. Define $w_T = \sum_{i \in T} w_i$, $S_T(X) = \sum_{i \in T} S_i(X)$ and $Q_T(X) = \{p \in P\colon \inf p[S_T(X) - w_T] \geqslant 0\}$.

An allocation X is said to be in the core of $\mathscr{E}$ if for all T, $w_T \notin S_T(X)$. With A_1–A_5, it may be shown that X is in the core if and only if for all T, $Q_T(X) \neq \varnothing$.

Consider the following three conditions:

(i)	$Q^j(X) = Q^0(X) \neq \varnothing, \quad j = 1, \ldots, n$	(NS)	Corollary 1.2
(ii)	$\bigcap_j Q^j(X) \neq \varnothing$	(WE)	Proposition 1
(iii)	$Q_T(X) \neq \varnothing, \quad \text{all} \quad T \subseteq I$	(CORE)	

Obviously, (i) $\Mapsto$ (ii) and (iii) $\Mapsto$ (iv), where $\Mapsto$ means "implies but is not implied by." Further, it is well known that (ii) $\Mapsto$ (iii).

The core criterion for $\mathscr{E}$ to be perfectly competitive is that *(iii)* $\Rightarrow$ *(ii)*. The NS criterion is (i), assuming the linear independence hypothesis in Corollary 1.2. The relation between the two criteria is summarized by

PROPOSITION 3. *For* $\mathscr{E}$, (i) $\Mapsto$ [(iii) $\Rightarrow$ (ii)].

The implication $\Rightarrow$ follows from the fact that (i) implies all of the other conditions. The following example demonstrates $\Mapsto$.

EXAMPLE 3. Let tastes be defined by the utility functions $u_i(x_i) = u_i(x_{i1}, x_{i2}, x_{i3})$, $i = 1, 2, 3$, and 4, and let initial and final allocations be given by W and X^*, respectively, where

$$u_1(x_1) = (x_{11}x_{12}x_{13})^{1/3},$$

$$u_2(x_2) = (x_{21}x_{22}x_{23})^{1/3},$$

$$u_3(x_3) = (x_{31}x_{32}x_{33})^{1/3},$$

$$u_4(x_4) = \frac{x_{41} + x_{42} + x_{43}}{3};$$

$$W = \begin{array}{c|ccc} i \diagdown c & & & \\ \hline & 3 & 0 & 0 \\ & 0 & 3 & 0 \\ & 0 & 0 & 3 \\ & 1 & 1 & 1 \end{array} ; \quad X^* = \begin{array}{c|ccc} i \diagdown c & & & \\ \hline & 1 & 1 & 1 \\ & 1 & 1 & 1 \\ & 1 & 1 & 1 \\ & 1 & 1 & 1 \end{array}$$

With little difficulty, the reader may verify that X^* is a WE allocation at the price vector $p^* = (r, r, r)$, $r > 0$, that yields the utility vector $u^* = (u_1^*, u_2^*, u_3^*, u_4^*) = (1, 1, 1, 1)$.

An outline of the argument that X^* is the only allocation in the core is as follows: The coalition $\overline{123}$ can achieve any (u_1, u_2, u_3) such that $u_1 + u_2 + u_3 = 3$, and 4 can achieve $u_4 = 1$ on its own. It follows that for any $u = (u_1, u_2, u_3, u_4)$ in the core, $u_1 + u_2 + u_3 = 3$ and $u_4 = 1$, and if $u_i < 1$, $i \neq 4$, then because $w_T \in \text{cl}\, S_T(X^*)$, $T = \overline{i4}$, the coalition $\overline{i4}$ will upset u. This means that $u = (1, 1, 1, 1)$ and this is achievable only by the allocation X^*.

X^* does not satisfy NS—$w_T \notin \text{cl}\, S_T(X^*)$ for $T \in \{\overline{124}, \overline{134}, \overline{234}\}$.

The core does not capture the monopoly power of individuals 1, 2, or 3. For example, without 1, the WE price vector for the economy $\mathscr{E}^1$, the economy without agent 1, is $p = (2r, r, r)$. In terms of prices, 1's contribution or marginal product may be measured by the decrease in the WE price of the commodity he supplies. From a noncooperative point of view, there does not appear to be any reason why 1 should surrender all of the surplus represented by this price decrease to the other traders.

Such a finding has already been reported for economies composed of a (nonatomic) continuum of agents and large or atomic agents. See Gabszewicz and Mertens [6] and Shitovitz [20]. They constructed classes of economies exhibiting Equivalence, i.e., [(iii) ⇒ (ii)], but the presence of the large traders precludes (i). Example 3 shows that such results also occur in purely atomic economies.

$\mathscr{E}_\infty$: Let I_k be the index set of agents in $\mathscr{E}_k$. The analogous conditions are quite similar.

(i) $Q^{j(\infty)}(X) = Q^0(X) \neq \varnothing, \quad j = 1, \ldots, n$ (NS) Theorem 1_∞
(ii) $\bigcap_j Q^{j(\infty)}(X) \neq \varnothing$ (WE) Proposition 1_∞
(iii) $Q_T(X) \neq \varnothing$, all $T \subsetneqq I_k$, $k = 1, 2, \ldots$ (CORE)

As in $\mathscr{E}$, (i) ⇒ (ii), and (ii) ⇒ (iii). Again, the core criterion for $\mathscr{E}_\infty$ to be perfectly competitive is [(iii) ⇒ (ii)] and NS_∞ is (i). Example 2 demonstrates that [(iii) ⇒ (ii)] ⇏ (i) and therefore we have the analog of Proposition 3 for $\mathscr{E}_\infty$.

The result [(iii) ⇒ (ii)], the core convergence theorem, was initiated by Edgeworth [5], revived by Shubik [21], and put into its current form by Debreu and Scarf [4]. (Hildenbrand [9] gives the considerable extensions of this result.) To establish the NS_∞ property requires a different method of proof than [(iii) ⇒ (ii)] and it does not hold in quite the same generality. However, *when* $Q^0(X) = \{p\}$, *the core convergence theorem is an immediate implication of* Lemma 2 in Section IV. In this respect, our approach is closely related to the simplication of the Debreu–Scarf result obtained by Hansen [8].

Remark 6. In the demonstration of (i), allocations were restricted to those satisfying equal treatment—allocations of the form $[X]^k$. With the core approach, $[X]^k$ may be derived as a conclusion. This can also be demonstrated via the NNS condition of Section II. In fact, the NNS condition may be used to exhibit the analog of (i) for sequences of economies in which agents are not drawn from a fixed, finite set of types. This result, in Ostroy [14], parallels a core convergence theorem of Nishino [13].

If we adopt a generic point of view, distinctions between the core criterion and NS vanish. For $\mathscr{E}_\infty$, they coincide when $Q^0(X) = \{p\}$ and for $\mathscr{E}$, even though endless cases such as Example 3 may be constructed, the fact remains that within the entire class of finite economies each criterion says that the set of perfectly competitive economies is negligible.

It is only at the conceptual level that differences emerge. With the core criterion, perfectly competitive equilibrium is obtained as the residual outcome after all groups of agents cooperate in the interests of improving upon any given allocation but no group is able to hold together to use its potential monopoly power to extract a more favorable outcome. With NS, just the opposite occurs. Only small groups (individuals) are able to form but they are relied upon to bargain as monopolists for the maximum they can possibly extract. It is remarkable that these two seemingly contradictory approaches should "almost always" yield the same conclusion. It appears that the economic forces permitted and precluded by the one are almost always counterbalanced by what is precluded and permitted by the other. However, a closer look at the discrepancies, such as Example 3 for $\mathscr{E}$, Example 2 for $\mathscr{E}_\alpha$ (due to Edgeworth!), or the results of Shitovitz [20] and Gabszewicz and Mertens [6] for so-called "mixed" economies show that what is permitted and precluded by the one is not always counterbalanced by the other. Which interpretation—the core or NS criterion—is the preferred description of what will frequently be equivalent mathematical conditions characterizing perfectly competitive equilibrium should be judged on its eventual connections to the theory of imperfect competition.

ACKNOWLEDGMENT

Much of this material was formulated in discussions with Louis Makowski. I have also benefited from many comments by Bryan Ellickson and Juan Urrutia. Felix Grafe pointed out several inaccuracies in an earlier version.

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JOURNAL OF ECONOMIC THEORY 22, 208-221 (1980)

A Characterization of Perfectly Competitive Economies with Production*

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Received January 10, 1979

This paper answers the question, what do you need for perfect competition with production? The answer given is that you need to realize an allocation in which no individual contributes a surplus to anyone else. This essentially amounts to giving each individual his "marginal product." In the model of perfect competition, profits are competitively determined rents: they reflect the marginal product of entrepreneurs. And the competitive theory of value derives from the competitive theory of distribution.

It is well known that a Walrasian equilibrium will generally not be a perfectly competitive equilibrium: heuristically, an equilibrium in which no single economic agent can influence the level of equilibrium prices. When will a Walrasian equilibrium be perfectly competitive?

For exchange economics, Ostroy in [4] has formulated the *no surplus* condition as a test for verifying whether a Walrasian equilibrium is perfectly competitive. It has been observed by Ostroy in [5] that the no surplus theory is a generalized marginal productivity theory of distribution.

In this paper, some of Ostroy's results are extended to general economies with production. The extension occurs in an Arrow-Debreu type production model with one important difference: production possibilities belong to individuals ("entrepreneurs") rather than to firms.

The main results of the paper may now be summarized.

(1) Perfectly competitive allocations are no surplus allocations (Theorems 1 and 2 below).

(2) Perfectly competitive entrepreneurs fully appropriate their contribution to aggregate consumers' surplus in the form of profits. Alternatively expressed, profits measure surplus contributed by entrepreneurs (Proposition 1 below). Thus,

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* This paper has grown out of conversations with Joseph Ostroy. He is virtually its co-author. Of course, any errors are solely my responsibility. Research for the paper was partly supported by NSF Grant SOC 74-17982, and partly by a grant from the (U.K.) Social Science Research Council. This support is gratefully acknowledged.

0022-0531/80/020208-14$02.00/0

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(3) Under perfect competition profits are competitively determined rents to entrepreneurial ability (Proposition 2 below).

(4) In the "long-run" when entrepreneurial abilities are not scarce and hence essentially redundant, there is product exhaustion with zero profits whether or not there are constant returns to scale (Theorem 3 and, in particular, Lemma 4).

(5) In a special case of the model, so called "simple production economies,"

(a) No surplus allocations are equivalent to allocations that give each agent his marginal product. Alternatively expressed, no surplus is equivalent to product exhaustion or "adding up" (Lemmas 6, 7). Thus, in view of (1),

(b) There is perfect competition if and only if there is adding up.

(c) The fact that profits are rents, (3), is reflected in the simple formula that for each individual his profits equal the marginal productivity of his entrepreneurial abilities (Theorem 4). Thus,

(d) In the long-run there is product exhaustion or adding-up with zero profits since the marginal productivity of entrepreneurs equals zero (Lemma 8).

The results under (5) perhaps deserve some elaboration. They effectively say that the no surplus condition yields a modern solution to the classical adding-up problem. In particular, it is remarkable that adding up is not only implied by perfect competition (as neoclassical economists knew), but also implies perfect competition.

1. COMPETITIVE ALLOCATIONS AS NO SURPLUS ALLOCATIONS

1A. *The Basic Model*[1]

There is a finite set of individuals I. The economy is described by a triple $(X_i, Y_i, A_i)_{i \in I}$ consisting of a trading possibility set $X_i \subset R^n$ for each individual $i \in I$, a production possibility set $Y_i \subset R^n$ for each individual, and an "at least as good as" correspondence A_i from X_i to X_i for each individual. A_i completely preorders X_i; i.e., for any x_i, y_i, z_i in X_i $x_i \in A_i(x_i)$ (reflexivity), $x_i \in A_i(y_i)$ and $y_i \in A_i(z_i)$ implies $x_i \in A_i(z_i)$ (transitivity), and $x_i \in A_i(y_i)$ or $y_i \in A_i(x_i)$ (completeness).[2]

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[1] Given sets A and B in R^n and points x and p in R^n, $px \leq pA$ means $px \leq py$ for all $y \in A$; conv A means the convex hull of A; ri A means the relative interior of A; and $A-B$ means the set of all points $y = y^1 - y^2$, where $y^1 \in A$ and $y^2 \in B$.

[2] Both the completeness and transitivity assumptions can be substantially weakened, at some cost in terms of complicating the exposition.

A pair $(x, y) \in \times_i X_i \times_i Y_i$ is a *feasible allocation* if $\sum_i x_i = \sum_i y_i$. Similarly, a pair

$$(x^i, y^i) \in \mathop{\times}_{\substack{h\in I\\ h\neq i}} X_h \mathop{\times}_{\substack{h\in I\\ h\neq i}} Y_h$$

is a *feasible allocation for the economy without individual i* if $\sum_{h\neq i} x_h^i = \sum_{h\neq i} y_h^i$. Henceforth, (x, y) and (x^i, y^i) will represent such feasible allocations respectively.

1B. *Competitive Equilibrium*

It will be convenient to define a "better than" correspondence B_i from X_i to X_i for each individual i: $B_i(x_i) = \{y_i \in X_i: y_i \in A_i(x_i)$ but $x_i \notin A_i(y_i)\}$.

Letting $p \in R^n$, (x, y, p) is a *Walrasian equilibrium* (written $(x, y, p) \in W$) if for each i, $px_i = py_i = \sup pY_i < pB_i(x_i)$. Similarly, (x^i, y^i, p) is a *Walrasian equilibrium for the economy without individual i* (written $(x^i, y^i, p) \in W^i$) if for each $h \neq i$, $px_h^i = py_h^i = \sup pY_h < pB_h(x_h^i)$.

Suppose $(x, y, p) \in W$. Individual *i has no monopoly power* in the equilibrium (written $(x, y, p) \in C^i$) if there exists an $(x^i, y^i, p) \in W^i$. That is, i has no monopoly power in (x, y, p) if the economy can, without i's supplies and demands, still achieve a Walrasian equilibrium *at the same prices p*. If i has no monopoly power in (x, y, p) then i cannot influence the level of Walrasian equilibrium prices, at least according to a basic test: If when all other individuals charge the prices in p, i tries to get higher prices than those in p for all the commodities he sells in the Walrasian equilibrium and tries to get lower prices than those in p for all the commodities he buys in the equilibrium he will be unsuccessful. Any individual h, $h \neq i$, will clearly prefer to buy (sell) all the commodities he wants to buy (sell) at the lower (higher) prices in p rather than at the higher (lower) prices offered by i. But if i has no monopoly power in (x, y, p) then all $h \neq i$ can buy (sell) all they want at p without i.

Aggregating over individuals, (x, y, p) is a *Competitive Equilibrium* (written $(x, y, p) \in C$) means $(x, y, p) \in W$ and for each i, i has no monopoly power in (x, y, p). Thus, if $(x, y, p) \in C$, then no one individual can influence the level of Competitive Equilibrium prices by withdrawing his supplies and demands from the market. Alternatively expressed, in a Competitive Equilibrium no one individual can influence the level of equilibrium prices by trying to sell all his goods at prices higher than p and buy all his goods at prices lower than p. Thus a Competitive Equilibrium satisfies a basic test for the existence of perfect competition in the heuristic sense.

Remark 1. It should be noted that there are other tests—perhaps second-order tests—for the existence of perfect competition in the heuristic sense that a Competitive Equilibrium may not satisfy. For example, if an individual tries to sell some—but not all—of his commodities for more than their equilibrium prices, he may be able to influence the level of prices, even in a Competitive Equilibrium. This objection will be overcome somewhat in the next section. Also, an individual may be able to influence the level of equilibrium prices by "cornering a market"—buying up all supplies of a commodity and then reselling the commodity as a monopolist.[3] In spite of these objections, I would argue that the relatively simple definition of a Competitive Equilibrium given above reflects a lot of what we intuitively mean by a competitive equilibrium.

1C. *The No Surplus Condition*

Let $A^i(x) = \{(x^i, y^i)\colon \text{for each } h \neq i,\ x_h{}^i \in A_h(x_h)\}$. Individual *i contributes no surplus in* (x, y) (written $(x, y) \in NS^i$) means $A^i(x) \neq \varnothing$ but for any $(x^i, y^i) \in A^i(x)$ and any $h \neq i$ $x_h{}^i \notin B_h(x_h)$. If $(x, y) \in NS^i$ then the set of all individuals except i can on their own achieve an allocation in which every individual in the set is as well off as with x. Alternatively expressed, if $(x, y) \in NS^i$ then x imputes to i his entire contribution to aggregate consumers' surplus.

Let the aggregate production set be $Y = \sum_i Y_i$; (x, y) is a *no surplus allocation* (written $(x, y) \in NS$) means $0 \notin \sum_i B_i(x_i) - Y$ and for each i, i contributes no surplus in (x, y). The first condition says that x is a Pareto optimum. The second says that every individual is inessential to the welfare of all others relative to the allocation x.

Note that the no surplus condition is purely technological; it does not depend on prices.

Let us assume

(A1) For each i and each $x_i \in X_i$: $x_i \in cl\, B_i(x_i)$ (local nonsatiation).

(A2) For each i and each $x_i \in X_i$: $px_i \leqslant pB_i(x_i), p \neq 0$, implies $px_i < pB_i(x_i)$.

(A3) $0 \notin \sum_i B_i(x_i) - Y$ implies $0 \notin ri\,(\text{conv}(\sum_i B_i(x_i) - Y))$.

Assumptions (A1)–(A2) are standard. The economic literature contains a number of sets of sufficient conditions guaranteeing (A2). Assumption (A3) will be satisfied if $\sum_i B_i(x_i)$ and Y are convex. We use a generalization of the standard convexity assumptions to allow for the possibility of nonconvexities and indivisibilities. We now prove that

THEOREM 1. (characterization of competitive equilibria). *Assume* (A1)–(A3). *Then* (x, y, p) *is a Competitive Equilibrium iff* (x, y) *is a no surplus allocation.*

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[3] I am indebted to Earl Thompson for pointing out to me the importance of "cornering the market" strategies.

Theorem 1 is an extension to economies with production of the basic characterisation result in Ostroy [4]. Since, given (A1), any Walrasian equilibrium is a Pareto optimum, the theorem is a direct consequence of the following two lemmas.

LEMMA 1. *Assume* (A1) *and suppose* $(x, y, p) \in W$. *Then for any individual* i, i *has no monopoly power in* (x, y, p) *iff* i *contributes no surplus in* (x, y).

Proof. If $(x, y, p) \in C^i$ then there exists $(x^i, y^i, p) \in W^i$. It is easy to verify that $(x^i, y^i) \in A^i(x)$. Hence, since (x, y) is in the core, $(x, y) \in NS^i$.

On the other hand, suppose $(x, y) \in NS^i$. We show that $(x^i, y^i, p) \in W^i$, where $(x^i, y^i) \in A^i(x)$. Since for all $h \neq i$, $px_h^i \geq px_h = py_h \geq py_h^i$, if there exists an h such that the above held with some inequalities then $\sum_h x_h^i = \sum_h y_h^i$ would be contradicted. So,

$$px_h^i = py_h^i = \sup pY_h < pB_h(x_h^i).$$

The last inequality follows from the fact that $x_h^i \in A_h(x_h)$ and $x_h \in A_h(x_h^i)$, so $B_h(x_h^i) = B_h(x_h)$. Q.E.D.

LEMMA 2 (existence of values). *Assume* (A1)-(A3). *Then if* (x, y) *is a no surplus allocation, there exists some* $p \neq 0$ *such that* $(x, y, p) \in W$.

Proof. $(x, y) \in NS$ implies $0 \notin \sum_i B_i(x_i) - Y$. So, in view of (A3), there exists $p \neq 0$ such that $p \sum_i B_i(x_i) - pY > 0 = \sum_i px_i - \sum_i py_i$. Or, in view of (A1), for each i

$$px_i \leq pB_i(x_i) \quad \text{and} \quad py_i \geq pY_i.$$

In view of (A2), it suffices to show that $px_i = py_i$. If not, for some i $\sum_{h \neq i} px_h > \sum_{h \neq i} py_h$. Consider $(x^i, y^i) \in A^i(x)$. $\sum_h px_h^i \geqslant \sum_h px_h > \sum_h py_h > \sum_h py_h^i$, contradicting $\sum_h x_h^i = \sum_h y_h^i$. Q.E.D.

2. PROFITS AS RENTS

In this section we want to characterize the nature of the profits in the model. To do so, we must isolate the contribution of an individual's production possibilities from that of his trading possibilities (which reflect his endowment); in general both will contribute to the individual's consumer surplus. To isolate these two contributions, we shall make the following, basically technical assumption:

(A4) For any $i \in I$ there exists an $\bar{i} \in I$ such that $X_{\bar{i}} = X_i$, $A_{\bar{i}} = A_i$, but $Y_{\bar{i}} = \{0\}$.

That is, if i has entrepreneurial abilities there is another individual with the same preferences and endowment but no entrepreneurial abilities.[4] Comparing i's equilibrium allocation with that of his double, $\bar{i}$, will allow us to isolate the contribution of Y_i *provided i* is a perfectly competitive entrepreneur, as well as a perfectly competitive consumer. To establish this proviso, we shall need a slightly stronger notion of competitive equilibrium and a corresponding, slightly stronger no surplus condition.

2A. *Perfectly Competitive Equilibrium*

First, two preliminary definitions. A pair $(\bar{x}, \bar{y}^i) \in \times_i X_i \times_{h \neq i} Y_h$ is a *feasible allocation for the economy without i as an entrepreneur* if $\sum_i \bar{x}_i = \sum_{h \neq i} \bar{y}_h{}^i$. Henceforth $(\bar{x}, \bar{y}^i)$ will represent such a feasible allocation. $(\bar{x}, \bar{y}^i, p)$ is a *Walrasian equilibrium for the economy without individual i as an entrepreneur* (written $(\bar{x}, \bar{y}^i, p) \in W(Y_i)$) means that for each $h \neq i$, $p\bar{x}_h = p\bar{y}_h{}^i = \sup pY_h < pB_h(\bar{x}_h)$ and $p\bar{x}_i = 0 < pB_i(\bar{x}_i)$.

Now suppose $(x, y, p) \in W$. Individual *i is a perfectly competitive entrepreneur* in (x, y, p) (written $(x, y, p) \in C^i(Y_i)$) if there exists a $(\bar{x}, \bar{y}^i, p) \in W(Y_i)$. The idea behind the definition can be explained. Think of individual i as playing two roles, entrepreneur and consumer. As noted earlier, for the purpose of this section we are interested in isolating i's ability to influence prices as an entrepreneur. If i is a perfectly competitive entrepreneur in (x, y, p) then the economy can, without i's supplies and demands as entrepreneur, still achieve a Walrasian equilibrium *at the same prices p*. Thus i as entrepreneur *cannot* influence the level of Walrasian equilibrium prices, at least according to a basic test: If when all other individuals charge the prices in p, i as entrepreneur tries to get higher prices than those in p for all the commodities he sells in the Walrasian equilibrium and tries to get lower prices than those in p for all the commodities he buys in the equilibrium he will be unsuccessful. Any individual h, $h \neq i$, will clearly prefer to buy (sell) all the commodities he wants to buy (sell) at the lower (higher) prices in p rather than at the higher (lower) prices offered by i as entrepreneur. Similarly, i as consumer will want to buy and sell at the prices in p. But if i is a perfectly competitive entrepreneur in (x, y, p) then all $h \neq i$ can buy (sell) all they want to at the prices p without i acting as entrepreneur. Similarly, i as consumer can buy and sell all he wants to at p.

Aggregating over individuals, (x, y, p) is a *Perfectly Competitive Equilibrium* (written $(x, y, p) \in PC$) if $(x, y, p) \in W$, and for each individual i, i has no monopoly power in (x, y, p) and also, in particular, i is a perfectly competitive entrepreneur in (x, y, p). Thus, if (x, y, p) is a Perfectly Competitive Equilibrium, no one individual or entrepreneur can influence the level of equilibrium prices by withdrawing his supplies and demands from the market. Alternatively expressed, in a Perfectly Competitive Equilibrium no one individual or entrepreneur can influence the level of equilibrium prices by trying to sell all his goods at prices higher than p and buy all his goods at prices lower than p.

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[4] Note that we identify i's "production possibilities" with his "entrepreneurial abilities". The validity of this identification follows from an interpretation of the current model as one with endogenous firm formation (see the Supplement to [3]). Entrepreneurial abilities may be heuristically thought of as abilities to *organize* (organ-ize) factors into productive units.

2B. *The Strong No Surplus Condition*

Let $A^{\bar{i}}(x, Y_i) = \{(\bar{x}, \bar{y}^i)$: for each $h \neq i$, $\bar{x}_h \in A_h(x_h)$ and $\bar{x}_i \in A_i(x_{\bar{i}})\}$. Individual *i contributes no surplus as an entrepreneur in* (x, y) (written $(x, y) \in NS(Y_i))$ means $A^{\bar{i}}(x, Y_i) \neq \varnothing$ but for any $(\bar{x}, \bar{y}^i) \in A^{\bar{i}}(x, Y_i)$ and any $h \neq i$ $\bar{x}_h \notin B_h(x_h)$ and $\bar{x}_i \notin B_i(x_{\bar{i}})$. That is, if i stops acting as an entrepreneur the economy can achieve an allocation in which everyone except i is as well off as with x and i is as well off as $\bar{i}$ is in x. The pair (x, y) is a *strong no surplus allocation* (written $(x, y) \in NS^*$) means $(x, y) \in NS$ and also in particular, for each individual i, i contributes no surplus as an entrepreneur in (x, y).

We now prove

THEOREM 2 (characterization of perfectly competitive equilibria). *Assume* (A1)–(A4). *Then (x, y, p) is a Perfectly Competitive Equilibrium iff (x, y) is a strong no surplus allocation.*

The theorem follows readily from Theorem 1 and the following lemma. The lemma is styled "full appropriation" for reasons explained in Remark 4 below.

LEMMA 3 (full appropriation). *Assume* (A1) *and* (A4), *and suppose $(x, y, p) \in W$. Then for any individual $i \in I$, i is perfectly competitive entrepreneur in (x, y, p) iff i contributes no surplus as an entrepreneur in (x, y).*

Proof. Suppose $(\bar{x}, \bar{y}^i, p) \in W(Y_i)$, then it is easy to verify that $(\bar{x}, \bar{y}^i) \in A^{\bar{i}}(x, Y_i)$. Also, $(\bar{x}, \bar{y}^i) \in A^{\bar{i}}(x, Y_i)$ and $\bar{x}_h \in B_h(x_h)$ for some $h \neq i$ or $\bar{x}_{\bar{i}} \in B_i(x_{\bar{i}})$ would imply $\sum_i p\bar{x}_i > \sum_h p\bar{y}_h{}^i$, contradicting the feasibility of $(\bar{x}, \bar{y}^i)$. So $(x, y) \in NS(Y_i)$.

Suppose $(x, y) \in NS(Y_i)$, so $(\bar{x}, \bar{y}^i) \in A^{\bar{i}}(x, Y_i)$. Then for all $h \neq i$

$$p\bar{x}_h \geqslant px_h = \sup pY_h \geqslant p\bar{y}_h{}^i$$

and

$$p\bar{x}_i \geqslant p\bar{x}_{\bar{i}} = 0.$$

Since $\sum_i \bar{x}_i = \sum_h \bar{y}_h^i$, the above inequalities must all be equalities. So

$$p\bar{x}_h = \sup pY_h = p\bar{y}_h^i < pB_h(x_h) = pB_h(\bar{x}_h),$$
$$p\bar{x}_i = 0 < pB_i(x_i) = pB_i(\bar{x}_i).$$

Observe that $B_h(x_h) = B_h(\bar{x}_h)$ since $\bar{x}_h \in A_h(x_h)$ and $x_h \in A_h(\bar{x}_h)$. Similarly, $B_i(x_i) = B_i(\bar{x}_i)$. Q.E.D.

2C. *Perfect Competition, Free Entry, and the Long Run*

The triple (x, y, p) is a *perfectly competitive long-run equilibrium* (written $(x, y, p) \in LPC$) if $(x, y, p) \in PC$ and for each individual $i \in I$, $py_i = 0$ (i.e., profits are zero).

Let $A(x, Y_i) = \{(\bar{x}, \bar{y}^i)$: for each $h \in I$ (including i), $\bar{x}_h \in A_h(x_h)\}$. Individual *i is a redundant entrepreneur in* (x, y) (written $(x, y) \in RE^i$) means $A(x, Y_i) \neq \emptyset$ but for any $(\bar{x}, \bar{y}^i) \in A(x, Y_i)$ and any $h \in I$ $\bar{x}_h \notin B_h(x_h)$. That is, everyone *including* i can do as well as in x if i stops being an entrepreneur. Aggregating over individuals, (*all*) *entrepreneurs are redundant in* (x, y) if for every $i \in I$, $(x, y) \in RE^i$.

Assume

(A5) for each individual $i \in I$, $0 \in Y_i$ (possibility of inaction).

We now prove

THEOREM 3 (characterization of perfectly competitive long-run equilibrium). *Assume* (A1)–(A5). *Then* $(x, y, p) \in LPC$ *iff* (x, y) *is a strong no surplus allocation and entrepreneurs are redundant in* (x, y).

The theorem follows readily from Theorem 2 and the following two lemmas, each of which has some interest in itself.

LEMMA 4 (sufficient condition for zero profits). *Assume* (A1) *and* (A5), *and suppose* $(x, y, p) \in W$. *Then if i is a redundant entrepreneur in* (x, y), *i earns zero profits in* (x, y, p) (i.e. $py_i = 0$).

Proof. By assumption there exists a $(\bar{x}, \bar{y}^i) \in A(x, Y_i)$. Therefore for all $h \neq i$,

$$p\bar{x}_h \geqslant px_h = py_h \geqslant p\bar{y}_h^i$$

and

$$p\bar{x}_i \geqslant px_i \qquad py_i \geqslant 0.$$

Since $\sum_{h\in I} p\bar{x}_h = \sum_{h\neq i} p\bar{y}_h^i$, $py_i = 0$. Q.E.D.

LEMMA 5 (profits as surplus). *Assume* (A1), (A4), *and* (A5), *and suppose* $(x, y, p) \in PC$. *Then for any* $i \in I$, *individual i earns zero profits in* (x, y, p) (i.e., $py_i = 0$) *iff he is a redundant entrepreneur in* (x, y).

Proof. In view of Lemma 4 we need only show the "only if" part. So assume $py_i = 0$. Thus, since $(x, y, p) \in W$, $px_i = 0 < pB_i(x_i)$ and $px_{\bar{i}} = 0 < pB_{\bar{i}}(x_{\bar{i}})$. Or, given completeness of preferences, $x_{\bar{i}} \in A_i(x_i)$. Also, i is a perfectly competitive entrepreneur in (x, y, p) implies there exists $(\bar{x}, \bar{y}^i) \in A^{\bar{i}}(x, Y_i)$. So $\bar{x}_i \in A_i(x_{\bar{i}})$. But then $\bar{x}_{\bar{i}} \in A_i(x_i)$, i.e., $(\bar{x}, \bar{y}^i) \in A(x, Y_i)$. To complete the proof observe that $(\bar{x}, \bar{y}^i) \in A(x, Y_i)$ and $\bar{x}_h \in B_h(x_h)$ for some $h \in I$ would imply $\sum_i p\bar{x}_i > \sum_h p\bar{y}_h{}^i$, contradicting the feasibility of $(\bar{x}, \bar{y}^i)$. Q.E.D.

Remark 2. The import of Lemma 4 is that it provides an answer to the classical question, when will a perfectly competitive entrepreneur's payments to his factors exhaust his product? The answer being that these payments will exhaust his product if he is a redundant entrepreneur. Note that this answer, in contrast to the more popular constant returns to scale answer, is consistent with the possibility of U-shaped cost curves. (Lemma 4 does not depend on any convexity assumptions.) Also note that according to Lemma 5, that an entrepreneur is redundant is not only sufficient but also necessary for product exhaustion *if the entrepreneur is a perfect competitor.*

Remark 3. We wish to argue that the Redundant Entrepreneurs Condition characterizes not only the final allocation in a perfectly competitive economy that is in long run (i.e., zero profits) equilibrium—as proved in Theorem 3—but also characterizes the final allocation in a perfectly competitive economy in which free entry by competitor firms has fully run its course, exploiting all profit potentials. So those long-run perfectly competitive equilibria resulting from free entry, like the long-run equilibria described in principles texts, may be characterized as long-run equilibria in which all entrepreneurs are redundant. To see this, think of a perfectly competitive equilibrium in which free entry has fully run its course as being an equilibrium in which for each active entrepreneur in the economy there is an inactive, potential entrepreneur with essentially the same production possibilities. The latter individual is indifferent between operating and not operating because the former entrepreneur is just breaking even in the equilibrium. Now clearly any active entrepreneur is redundant because if any active entrepreneur were to stop being an entrepreneur, his inactive "double" could just take his place and the equilibrium allocation could be achieved without that active entrepreneur.

Remark 4. Lemma 5 may be equivalently expressed: a perfectly competitive entrepreneur earns positive profits iff he contributes a positive surplus to somebody. We wish to establish the stronger proposition that a perfectly competitive entrepreneur's profits *measure* his contribution to surplus. This is a generalized "adding-up" proposition.

To establish the proposition recall that if an individual i is a perfectly competitive entrepreneur then, by definition, whether or not he acts as an entrepreneur will not affect the level of equilibrium prices. So, for each individual $h \neq i$, whether or not i is an entrepreneur does not matter to h: he faces the same budget constraint with or without i as entrepreneur. Summarizing,

PROPOSITION 1A. *A perfectly competitive entrepreneur contributes nothing to the consumer surplus of anyone besides himself.*

This, of course, is the import of Lemma 3 above. It may be alternatively expressed by saying that a "full appropriation condition" characterizes any entrepreneur who is a perfect competitor: an entrepreneur who is a perfect competitor fully appropriates from all other individuals, *in the form of profits*, any contribution he may make to their consumer surplus. For a similar observation in a somewhat different context, see Hart [1].

On the other hand, if i acts as an entrepreneur he may earn entrepreneurial profits; if he stops acting as an entrepreneur he will lose these profits. And this potential wealth loss is the *only* effect that the cessation of i's entrepreneurial activities will have on i's budget line or welfare since he will face the same prices whether or not he acts as an entrepreneur: The wealth loss will result in a *parallel* shift downward in i's budget line, the magnitude of the shift being measured by i's profits. Since his new optimum will be on the new lower budget line, we have,

PROPOSITION 1B. *The contribution of a perfectly competitive entrepreneur to his consumer surplus (and aggregate consumer surplus) is measured by his entrepreneurial profits.*

To bring Proposition 1 into focus by way of contrast, we observe that in an arbitrary Walrasian equilibrium profits do *not* in general measure surplus. For example, an entrepreneur with a constant returns-to-scale technology will earn zero profits in any Walrasian equilibrium, but he may be the only individual in the economy with production possibilities (!) and thus be contributing a substantial surplus in any Walrasian equilibrium.

Remark 5. A second proposition is implicit in the first. Since a perfectly competitive entrepreneur cannot affect prices, he appropriates his entire contribution to consumers' surplus in the form of profits. Thus,

PROPOSITION 2. *For a perfectly competitive entrepreneur, his profits are really competitively determined rents on his entrepreneurial abilities. (Or, alternatively expressed, his profits are really rents that reflect the contribution to aggregate consumers' surplus of his production possibilities.)*

3. A SPECIAL CASE OF THE MODEL: A SIMPLE PRODUCTION ECONOMY

Following Ostroy [5], we observed that the no surplus theory may be viewed as a generalized marginal productivity theory of distribution. And it becomes exactly a marginal productivity theory of distribution in what he calls "simple production economies." This seminal result, which is discussed in detail in [5], is generalized below only insofar as we include the possibility of entrepreneurs and characterize entrepreneurs' marginal productivity.

The result may be alternatively expressed. Perfectly competitive economies, which we have shown earlier are characterized by a no surplus condition, are also characterized by the marginal productivity theory of distribution: There is perfect competition *if and only if* there is adding-up! And furthermore, this theory of distribution corresponds to an implicit theory of value in an appropriate dual price space (e.g., see Lemma 2 above). In this view, the competitive theory of value derives from the competitive theory of distribution. Indeed, the appropriate dual price space to support some (nonconvex) no surplus allocations may involve nonlinear—but perfectly competitive (!)—pricing. This is illustrated in [2].[6]

Suppose each individual i has an endowment of resources $r_i \in R_+^{n-1}$ which can be transformed into a single output q. And suppose individuals desire only q's. Specifically, for each i

$$X_i = \{x_i \in R^n : x^i = (-r_i, q_i), \text{ where } q_i \in R_+\},$$

and i's "utility function" is given by $U_i(-r_i, q_i) = q_i$. (It simplifies matters to not allow for free disposal in characterizing X_i. It is quite harmless.)

Also suppose that each individual i has a production function f_i from R_+^{n-1} to R_+. The aggregate production function is f from R_+^{n-1} to R_+, and the aggregate production function without i acting as an entrepreneur is f^i from R_+^{n-1} to R_+. (The functions f_i, f, and f^i are functional representations of Y_i, $\sum_i Y_i$, and $\sum_{h \neq i} Y_h$, respectively.) We assume that if $r^1 \geqslant r^2$, r^1 and r^2 in R_+^{n-1}, then $f_i(r^1) \geqslant f_i(r^2)$, and similarly $f(r^1) \geqslant f(r^2)$ and $f^i(r^1) \geqslant f^i(r^2)$, so that all resources can be utilized in production at no cost in terms of output forgone.

Since individuals always supply all their resources to production, there is a one-to-one correspondence between points $x_i \in X_i$ and points $q_i \in R_+$, namely $x_i = (-r_i, q_i)$. Thus we can, for simplicity, regard a (consumption) allocation in our simple production economy as a point $(q_i) \in R_+^{\#I}$. Using this simplification, the reader will easily verify that our no surplus definitions can be given the following special forms. (Henceforth (q_i) will represent a Pareto optimal allocation, i.e., one satisfying $\sum_i q_i = f(r)$, where $r = \sum_i r_i$.)

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[6] The equilibrium allocation in the firm-specific labor example in [2] is a no surplus allocation that can only be supported by a nonlinear (but perfectly competitive) price system. See [2] for details.

$$(q_i) \in NS \quad \text{iff} \quad \text{for each } i,\ q^i = f^i(r^i),$$

where $q^i = \sum_{h \neq i} q_h$ and $r^i = \sum_{h \neq i} r_h$;

$$\begin{aligned} (q_i) &\in NS(Y_i) && \text{iff} \quad q^i + q_i = f^i(r); \\ (q_i) &\in NS^* && \text{iff} \quad (q_i) \in NS \text{ and for each } i,\ (q_i) \in NS(Y_i); \\ (q_i) &\in RE^i && \text{iff} \quad f(r) = f^i(r). \end{aligned}$$

The following definitions of marginal productivity concepts should require no elaboration; they are completely straightforward:

$$MP_i = f(r) - f^i(r^i),$$

where MP_i represents the marginal productivity of individual i.

$$MP(Y_i) = f(r) - f^i(r),$$

where $MP(Y_i)$ represents the marginal productivity of i's entrepreneurial abilities.

$$MP(r_i) = f(r) - f(r^i),$$

where $MP(r_i)$ represents the marginal productivity of i's resources.

Note in particular that while i's entrepreneurial abilities are in some sense indivisible, $MP(Y_i)$ can be defined in a straightforward manner. It depends only on the economy's aggregate production possibilities, summarized in f, and its aggregate production possibilities without i, summarized in f^i. We now prove some equivalences between the above no surplus concepts and marginal productivity concepts. Before proceeding note that in this simple production economy assumption (A4) can be given the following special form:

(A4′) For any individual $i \in I$ there exists an $\bar{i} \in I$ such that $r_{\bar{i}} = r_i$, and for any $r \in R^{n-1}_+$ $f^{\bar{i}}(r) = f(r)$.

Lemma 6 (marginal productivity characterization of no surplus). $(q_i) \in NS$ *iff for each* i, $q_i = MP_i$.

Proof. If for each i $q_i = MP_i$ then for each i $q_i - f^i(r^i) = (q^i + q_i) - f^i(r^i)$. Or, $f^i(r^i) = q^i$.

If $(q_i) \in NS$ then for each i $f^i(r^i) = q^i$. Or, for each i, $\sum_i q_i - q^i = q_i = f(r) - f^i(r^i) = MP_i$. Q.E.D.

LEMMA 7 (marginal productivity characterization of strong no surplus). *Assuming* (A4′), $(q_i) \in NS^*$ *iff for each* i, $q_i = MP_i = MP(r_i) + MP(Y_i)$.

Proof. If for each i, $q_i = MP_i = MP(r_i) + MP(Y_i)$ then by Lemma 6, $(q_i) \in NS$, and

(a) $f(r) - f^i(r^i) = (f(r) - f(r^i)) + (f(r) - f^i(r))$

But

(b) $f^i(r^i) = q^i$ since $(q_i) \in NS$ and

(c) $f(r) - f(r^i) = MP(r_i) = MP(r_i) = MP_i = q_i$.

Therefore, substituting yields

(d) $f(r) - q^i = q_i + f(r) - f^i(r)$.

Or, $q^i + q_i = f^i(r)$, i.e., $(q_i) \in NS(Y_i)$.

If $(q_i) \in NS^*$ then by Lemma 6 for each i, $q_i = MP_i$, and also $q^i + q_i = f^i(r)$. Since (b) and (c) of above still hold, and since (d) also still holds, substituting (b) and (c) into (d) yields (a). Or, $MP_i = MP(r_i) + MP(Y_i)$. Q.E.D.

LEMMA 8 (marginal productivity characterization of redundant entrepreneurs). *Let* $(q_i) \in NS$ *and assume* (A4′). *Then* $(q_i) \in RE^i$ *iff* $MP(Y_i) = 0$.

Proof. If $(q_i) \in RE^i$ then $f(r) - f^i(r) = 0 = MP(Y_i)$. Alternatively, if $MP(Y_i) = 0$ then, by definition, $f(r) - f^i(r) = 0$, i.e., $(q_i) \in RE^i$. Q.E.D.

The lemmas can be used to show the following generalized adding-up result:

THEOREM 4 (generalized adding-up). *Assume* (A4′). *If* $(q_i) \in NS$ *then* $f(r) = \sum_i MP_i$. *Furthermore, if* $(q_i) \in NS^*$ *then*

$$f(r) = \sum_i MP(r_i) + \sum_i MP(Y_i).$$

In addition, if $(q_i) \in NS^*$ *and for all* i, $(q_i) \in RE^i$ *then*

$$f(r) = \sum_i MP(r_i),$$

where for each i, $MP(Y_i) = 0$.

Finally, suppose (x, y, p') *is a Perfectly Competitive Equilibrium for the simple production economy, where* $p' = (p, 1) \in R^{n-1} \times R$ *(i.e.,* q *is the numeraire commodity) and* $x_i = (-r_i, q_i)$ *(i.e.,* (q_i) *is the equilibrium consumption allocation). Then, for each* i, $pr_i = MP(r_i)$ *and* $\pi_i = \sup p'Y_i = MP(Y_i)$.

Proof. Given Lemmas 5–7, we need only verify that $pr_i = MP(r_i)$ and $\pi_i = MP(Y_i)$. But if $(x, y, p') \in PC$ then for each i, $p'x_i = -pr_i + q_i = \pi_i$ and $p'x_{\bar{i}} = -pr_{\bar{i}} + q_{\bar{i}} = 0$. So $pr_i + \pi_i = q_i$ and $pr_{\bar{i}} = pr_i = q_{\bar{i}}$. Since $(x, y, p') \in PC$ implies $(q_i) \in NS^*$ by Lemmas 1 and 3, and since $MP(Y_{\bar{i}}) = 0$, Lemma 7 implies $q_{\bar{i}} = MP_{\bar{i}}(r_{\bar{i}}) = MP_i(r_i) = pr_i$ and $q_i = MP(r_i) + MP(Y_i) = pr_i + MP(Y_i)$. Or $MP(Y_i) = \pi_i$. Q.E.D.

Remark 6. Theorem 4 amounts to a complete adding-up result. First, under perfect competition:

$$\underbrace{f(r)}_{\substack{\text{total}\\ \text{product}}} = \underbrace{\sum_i MP(r_i)}_{\substack{\text{factor}\\ \text{earnings}}} + \underbrace{\sum_i MP(Y_i)}_{\text{profits}}.$$

And second, in the long run Euler's theorem holds—at least locally—so that factor payments exhaust the product (entrepreneurs are redundant so profits are zero):

$$\underbrace{f(r)}_{\substack{\text{total}\\ \text{product}}} = \underbrace{\sum_i MP(r_i)}_{\substack{\text{factor}\\ \text{earnings}}}.$$

Note in particular, that under perfect competition profits are rents to entrepreneurial ability, as observed in Remark 5 above: for each individual i,

$$\pi_i = MP(Y_i).$$

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JOURNAL OF ECONOMIC THEORY 22, 222–242 (1980)

Perfect Competition, the Profit Criterion, and the Organization of Economic Activity

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Received August 6, 1979

Suppose that profit-seeking firms endogenously determine the set of traded commodities (the "commodity space"). If firms have monopoly power, it is well known that profit seeking will generally result in the marketing of an inefficient set of commodities. However, if firms are perfect competitors, it is widely believed that self-interested profit seeking will result in an efficient set of marketed commodities. That is, an economic system founded on the Profit Criterion is efficient, or "the Invisible Hand works." This moral is sometimes inferred from the fact that any standard competitive equilibrium is a Pareto optimum. But since in any standard competitive model the set of traded commodities is not a variable, the optimality of standard competitive equilibria neither proves nor disproves the efficiency of the Profit Criterion.

In the sequel we construct a competitive model which *endogenously* determines the set of traded commodities. In some sense this model completes the standard competitive model, which may be interpreted as a model in which the set of markets—whether complete *or incomplete*—is exogenously given. We use our model to examine the efficiency of the Profit Criterion. Sufficient conditions for efficiency are provided. But some counterexamples are also provided. In the latter context, themes that appear in the development literature, about the difficulties in evolving a complex well-developed economy, reappear.

To summarize the sequel, we show that:

(1) Any perfectly competitive firm will profit maximize if it is operating in accordance with its shareholders' interests (Theorem 1, Section 2).

(2) A perfectly competitive firm will always perceive positive profits to any unilateral marketing decision, e.g., innovation, which improves social welfare (Theorem 2, Section 3).

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* Research for the paper was partially supported by a (U.K.) Social Science Research Council grant, which is gratefully acknowledged.

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However,

(3) Perfectly competitive firms will not, in general, introduce an efficient set of commodities, the source of market failure being the existence of inter-firm complementarities in innovation. Such complementarities give rise to potential, favorable "pecuniary externalities" in the innovation process (Theorem 3, Section 4A, and Section 4B).

(4) Perfectly competitive firms will introduce an efficient set of commodities if

(a) there are no intermediate innovations and the economy's aggregate technology is convex and differentiable (Theorem 4, Section 4C); this effectively rules out all inter-firm complementarities; or

(b) firms can become sufficiently large (e.g., by mergers or joint ventures) to internalize all potential inter-firm complementarities (Corollaries 1 and 2, Section 5; this provides a rationale for the structures of firms (Corollary 3, Section 5).

This paper is to a large degree concerned with constructing a simple model which reflects some of the principles underlying Oliver Hart's work on "monopolistic competition," principles that are obfuscated by Hart's desire to *simultaneously* ensure perfect competition by replication *and* exposit the consequences of perfect competition. It is the latter consequences that we wish to highlight here. Thus, (a) our characterization of a "perfectly competitive firm" in Section 2 below is an idealization of Hart's "small firm" in his limiting economy; (b) the unanimity result, (1), is a reformulation of Hart's unanimity result in [3], showing that it follows from firms being perfectly competitive; and (c) the optimality result, (4a), is a simplified version of Hart's optimality theorem in [2], showing that it depends crucially on convexity and differentiability assumptions.

The results mentioned in (2), (3), and (4b) are somewhat more original. A few remarks may be helpful: The optimality result, (2), may be viewed as an extension of the first basic theorem of welfare economics to competitive economies with endogenous commodity spaces. And (4b) is a corollary of (2). The nonoptimality result, (3), shows that any validation of the Profit Criterion must be qualified. Recently, Hart independently found results similar to (2) and (3); see [4]. It should also be mentioned that the model and results below are somewhat more general than those in [2–4] because they allow for innovation of intermediate inputs as well as consumption goods, and they allow for the possibility of multiproduct innovations within a firm (e.g., see Theorem 2 below).

Before closing, Ostroy's and my complementary work on the no-surplus condition should be mentioned. The explicit connections between this work and the current work will form the subject of a future paper. There it will be shown that a no-surplus condition characterizes perfectly competitive innovators. In other words, the theme in [7, 8], that no surplus (in quantity space) is the dual to perfect competition (in price space), extends to perfect competition with innovation.

1. THE PRELIMINARIES: THE BASIC MODEL[1]

The model is basically an Arrow–Debreu model with the possibility of product innovation added. There is a large, finite set of potential commodities C and a finite set of agents J. J consists of a set of individuals I and a set of firms F (i.e., $J = I \cup F$). Each economic agent $j \in J$ has a trading possibility set $X^j \subset R^n$, where $n = {}^{\#}C$. Also each individual $i \in I$ has an "at least as good as" correspondence A^i from X^i to X^i that completely preorders X^i, and a "better than" correspondence B^i from X^i to X^i satisfying $y^i \in B^i(x^i)$ iff $y^i \in A^i(x^i)$ and $x^i \notin A^i(y^i)$. Finally, there are nonnegative share ownerships in the firms, $s(i, f)$, satisfying, for each $f \in F$, $\sum_i s(i, f) = 1$.

The set of potential commodities that is actually marketed is some (to be determined) subset of C, T. For any $T \subset C$, let $X^j(T) = \{x^j \in X^j: x_c^j = 0$ for all $c \notin T\}$; it represents j's trading possibilities in T. A point $x = (x^j)$ is a *T-feasible allocation* if $x \in \times_j X^j(T)$ and $\sum_j x^j = 0$.[2] For any $x^i \in X^i$ let $B^i(x^i, T) = B^i(x^i) \cap X^i(T)$; it represents i's better than x^i set in T. (So, $X^j = X^j(C)$ and $B^i = B^i(\ldots, C)$.) An allocation y *dominates* an allocation x if $y^i \in A^i(x^i)$ for all i and $y^i \in B^i(x^i)$ for some i. A T-feasible allocation x is an (unconstrained) *Pareto optimum* if there is no C-feasible allocation that dominates it.

We assume that each firm markets only personalized commodities and is sole supplier of its commodities. Specifically, letting $C(f) \subset C$ represent the set of *commodities personalized to* f, we assume $x^f \in X^f$ and $x_c^f < 0$ implies (a) $c \in C(f)$ and (b) $x_c^j \geqslant 0$ for all $x^j \in X^j$ and all $j \in J$, $j \neq f$. Note that this is not very restrictive since it is possible for different firms to market perfect substitutes.

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[1] Some notation and terminology: Let $z \in E^n$, $p \in R^n$, and $B \subset R^n$. Then $pz \leq pB$ means $pz \leq py$ for all $y \in B$; cl B means the closure of B; ri B means the relative interior of B. If A is a correspondence from B to B then A "completely preorders" B means for any x, y, z in B $x \in A(x)$ (reflexivity), $x \in A(y)$ or $y \in A(x)$ (completeness), and $x \in A(y)$, $y \in A(z)$ implies $x \in A(z)$ (transitivity).

[2] More conventionally, (x^j) is written in the form $((x^i), (y^f))$, where for each f $y^f = -x^f$. Under the latter notation, py^f rather than $-px^f$ measures f's profits under prices $p \in R^n$. And the feasibility condition takes on the more usual form $\sum_i x^i = \sum_f y^f$; or $\sum_i \bar{x}^i = \sum_i w^i + \sum_f y^f$, if each "net trade set" X^i is translated to the "consumption set" $\bar{X}^i = X^i + \{w^i\}$, where $w^i \in R^n$ is i's endowment. Since we shall often find it convenient to treat firms and individuals symmetrically, we find our notation more convenient.

$X^i(x, p, T) = \{x^i \in X^i(T): px^i \leq -\sum_f s(i, f)\, px^f < pB^i(x^i, T)\}$; it represents i's demand correspondence in T given incomes derived from the allocation $x \in \times_J X^j$ and prices $p \in R^n$. Similarly, $X^f(x, p, T) = \{x^f \in X^f(T): -px^f \geqslant -pX^f(T)\}$ represents f's demand correspondence in T. A *Walrasian equilibrium in* T ($W(T)$) is a pair (x, p) consisting of a T-feasible allocation and a price vector $p \in R^n$ such that for all $j \in J$ $x^j \in X^j(x, p, T)$. If (x, p) is a $W(T)$ we write $(x, p) \in W(T)$.

Since, for simplicity, in this paper we shall only allow for product innovation by firms, we shall assume that trading in commodities other than those personalized to firms is always possible. That is, letting $\mathbf{T} = C \backslash (\bigcup_f C(f))$, we shall assume throughout that $\mathbf{T} \subset T$. But there exists a market for $c \in C(f)$ only if f actually markets c. That is, letting $C(x^f) = \{c \in C(f): x_c^f < 0\}$ represent the *commodities personalized to* f *marketed in* x^f, we shall assume that $(x, p) \in W(T)$ implies $T \cap C(f) = C(x^f)$. Finally, we shall be assuming throughout that individuals are locally nonsatiated in $\mathbf{T}$. That is, for any i and any $x^i \in X^i$ there exists a $y^i \in B^i(x^i)$ such that y^i is arbitrarily close to x^i and $y_c^i = x_c^i$ for all $c \notin \mathbf{T}$.

2. The Firm's Objective and the Concept of Equilibrium

2A. *Profit Maximizing as the Perfectly Competitive Frm's Objective*

We are now in a position to observe that an arbitrary $W(T)$ has no claims to being a stable equilibrium since any firm f can move the economy from that "equilibrium" to another position by altering the set of commodities it markets. But how will a firm decide what commodities to market?

We first prove that if a firm is a perfect competitor, it will—assuming it is being operated in accordance with its shareholders' interests—make this decision based on a profit-maximizing objective.

Let $(x, p) \in W(T)$, $y^f \in X^f$, and $T(y^f) = (T \backslash C(f)) \cup C(y^f)$. A *quasi-equilibrium for* y^f ($Q(y^f)$) is a pair (y, p^f) consisting of a $T(y^f)$-feasible allocation and a price vector $p^f \in R^n$ such that $y^j \in X^j(y, p^f, T(y^f))$ for each $j \in J$, $j \neq f$. If (y, p^f) is a $Q(y^f)$, we write $(y, p^f) \in Q(y^f)$. We also write $p^f \in Q(y^f)$ to indicate $(y, p^f) \in Q(y^f)$ for some y; no confusion should result. The concept of a quasi-equilibrium is useful for judging the stability of a $W(T)$. To see this, observe that if $(x, p) \in W(T)$, a firm f may try to manipulate the set of traded commodities T and/or the equilibrium prices p by altering its trade demand from x^f to some y^f. If it assumes that it cannot effect the commodities marketed by other firms (as we shall suppose that it does assume) then the change to y^f is feasible for f only if $y_c^f > 0$ only for $c \in T \backslash C(f)$. If y^f is a feasible change for f and f decides to change from x^f to y^f then the set of marketed commodities changes to $T(y^f)$. If the firm f also assumes that all other agents will still act as price takers (as we shall suppose that it does assume), then under "rational expectations" firm f will expect the change to y^f to alter prices from p to p^f, where $p^f \in Q(y^f)$. To summarize, a $W(T)$ is "stable" if no firm f would be interested in moving from it to some quasi-equilibrium $Q(y^f)$, where $y^f \in X^f$ and $y_c^f > 0$ only for $c \in T \backslash C(f)$. The set $Y^f(T) = \{y^f \in X^f : y_c^f > 0$ only for $c \in T \backslash C(f)\}$ represents the set of *feasible marketing strategies* for f in $W(T)$. We assume that if $y^f \in Y^f(T)$ then $Q(y^f) \neq \emptyset$. This is obviously a strong assumption. However, without it, it is possible that a firm may not introduce an efficient innovation because the consequences of its introduction are not well defined.

A firm f is a *perfect competitor* in a $W(T)$, (x, p), if (a) it cannot influence the prices of any commodities in $T \backslash C(f)$ and (b) it cannot influence the prices of its own commodities, in the sense that it always receives the economy's reservation prices for its own commodities. That is, given any $y^f \in Y^f(T)$ and any $(y, p^f) \in Q(y^f)$, (a) $p_c^f = p_c$ for each $c \in T \backslash C(f)$ and (b) for each $j \neq f$ there exists a $y(f)^j \in X^j(y, p^f, T(y^f))$ satisfying $y(f)_c^j = 0$ for all $c \in C(f)$. The interpretation of (b) is that the prices in p^f for the commodities in $C(y^f)$ leave all agents $j \neq f$ with "no surplus" from purchasing them.

A note on notation: henceforth, given *any* allocation y, any prices $p \in R^n$, and any $T \subset C$, if $y^j \in X^j(y, p, T)$ then we write $y(f)^j$ to indicate a point in $X^j(y, p, T)$ satisfying $y(f)_c^j = 0$ for all $c \in C(f)$.

Remark 2.1. The above definition of a perfectly competitive firm is an idealization of Hart's seminal observation in [2] that if a firm is *small relative to its market* then it can essentially get the economy's reservation prices for its commodities (it faces perfectly elastic demands for its commodities at the economy's reservation prices) even if it is the *only* seller of the commodities and they have *no* close substitutes. To illustrate, suppose only one firm can produce Q's, and it can produce up to 100 Q's before its costs become prohibitive. Suppose also that Q's have no good substitutes. They are indivisible goods, and there are 10,000 people willing to pay £10 each for a Q; no one is willing to pay more than £10. Then the firm within its effective domain (100 Q's) can demand the highest-valued user's price (£10) for each of the Q's it produces. Alternatively expressed, it faces a perfectly elastic demand for Q's at £10, the price that would result from competitive bidding among the potential users. See [2] for further discussion.

We now prove a simplified version of Hart's [3] basic result.

THEOREM 1 (shareholder unanimity). *If* $(x, p) \in W(T)$ *and a firm* f *is a perfect competitor in the equilibrium then* (a) *shareholders in* f *unanimously favor profit maximizing as the firm's sole objective and* (b) *nonshareholders in* f *do not care about* f*'s behavior. That is, if* $(x, p) \in W(T)$, $y^f \in Y^f(T)$, *and* $(y, p^f) \in Q(y^f)$ *then*

(a) *for each i such that $s(i, f) > 0$:*

$$\left\{\begin{array}{l} y^i \in B^i(x^i) \\ y^i \in A^i(x^i) \text{ and } x^i \in A^i(y^i) \\ x^i \in B^i(y^i) \end{array}\right\} \quad \text{as} \quad -p^f y^f \left\{\begin{array}{c} > \\ = \\ < \end{array}\right\} - px^f$$

and

(b) *for each i such that $s(i, f) = 0$:*

$$y^i \in A^i(x^i) \quad \text{and} \quad x^i \in A^i(y^i)$$

(i.e., i is indifferent between x^i and y^i).

Proof. Let $(x, p) \in W(T)$, $y^f \in Y^f(T)$, and $(y, p^f) \in Q(y^f)$. Since f is a perfect competitor

$$p_c^f = p_c \qquad \text{for each } c \in T \backslash C(f). \tag{2.1}$$

Using (2.1) we know that for all $f' \in F$, $f' \neq f$:

$$p^f y(f)^{f'} = p^f y^{f'} = px(f)^{f'} = px^{f'}. \tag{2.2}$$

We also know that for all i

$$x^i \in A^i(x(f)^i) \quad \text{and} \quad x(f)^i \in A^i(x^i) \tag{2.3}$$

and

$$y^i \in A^i(y(f)^i) \quad \text{and} \quad y(f)^i \in A^i(y^i). \tag{2.4}$$

Finally, we know that

$$x(f)^i \in X^i(T(y^f)) \quad \text{and} \quad y(f)^i \in X^i(T). \tag{2.5}$$

For each i such that $s(i, f) = 0$, $x^i \in X^i(x, p, T)$ and $y^i \in X^i(y, p^f, T(y^f))$ implies by virtue of (2.2) and (2.5) that

$$x^i \in A^i(y(f)^i) \quad \text{and} \quad y^i \in A^i(x(f)^i). \tag{2.6}$$

Using (2.3) and (2.4) plus transitivity, (2.6) implies

$$x^i \in A^i(y^i) \quad \text{and} \quad y^i \in A^i(x^i), \tag{2.7}$$

which proves part (b) of the theorem.

Now suppose $-p^f y^f = -px^f$. Then even if $s(i, f) > 0$, $x^i \in X^i(x, p, T)$ and $y^i \in X^i(y, p^f, T(y^f))$ implies by virtue of (2.2) and (2.5) that (2.6) holds. And, as above, (2.6) implies (2.7) for each i such that $s(i, f) > 0$.

Suppose $-p^f y^f > -px^f$. Then if $s(i, f) > 0$, $px(f)^i = p^f x(f)^i < -\sum_{f' \in F} s(i, f') p^f y^{f'}$. So $y^i \in X^i(y, p^f, T(y^f))$ implies by virtue of (2.5) that $y^i \in A^i(x(f)^i)$. Or, using (2.3), $y^i \in A^i(x^i)$. But *not* $x^i \in A^i(y^i)$. To see this, observe that $p^f x(f)^i < -\sum_{f'} s(i, f') p^f y^{f'}$. And by local nonsatiation there is an $\mathbf{x}^i \in X^i(T \setminus C(f))$ such that $\mathbf{x}^i \in B^i(x(f)^i)$ and $-p^f \mathbf{x}^i < -\sum_{f'} s(i, f') p^f y^{f'}$. So, if $x^i \in A^i(y^i)$ then $\mathbf{x}^i \in B^i(y^i)$, which is impossible since $y^i \in X^i(y, p^f, T(y^f))$.

The demonstration that $-p^f y^f < -px^f$ implies $x^i \in B^i(y^i)$ for each i such that $s(i, f) > 0$ is exactly analogous to the above. Q.E.D.

2B. *A Complete Competitive Equilibrium Concept*

We have observed that a $W(T)$ is not a complete competitive equilibrium concept since firms endogenously determine T, the set of marketed commodities.

Consider instead the concept of a "Full Walrasian Equilibrium" (FWE): Suppose $(x, p) \in W(T)$. We shall say *there exist profits to innovation* by f in (x, p) if there exists a $y^f \in Y^f(T)$ and $p^f \in Q(y^f)$ such that $-p^f y^f > -px^f$. A pair (x, p) is a *Full Walrasian Equilibrium* (in T) (written $(x, p) \in$ FWE (in T)) if $(x, p) \in W(T)$ and there exist no profits to innovation by any firm $f \in F$.

A FWE is a *complete* competitive equilibrium concept insofar as it characterizes the outcome of a decentralized *innovation process* whereby the set of traded commodities is determined. Given some initial set of traded commodities (e.g., some rudimentary set of traded commodities T), commodities are introduced into the marketplace as long as they promise the introducing firm increased profits. When no firm sees that it can increase its profits by altering its trade decision assuming the set of marketed commodities other than its own will remain the same (and all other economic agents will act as perfect competitors, i.e., price takers), the innovation process ends. Alternatively viewed, in a FWE no firm can individually manipulate prices to its advantage by altering its trading decision, either the quantities it markets or the commodities it markets. Theorem 1 characterizes the relevance of the concept of a FWE for competitive economies: in a FWE perfectly competitive firms have endogenously determined the set of marketed commodities in accordance with their shareholders' interests.

Remark 2.2. A special case of the above innovation process involves two steps at each stage of the process. In Step 1 a firm introduces a new product or set of products. Then, with a lag, in Step 2 entrants appear, introduce perfect substitute products, and compete away the profits. The two steps are repeated until no firm sees any potential profits to introducing new products. This is a Schumpeterian version of the process. In the spirit of Remark 2.1, we observe that Step 2 is not essential, in general, for perfect competition.

We should also mention that our model, with its "sneaky dynamics," is not flexible enough to permit analysis of a possibly important aspect of any Schumpeterian innovation process: In such a process there may exist "wasteful competition"[3] resulting from firms competing to innovate a product first, to appropriate monopoly profits before entrants compete them away. The inefficiency of such competition is exposed in the well-known theory of common property resources.

3. PROFITS TO INDIVIDUALLY IMPROVING WELFARE

Since we have assumed local nonsatiation, as is well known any $W(C)$ is a Pareto optimum. And any $W(T)$ is an "optimum relative to T:" if $(x, p) \in W(T)$ then no y can dominate x provided y is T-feasible. But an arbitrary $W(T)$ is generally not a Pareto optimum.

In the rest of the paper we shall be concerned with the question, under what conditions will a FWE be a Pareto optimum? We shall be assuming throughout the sequel that all firms are perfect competitors.

Suppose (x, p) is a FWE in T. And suppose T' is an arbitrary subset of C containing T, i.e., $T \subset T' \subset C$. We shall first show that if some allocation y dominates x, where y is T'-feasible, then more than one innovating firm is needed to move the economy from T to T'. That is, y involves innovations by more than one firm. Or, in other words, it is not the case that $T' \backslash T$ is contained in $C(f)$ (for some f). This is the import of the following benchmark result, which may be viewed as an extension of the first basic theorem of welfare economics to competitive economies with endogenous commodity spaces.

THEOREM 2 (profits to individually improving welfare). *Suppose* $(x, p) \in W(T)$ *and suppose some* T'*-feasible allocation* y *dominates* x*. Then if* $T' \backslash T \subset C(f)$ *for some* $f \in F$*, there are profits to innovation by* f*; i.e.* (x, p) *is not a FWE.*

Proof. Clearly for all j, $y^j \in X^j(T(y^f))$ and $y^f \in Y^f(T)$. Let $(\bar{y}, p^f) \in Q(y^f)$, so $\bar{y}^f = y^f$. We show that $-p^f y^f > -p x^f$.

Assume the contrary. Then, by Theorem 1, for all i, $x^i \in A^i(\bar{y}^i)$. Also, since y dominates x, for all i, $y^i \in A^i(x^i)$ and for some i, $y^i \in B^i(x^i)$. So, using transitivity, for all i, $y^i \in A^i(\bar{y}^i)$ and for some i, $y^i \in B^i(\bar{y}^i)$. Now this implies by a standard argument that for all i, $p^f y^i \geqslant p^f \bar{y}^i$ with strict inequality for some i. Or, using the fact that for all $f' \neq f$, $\bar{y}^{f'} \in X^{f'}(\bar{y}, p^f, T(y^f))$, $\sum_i p^f \bar{y}^i > \sum_i p^f \bar{y}^i = -\sum_{f' \in F} p^f \bar{y}^{f'} \geqslant -\sum_{f' \in F} p^f y^{f'}$. But $\sum_i y^i = -\sum_{f'} y^{f'}$, a contradiction. Q.E.D.

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[3] The term is Ostroy's.

4. THE EFFICIENCY OF FWE IN THE ABSENCE OF PECUNIARY EXTERNALITIES

4A. *A Second Benchmark Result*

Suppose (x, p) is a FWE in T. Several innovators may be needed to move the economy from T to some $T' \supset T$. So one cannot determine from Theorem 2 whether in general x is a Pareto optimum. We now show that it is provided there would be no potential, favorable "pecuniary externalities" in the innovation process from T.

It will be convenient to first introduce some notation. Let $C^* = C \backslash T$ represent the set of potential innovations from T. Then $F^* = \{f: C^* \cap C(f) \neq \varnothing\}$ represents the set of potential innovators from T. And $J^* = J \backslash F^*$ represents the set of potential buyers of innovated commodities. (Condition (C1) below ensures there are no potential buyers in F^*.)

Finally, $P^* = \{q \in R^n: q_c = p_c$ for all $c \in T$ and for each $j \in J^*$, $x^j \in X^j(x, q, C)\}$ represents the set of price vectors which extend p to C^* so as to reflect or exceed the economy's reservation prices for the potential innovations C^* when *all* potential innovations are marketed *simultaneously* beginning from (x, p). The "minimum" $q \in P^*$ may be thought of as just reflecting the economy's reservation prices for C^*.[4]

There would be *no potential (favorable) pecuniary externalities in the innovation process from* T if

(C1) for each $f \in F^*$ $y^f \in X^f$ implies $y_c^f = 0$ for all $c \in C^*$ ("no intermediate innovations")

and

(C2) there exists a $p^* \in P^*$ with the property: for all $f \in F^*$, all $y^f \in Y^f(T)$, and all $p^f \in Q(y^f)$, $p_c^f \geq p_c^*$ for all $c \in C(y^f)$ ("no reservation price externalities").

The heuristics that justify giving this technical condition the title "no pecuniary externalities in the innovation process" may be explained.

Condition (C1) says that potential customers of innovators are not innovators. It rules out the possibility of "intermediate innovations": innovations by one firm that are useful to the production of other innovations by other firms. More specifically, it rules out the possibility that an innovation $c_1 \in C(f_1) \backslash T$ may be useful to the production of another innovation, $c_2 \in C(f_2) \backslash T$, where $f_2 \neq f_1$. If this possibility were not ruled out there could be favorable pecuniary externalities in the innovation process since if f_1 innovated c_1 it would reduce the shadow price of c_1 perceived by f_2 from infinity to some finite amount, aming c_1 available to him and thus perhaps enabling him to innovate c_2.[5]

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[4] If the aggregate technology is convex and differentiable, then there exists a well-defined minimum $q \in P^*$, as will be shown in Section 4C below.

Consider (C2) next, and suppose that all potential innovators cooperated and introduced all possible innovations simultaneously beginning from (x, p). Suppose further that this joint action is at a sufficiently small scale so that each $f \in F^*$ earns the economy's reservation prices for its innovations *when all other possible innovations are available.* Then each $f \in F^*$ can expect to receive the prices in p^* for its innovations, where p^* is the "minimum" $q \in P^*$. But if any $f \in F^*$ contemplates innovating a $y^f \in Y^f(T)$ independently beginning from (x, p), then it can expect to earn only the prices in p^f for its innovations, where $p^f \in Q(y^f)$.

If $p_c^f < p_c^*$ for some $c \in C(y^f)$ then there would be favorable pecuniary externalities in the innovation process since if all potential innovations were available some potential innovator could earn more for some innovation than he perceives he can earn operating independently. Or, alternatively expressed, if (C2) were not satisfied some innovator f_1 by innovating a $c_1 \in C(f_1) \backslash T$ may be able to favorably effect the reservation price perceived by another innovator, f_2, for some $c_2 \in C(f_2) \backslash T$.

THEOREM 3. *If (x, p) is a FWE in T and there are no potential pecuniary externalities in the innovation process from T then x is a Pareto optimum. Indeed, there exists a $p^* \in P^*$—namely, the p^* satisfying* (C2)*—such that $(x, p^*) \in W(C)$.*

Proof. We need only verify that $(x, p^*) \in W(C)$, since then obviously x is a Pareto optimum. Let p^* satisfy the property in (C2). Since for any $j \in J^*$ $x^j \in X^j(x, p^*, C)$, to complete the proof we need only verify that for any $f \in F^*$ and any $y^f \in X^f$, $-p^*x^f \geqslant -p^*y^f$. Since $y^f \in Y^f(T)$ by (C1), $-px^f \geqslant -p^fy^f$, where $p^f \in Q(y^f)$. But $-px^f = -p^*x^f$, so $-p^*x^f \geqslant -p^fy^f$. Since $p_c^f \geqslant p_c^*$ for all $c \in C(y^f)$, $-p^fy^f \geqslant -p^*y^f$; so, $-p^*x^f \geqslant -p^*y^f$.

Q.E.D.

Remark 4.1. The role that "no reservation price externalities" plays in this model is analogous to the role played by the "simultaneous reservation price property" in Hart [4]. Analogous to [4], we provide sufficient conditions for guaranteeing (C2) in Section 4C below. (C1) is satisfied automatically in Hart [2–4] since he studies the special case in which only consumption goods are innovated.

Violations of (C1) are quite pernicious to the optimality of any FWE, as will be illustrated in Example 4.1 below. To find sufficient conditions to guarantee (C1) we will have to allow the structure of firms to vary endogenously; this is the program of Section 5.

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[5] Condition (C1) is slightly stronger than a perhaps more intuitive formulation of "no intermediate innovations from T": that for each $f \in F^*$ $Y^f(T) = X^f$. It is equivalent to this formulation if each potential innovator's trading possibility set is convex. The formulation in the text is favored because it greatly simplifies the exposition.

4B. *Some Examples of Inefficient FWE*

The following examples are intended to give some perspective to the sequel. They illustrate the intuitive proposition that there will be no potential favorable pecuniary externalities in the innovation process from T iff there are no interfirm complementary innovations in C^*; so the availability of some $c_1 \in C(f_1)\setminus T$ does not raise the economy's demand price (or lower its supply price) for any $c_2 \in C(f_2)\setminus T$, where $f_1 \neq f_2$. That is, interfirm complementary innovations (in quantity space) correspond to favorable pecuniary externalities (in price space). Alternatively expressed, interfirm innovation complementarities and favorable pencuniary externalities are "dual" phenomena. This intuitive proposition will be useful for understanding the sufficient conditions for (C2) provided in Section 4C below.

So, we shall now provide four illustrations of how an economy can get stuck at a suboptimal FWE in the presence of interfirm complementarities. The first example involves a failure of (C1) and the other three involve failures of (C2). In all examples (x, p) is a FWE in which neither c_1 nor c_2 is traded, where $\{c_1\} = C(f_1)$ and $\{c_2\} = C(f_2)$ (i.e., there is no joint production of c_1 and c_2); and y dominates x, where y involves trading in both c_1 and c_2.

Example 4.1 (intermediate cost-saving innovations or an example of a complementary input–output innovation). Suppose c_1 is a consumption good and c_2 is an input that significantly reduces f_1's cost of producing c_1. The unit cost of producing c_1 without c_2 exceeds any individual's reservation price for c_1. But c_2 has no significant uses other than to produce c_1; so, the reservation price of all firms including f_1 for c_2—given that the set of traded commodities T excludes c_1—is less than f_2's minimum unit cost of producing c_2.

This sort of example may occur when a new technique or, more typically, a number of new techniques are needed to economically produce a new product. This is a common occurrence. For example, consider the number of new "components" needed to produce a new type of airplane or a new type of instant-picture photo camera.

Example 4.2 (complementary consumption innovations). Suppose that that c_1 and c_2 are consumption goods that individuals view as perfect complements (i.e., they have right angle indifference curves between the goods). An example may be nuts and bolts, as in Hart [4]. Then any individual's reservation price for c_1 in the absence of c_2 is zero, and his reservation price for c_2 in the absence of c_1 is zero. So if both goods are costly to produce, firms f_1 and f_2 will remain out of business. Yet the minimum unit cost of a c_1–c_2 combination may be significantly less than the sum of individuals' reservation prices for c_1 *and* c_2 in one-to-one combination.

EXAMPLE 4.3 (consumption complementarities arising from overhead relocation costs, or an example of underdeveloped land). Suppose c_1 (respectively c_1') is housing in location L (respectively L'), and c_2 (c_2') is a recreational service—golfing—in location L (L'). Suppose also that L is an area of undeveloped vacant land outside a city located at L', and that housing and golfing are very expensive in L' because the opportunity cost of land is high in the city. So, equivalent housing and golfing can be supplied much more cheaply at L. But if an individual only resides (golfs) in L, he has to commute to the city to work and golf (work and live), so any potential resident's reservation price for c_1, given that he must work and golf in L', is less than the unit cost of housing in L; and, similarly, any potential golfer's reservation price for such recreation at L exceeds the minimum unit cost of providing golfing services at L. So, L remains undeveloped. But the sum of the unit costs of c_1 and c_2 may be significantly less than the sum of individuals' reservation prices for c_1 and c_2 when jointly supplied.

A numerical illustration may help fix the idea underlying this example. Suppose a representative individual i has a Cobb–Douglas utility function given by $(c_0'^i)^\alpha(c_1^i + c_1'^i)^\beta(c_2^i + c_2'^i)^\gamma$, where $c_0'^i$ represents i's consumption of leisure, and the exponents α, β, γ are positive and sum to one. Note that i views c_1 and c_1' (c_2 and c_2') as perfect substitutes. Let p_1 and p_2 (p_1' and p_2') represent the prices of housing and golfing at $L(L')$. Let p_0' represent the price of leisure, i.e., the wage rate at L'; we assume that i must work in the city. Then it is easy to verify that if i lives and golfs in the city, he can attain utility

$$U' = \left(\frac{\alpha Y}{p_0'}\right)^\alpha \left(\frac{\beta Y}{p_1'}\right)^\beta \left(\frac{\gamma Y}{p_2'}\right)^\gamma,$$

where Y is the value of i's endowment (i.e., his "income"). Now also suppose that if i golfs or resides at L he must suffer a relocation cost which we assume to be merely a lump-sum loss of leisure, $\bar{c}_0'$. So, if i lives and/or golfs at L, he can attain utility

$$U(p_1, p_2) = \left(\frac{\alpha(Y - p_0'\bar{c}_0')}{p_0'}\right)^\alpha \left(\frac{\beta(Y - p_0'\bar{c}_0)}{\min(p_1, p_1')}\right)^\beta \left(\frac{\gamma(Y - p_0'\bar{c}_0')}{\min(p_2, p_2')}\right)^\gamma.$$

Individual i's reservation price for c_1 in the absence of golfing at L is given by $p_1{}^R$, where $U(p_1{}^R, \infty) = U'$. Similarly, his reservation price for c_2 in the absence of housing at L is given by $p_2{}^R$, where $U(\infty, p_2{}^R) = U'$. It is easy to verify that if $p_0\bar{c}_0 > 0$, then $p_1{}^R < p_1'$ and $p_2{}^R < p_2'$. So, if AC_1 (AC_2)

represents the minimum unit cost of c_1 (c_2), it is possible that $p_1^R < AC_1 < p_1'$ and $p_2^R < AC_2 < p_2'$; and, consequently, neither c_1 nor c_2 is produced. But if AC_1 and AC_2 are sufficiently close to p_1^R and p_2^R, respectively, then clearly $U(AC_1, AC_2) > U'$ so c_1 and c_2 should be jointly produced.

It is of some interest to observe that the complementarity between c_1 and c_2 arises in this illustration not because of lack of differentiability or convexity in i's utility function, but because of an implicit nonconvexity in i's trading possibility set X^i: i can "spread" the overhead relocation cost $\bar{c}_0$ over both c_1 and c_2 if both are available at L.

EXAMPLE 4.4 (input complementarities arising from transportation costs, or an example of an underdeveloped region): Suppose c_1 is steel in location L and c_2 is railroad transporting services from L to L', where L is a region of largely undeveloped land (perhaps a sleepy community) and L' is a major automobile manufacturing center. Suppose also that L is an "ideal" location for producing steel because of favorable natural resources; but there is no railroad connecting L with L', so transporting steel from L to L' is prohibitively costly. As a consequence, the reservation price of the car industry, centered at L', for c_1 is significantly less than the minimum unit cost of c_1. And without a steel industry at L the economy's reservation price for c_2 is insignificant, so f_2 sees no incentive to build a road connecting L with L'. Yet if c_2 were supplied by f_2 at cost, firm f_1 could achieve significant profits from supplying c_1.

Note that in this example, a car manufacturer at L' views a unit of c_1 and a unit of (some) transporting service from L to L' as perfect complements. The above problem occurs when rail transport would be significantly cheaper than any available means of transporting steel from L to L'.

Remark 4.2. The reader may recognize Example 4.4 from [11] or [5]. Our analysis of it is consistent with that appearing there. Note that if f_2 needs c_1 to economically produce c_2, then the example involves a failure of (C1) rather than (C2). Also note that we have strained our analysis somewhat in applying it to railroad transporting services, which generally would not be small relative to f_2's market.

Remark 4.3. In addition to examples of underdeveloped regions, the above "methodology" for constructing examples of inefficient FWE can be applied to constructing examples of badly developed regions. Apropos, see the interesting "shopping center example" in [4].

4C. *Sufficient Conditions for No Reservation Price Externalities: Aggregate Convexity and Differentiability*

We show that if the aggregate technology is "convex and differentiable," assumption (A1) below—and some essentially technical assumptions, (A2)–

(A5), also hold—then (C2) will be satisfied. Thus, in view of Theorem 3, in economies satisfying (C1) (e.g., economies in which only consumption goods can be innovated), (A1)-(A5) are sufficient for a FWE to be a Pareto optimum.

To put the role of convexity and differentiability into perspective recall that all our examples of inefficient FWE (except Example 4.1 which involves a failure of (C1)) involve failures of one or the other, and that failures of differentiability are intimately connected with the existence of complementarities.

The importance of differentiability may be explained mathematically with reference to Example 4.2. As we are about to see, the key mathematical fact for intuitive understanding is that only differentiable functions have the property that their directional derivates are linear functions of their partial derivatives. Now consider Example 4.2 and, in particular, consider an individual's (nondifferentiable) utility function. At his individual optimum, being at a "corner," the sum of his partial derivatives (marginal utilities) with respect to c_1 and c_2 is less than his directional derivative for a package consisting of a unit each of c_1 and c_2. The partial derivatives (which equal zero) reflect his reservation prices for c_1 and c_2 alone, whereas the directional derivative (which is positive) reflects his reservation price for c_1 and c_2 in combination. Thus, in this example, there would be favorable pecuniary externalities to the producer of one of the commodities from the introduction of the other commodity. Alternatively expressed, only for differentiable functions, which have the property mentioned earlier, is it in general the case that experimenting in a vertical and horizontal direction (thinking about introducing one commodity at a time) gives one correct information about the profit consequences of "diagonal" experiments (thinking about simultaneously introducing several commodities).[6]

For our purpose the set-theoretic analog of differentiability at a point, at least for quasi-concave functions, is a unique support; e.g., a right angle indifference curve has an infinite number of supports at its corner. In the sequel we shall use this analog of differentiability and convexity rather than introduce quasi-concave, differentiable utility and production functions. The formal characterization follows. Let $E \subset R^n$ and $z \in R^n$. The *supports of E at z* are $S_z(E) = \{q \in R^n : qz \leqslant qE\}$. $S_z(E)$ is obviously a nonempty (contains zero) convex cone. E is a *smooth, convex set* if for any $z \in \text{cl } E$ there exists a $q' \in R^n$, $q' \neq 0$, such that $S_z(E) \subset \{q : q = \lambda q' \text{ where } \lambda \in R\}$. That is, there exists a *unique* support to E at z (up to a multiplicative constant).

Let $B \equiv \sum_i B^i(x^i) + \sum_{f \in J^*} X^f$; it represents the aggregate "better than x set" net of what is realizable *without* F^*. And let $B(T') \equiv B \cap \{z \in R^n$:

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[6] Differentiability is crucial to proving Grossman's characterization result [1], for exactly analogous reasons. It is also crucial for perfect competition, as shown by Ostroy in [8].

$z_c = 0$ for all $c \notin T'\}$; it represents the aggregate better than x set when trading is restricted to the subset of commodities T'. We assume

(A1) There exists a smooth, convex set $B^* \subset R^n$ such that $B = B^* \bigcap_{c \in C(F^*)} N(c)$, where $N(c) = \{z \in R^n : z_c \geq 0\}$ and $C(F^*) = \bigcup_{f \in F^*} C(f)$ ("aggregate convexity and differentiability").

(A2) For any $T' \supset T$, ri $B(T') \neq \varnothing$.

(A3) For any $(x, p) \in W(T)$, $\sum_{j \in J^*} x_c^j > 0$ for all $c \in T \cap C(F^*)$.

(A4) For any $q \in R^n$ and $i \in I$, $qx^i \leq qB^i(x^i)$ implies $qx^i < qB^i(x^i)$ provided $q_c = p_c$ for all $c \in T$.

(A5) For any $f \in F^*$, $i \in I$, $y^f \in y^f(T)$, and $(y, p^f) \in Q(y^f)$: $s(i, f) > 0$ implies there exists an $\bar{i} \in I$ such that $s(\bar{i}, f) = 0$ but $X^{\bar{i}} = X^i$, $B^{\bar{i}} = B^i$, and $\sum_{f' \in F} s(\bar{i}, f') p^f y^{f'} = \sum_{f' \in F} s(i, f') p^f y^{f'}$ ("individuals are typical in any quasi-equilibrium").

Let $(x, p) \in W(T)$ and let $\bar{x} = \sum_{j \in J^*} x^j$. Observe that given (A4) $q \in S_{\bar{x}}(B)$ implies $q \in P^*$ (provided $q_c = p_c$ for all $c \in T$). With this background, the role of B^* in the key assumption, (A1), may be explained. B^* "extends" B to the negative orthants in $C(F^*)$, permitting a characterization of J^*'s reservation prices for untraded commodities in $C(F^*)$, in particular for C^*. (Since the commodities in $C(F^*)$ are personalized to F^*, B does not include points in these orthants.) To illustrate, suppose $T = \{x_1\}$ and $C = \{c_1, c_2\}$. Then if B is as in Fig. 1, B may be extended to a smooth, convex B^* (see dashed line) the slope of whose *unique* support at $\bar{x}$ reflects J^*'s reservation price for c_2 in terms of c_1.

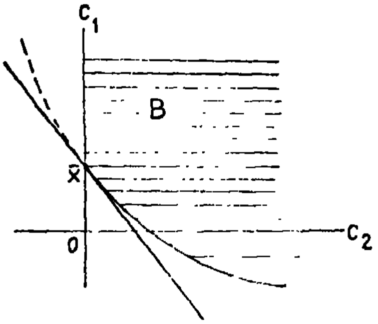

FIG. 1. Extending B to a smooth, convex set.

Consider this analogy to the role of B^*: If an individual's consumption set is R_+^n and x^i involves zero consumption of c, then the individual's utility function defined on R_+^n may have a right-hand partial derivative with respect to c at x^i but no left-hand partial derivative unless the function can be extended to a differentiable function defined on an open set containing R_+^n.

Regarding the technical assumptions, (A2)–(A5), we make three remarks. First, (A3) and (A4) could obviously have been derived from more primitive restrictions on the technology. Second, (A3) says that any $f \in F^*$ who markets some $c \in T$ has some customers in J^* (i.e., not all buyers of any $c \in C(f) \cap T$ are potential innovators). Third, (A5) says that in any quasi-equilibrium $Q(y^f)$, any shareholder i in f has a "double" who is not a shareholder in f but has the same trading possibilities, preferences, and wealth as i. Its role is to eliminate small income effects. In particular, it ensures that when an $f \in F^*$ switches from x^f to $y^f \in Y^f(T)$ and its profits change, the resulting change in the incomes to f's shareholders will not influence the economy's reservation prices for f's innovations.

We shall now prove

THEOREM 4. *Assume the economy satisfies* (C1) *and* (A1)–(A5), *and let* $(x, p) \in W(T)$. *Then there exists a* $p^* \in P^* \cap S_{\bar{x}}(B)$ *satisfying* (C2); *i.e., there would be no reservation price externalities in the innovation process from* T. *Thus, if* $(x, p) \in FWE$ *then* $(x, p^*) \in W(C)$ *and* x *is a Pareto optimum.*

Proof. To prove the theorem we utilize the following Fact, proved in [9] as Corollary 23.8.1:

> Let $B_1 \cdots B_m$ be convex sets in R^n whose relative interiors have a point in common. Then for any $z \in R^n$
>
> $$S_z(B_1 \cap \cdots \cap B_m) = S_z(B_1) + \cdots + S_z(B_m).^7$$

To proceed, observe that $\bar{x} \in \text{cl}\, B(T)$. So, since $p \in S_{\bar{x}}(B(T))$, (A1) plus the Fact implies

$$p_c = p_c^* + \lambda_c \text{ for all } c \in T, \text{ where } p^* \in S_{\bar{x}}(B^*) \text{ and where}$$
$$\lambda_c \geq 0, \lambda_c = 0 \text{ if } c \notin C(F^*) \cap \{c: \bar{x}_c = 0\}. \tag{4.1}$$

Given (A3) and (A4), clearly $p_c^* = p_c$ for all $c \in T$ and $p^* \in P^*$. Thus, in view of Theorem 3, to complete the proof we need only verify that p^* satisfies the property in (C2).

Let $f \in F^*$, $(y, p^f) \in Q(y^f)$, and $x(f) = \sum_{j \in J^*} x(f)^j$, where each $x(f)^j \in X^j(x, p, T)$. For any i such that $s(i, f) = 0$, $p^f x(f)^i = p^f y^i \leq p^f B^i(y^i, T(y^f)) = p^f B^i(x^i, T(y^f))$. And for any $f' \in J^*$, $p^f x(f)^{f'} = p^f y^{f'} \leq p^f X^{f'}(T(y^f))$. Thus, given (A5), $p^f \in S_{x(f)}(B(T(y^f)))$. So, (A1) plus the Fact implies:

$$p_c^f = q_c + \lambda_c^f \text{ for all } c \in T(y^f), \text{ where } q \in S_{x(f)}(B^*) \text{ and where}$$
$$\lambda_c^f \geq 0, \lambda_c^f = 0 \text{ if } c \notin C(F^*) \cap \{c: x(f)_c = 0\}. \tag{4.2}$$

---

[7] This Fact also plays a crucial role in a proof of the Kuhn–Tucker theorem; see [6]. This is no accident. An alternate proof of Theorem 4 relies on the Kuhn–Tucker theorem. But for this proof one must assume convexity and differentiability for each agent (analogous to Hart [2]), not just aggregate convexity and differentiability.

But $p^*x(f) = p^*\bar{x}$, so $p^* \in S_{x(f)}(B^*)$. Thus, given the smoothness of B^* and the fact that $p_c^f = p_c \neq 0$ for some $c \in \mathbf{T}$ (recall $(x, p) \in W(T)$ and individuals are locally nonsatiated in $\mathbf{T}$), $q = p^*$. Substituting into (4.2) yields $p_c^f \geq p_c^*$ for all $c \in C(y^f)$. Q.E.D

5. THE STRUCTURE OF FIRMS

In this section we adopt another strategy for achieving optimality. We do not restrict the economy's technology so as to rule out interfirm complementarities, e.g., with (C1) and (A1). Now we allow complementarities, but we also allow firms to *integrate* or to *cooperate* in their innovation plans, i.e., to effectively change the structure of firms (as decision-making units).

It is widely believed that if there is perfect competition and constant returns to scale (or additive production sets) then the structure of firms is completely indeterminate. In particular vertical and horizontal integration, e.g., through mergers or joint ventures, is inexplicable. We show that "complete markets," or absence of interfirm complementarities in innovation, are crucial to the validity of this belief. In particular, we show that there exist incentives to vertical and/or horizontal integration—even given that there are constant returns to scale and additivity—whenever there are interfirm complementarities preventing the innovation process (under an arbitrary, fixed firm structure) from reaching an optimum.

Let us suppose that F now represents the set of basic firms in the economy, before any vertical or horizontal integration. And let us suppose that if in some $W(T)$, (x, p), firms $F' \subset F$ cooperate (e.g., by merger or joint venture) then the cooperative "firm" $f(F')$ will have trading possibilities $X^{f(F')} = \sum_{f \in F'} X^f$—so there are no economies or diseconomies from cooperation—and $f(F')$'s profits will be distributed to shareholders in F' according to the "ownership shares in $f(F')$"

$$s(i, f(F')) = \frac{\sum_{f \in F'} s(i, f)\, px^f}{\sum_{f \in F'} px^f}.$$

Let $\mathscr{F}^*$ be the family of *all* subsets $F' \subset F$, and let $\mathscr{F}$ be a given subset of $\mathscr{F}^*$. $\mathscr{F}$ represents the sets of basic firms $F' \subset F$ that can cooperate in (x, p). Thus, vertical and horizontal integration among firms is always (respectively, not) possible in (x, p) if $\mathscr{F} = \mathscr{F}^*$ (respectively, $\mathscr{F} = \mathscr{F}_* = \bigcup_{f \in F} \{f\}$). And thus, $(x, p) \in$ FWE now only implies that (x, p) is a full Walrasian equilibrium when cooperation among firms is not possible in (x, p). Let us assume $\mathscr{F} \supset \mathscr{F}_*$. What happens when $\mathscr{F} \neq \mathscr{F}_*$?

Let $\mathscr{E}(F')$ represent the economy consisting of, as before, (a) individuals $i \in I$ with trading possibilities X^i and preferences A^i, but also, (b) firms

$f' \in F(F') \equiv F \backslash F' \cup f(F')$ with trading possibilities $X^{f'}$ and (c) ownership shares defined by the function s above from $I \times F(F')$ to R_+. Obviously our formal model of Sections 1–3 applies to any $\mathscr{E}(F')$, i.e., to *any* given firm structure $F(F'), F' \in \mathscr{F}^*$. Furthermore, it is easy to verify that if $(x, p) \in W(T)$, then for any $F' \in \mathscr{F}^*$ $(x(F'), p) \in W(T)$ for the economy $\mathscr{E}(F')$, where $x(F')^j \equiv x^j$ for all $j \notin F'$ and $x(F')^{f(F')} \equiv \sum_{f \in F'} x^f$.[8] This is consistent with the widely held belief mentioned earlier. But the same invariance does *not* in general apply to FWE under various firm structures, as we shall shortly see.

If $(x(F'), p) \in W(T)$ for $\mathscr{E}(F')$ and F' is a perfect competitor in $(x(F'), p)$ —i.e., when F' operates as the cooperating unit $f(F')$—then Theorem 1 says that shareholders in F' (or, equivalently, in $f(F')$) unanimously favor $f(F')$ to profit maximize in $\mathscr{E}(F')$. And, given the definition of $s(\ldots, f(F'))$ each shareholder in $f, f \in F'$, is at worst indifferent whether or not f joins F' to form $f(F')$ in the $W(T)$ (for $\mathscr{E}(\{f\})$), (x, p). Thus, given the possibilities for cooperation in the "super economy" with cooperation (these possibilities being summarized by $\mathscr{F}$), it is natural to define a *full Walrasian equilbrium given the possibilities for horizontal and vertical integration*, a FWE ($\mathscr{F}$), as an $(x, p) \in W(T)$ (for some $T \subset C$) satisfying for *each* $F' \in \mathscr{F}$ $(x(F'), p) \in$ FWE in T for the economy $\mathscr{E}(F')$. That is, there are no profits to innovation for any firm or possible coalition of firms in (x, p).

It may be helpful to remark at this point, before proceeding, that it is easiest to visualize the innovation process when $\mathscr{F} \neq \mathscr{F}_*$ in terms of the two-step sequence mentioned in Remark 2.2. Thus, if a firm leaves one coalition to enter another, it can assume that all firms outside the new coalition—including firms in the old coalition that do not enter the new one—will not change their marketed commodities (recall the definition of $T(y^f)$, and the consequent definition of $T(y^{f(F')})$): The firms in the old coalition may be visualized as being in Step 2 while the new coalition is embarking on Step 1. So, the firms in the old coalition that are not in the new one do not lose any necessary inputs by the old coalition breaking up; in Step 2 there are many available suppliers of all necessary inputs to these firms.

For the sequel, we extend our earlier assumption of perfect competition to assume that any possible coalition $F' \in \mathscr{F}$ would be a perfect competitor in any $W(T)$ in $\mathscr{E}(F')$.

COROLLARY 1 (profits to integration). *Suppose* $(x, p) \in W(T)$ *and some* T'*-feasible allocation* y *dominates* x. *Then, if* $T' \backslash T \subset \bigcup_{f \in F'} C(f)$ *for some*

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[8] In the current context, when we simply write $(x, p) \in W(T)$, it should be understood that we mean $(x, p) \in W(T)$ for the economy $\mathscr{E}(\{f\}]$, where $f \in F$. This is consistent with our previous notation.

$F' \in \mathscr{F}$, there are profits to innovation by $f(F')$ in $\mathscr{E}(F')$; i.e., (x, p) is not a FWE($\mathscr{F}$).

Proof. Follows immediately from Theorem 2. Q.E.D.

COROLLARY 2. *If $(x, p) \in$* FWE($\mathscr{F}$) *and integration is always possible (i.e., $\mathscr{F} = \mathscr{F}^*$) then x is a Pareto optimum.*

Proof. Immediate from Corollary 1. Q.E.D.

COROLLARY 3 (determinacy of firm structure). *Suppose $(x, p) \in$* FWE($\mathscr{F}$) *when integration is not possible (i.e., when $\mathscr{F} = \mathscr{F}_*$). Then $(x, p) \in$* FWE($\mathscr{F}$) *even when any integration is possible (i.e., when $\mathscr{F} = \mathscr{F}^*$) iff x is a Pareto optimum in $\mathscr{E}(\{f\})$.*

Proof. Given Corollary 2, we need only show that if x is a Pareto optimum then there are no profits to innovation for any $f(F')$ in $(x(F'), p)$, a $W(T)$ for $\mathscr{E}(F')$. Assume the contrary, i.e., that there exists a $y^{f(F')} \in Y^{f(F')}(T)$ such that $p^{f(F')}y^{f(F')} > -px(F')^{f(F')} = -p\sum_{f \in F'} x^f$, where $(y, p^{f(F')}) \in Q(y^{f(F')})$ in the economy $\mathscr{E}(F')$. By Theorem 1, y dominates $x(F')$ in the economy $\mathscr{E}(F')$. So, clearly, there exists a feasible allocation y' in $\mathscr{E}(\{f\})$ that dominates x. Q.E.D.

Corollary 3 says that the firm structure is completely indeterminate (in the sense that there are no potential profits to forming any cooperative ventures) iff the innovation process without cooperation leads to a Pareto optimum. But if there are interagent complementarities leading to inefficient FWE (without cooperatives), then Corollary 1 says there exist potential profits to cooperative efforts. In particular, when there are cost-saving intermediate innovation as in Example 4.1, then there exist incentives to *vertical* integration, and when there are complementary consumption innovation as in Examples 4.2 and 4.3, then there exist incentives to *horizontal* integration. Corollary 2 says that if agents can form cooperatives that are sufficiently large to internalize all interagent innovation complementarities then the innovation process will lead to an optimum.

Remark 5.1. It is a well-known empirical fact that many firms are "too large" to be explained by economies of scale. This theory provides one explanation consistent with competitive behavior. It explains a firm's simultaneous development of several, complementary processes or innovations even in the absence of joint economies of scale. See [13, esp. Sect. III], for a somewhat different model leading to a very similar explanation.

Remark 5.2. The large, multiproduct conglomerate, replete with a massive R & D department, may be viewed as a social invention for solving potential problems involving interfirm complementarities—producing, in some sense, profit signals that the price system fails to provide. But this does not imply that complex industrialized economies, possessing this invention, have solved the underdevelopment problem (in the sense of Pareto). Since agents in one society typically innovate "ideas" that follow a common known line of development, there may exist *completely different*, Pareto dominating lines of development (e.g., different life-styles with associated organizations) that remain nonexistent just because their development requires the simultaneous introduction of *many* complementary ideas/innovations, too many for any organizer—*including* any government—to figure out. They may appear in the future as a result of "lucky coincidences."

Remark 5.3. It is well known that joint ventures and multipurpose organizations are empirically common in the early phases of innovation and development; e.g., see [14, 12].

Remark 5.4. The process of forming cooperative ventures obviously deserves further analysis beyond the casual treatment of it given here. Our approach was adopted only to make the simple point that if an economy gets stuck at a suboptimal FWE then (a) there are unexploited potential profits to innovation for some cooperative; and (b) there is at least one distribution of these profits such that all shareholders in firms constituting the cooperative benefit.

ACKNOWLEDGMENTS

It is a pleasure to acknowledge that Earl Thompson, several years ago, pointed out to me the importance of complementarities for the innovation process, and pointed out the possibility of underdevelopment. He explicitly discusses this possibility [13]. I am also indebted to Thompson for the empirical fact mentioned in Remark 5.1. and the first observation in Remark 5.2. The empirical fact in Remark 5.1 was more recently mentioned to me by Richard Gilbert in a helpful conversation. Conversations with Joseph Ostroy were also very helpful. Finally, the work of Oliver Hart on monopolistic competition must be sighted as seminal, but for this see the text. All shortcomings of the paper are, of course, my responsibility.

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JOURNAL OF ECONOMIC THEORY 22, 243–255 (1980)

Small Efficient Scale as a Foundation for Walrasian Equilibrium*

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Received June 6, 1979

INTRODUCTION

We are concerned with a formal description of perfect competition in which small efficient scale[1] and the entry of firms occupy a central role. The set of aggregate technological possibilities for the economy is obtained by summing the production sets of a very large number of productive units, interpreted here as firms. These units are most efficient when their output is small (infinitesimal) relative to demand, and the classical case of *U*-shaped average cost is admitted in the analysis. Also, because efficient scale is small, firms have only an infinitesimal effect on price when confined to the region in which they make positive profit. This enables one to capture the notion that the demand curve appears flat to a firm, while at the same time demand price may change substantially with substantial changes in aggregate quantity. The mass of firms active in an equilibrium is determined by the conditions of supply and demand. Small changes in aggregate demand will typically change the list of firms which are present in an equilibrium. Since changes in taste cause some firms to leave the market and others to enter, and since there are usually firms "on the margin" of entry, entry plays an important role in the explanation of value.

Despite the fact that our description of a perfectly competitive economy fits quite nicely with the ordinary neoclassical conception, the reader will see

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* We wish to acknowledge the helpful comments of the referee and also to thank the National Science Foundation for financial support.

[1] The efficient scale of a firm is small if it takes a large number of such firms, each producing at minimum average cost (c), to satisfy demand at a market price equal to c.

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that both in detail and interpretation it differs in some important ways from modern formal competitive theory (see, e.g., Debreu [3]). For example, we will argue that the convexity of the set of aggregate production possibilities is irrelevant for the existence of competitive equilibrium. With small efficient scale and the absence of externalities, the aggregate production set is necessarily convex (by Richter's theorem, which asserts that integration with respect to a nonatomic measure yields a convex-valued correspondence), and with standard assumptions equilibrium exists. With small efficient scale and externalities, the aggregate production set is in general *not* convex, but again under general conditions equilibrium will exist. Small efficient scale, which only in the absence of externalities guarantees convexity, is the proper requirement for the existence of perfectly competitive equilibrium.

The argument of the paper proceeds as follows. We begin by defining a private ownership perfectly competitive economy $\mathscr{E} = (X_i, \omega_i, \succsim_i, Z(\cdot), s_i(\cdot))$ and defining its equilibria. Throughout the analysis, for the m consumers, preferences $\succsim_i$ are convex and initial endowments ω_i are interior to the convex consumption sets X_i.[2] The framework differs from that of standard competitive theory in that there are a continuum of productive units whose output must be integrated to generate a quantity that is significant when compared with demand. The set $Z(\beta)$ is the technology available to unit β, and $s_i(\beta)$ is the share of firm β owned by i. The notion of a variable profit assignments economy $\mathscr{E}' = (X_i, \omega_i, \succsim_i, Z, w(\cdot))$ is then introduced. In addition to the m consumers this requires a specification of the aggregate technology Z, and for each p, m numbers $w_i(p)$, which indicate how $\max p \cdot Z$ is shared among consumers: $\sum w_i(p) = \max p \cdot Z$. Variable profit assignments economies are of interest in their own right as they capture the notion of a regime in which the aggregate rent from the process of production is distributed to consumers according to a rule which depends on prices. (For example, an individual's share of $\max p \cdot Z$ may depend on his wage.) Standard techniques from competitive analysis are sufficient to showthat (loosely) every variable profit assignments economy $\mathscr{E}' = (X_i, \omega_i, \succsim_i, Z, w(\cdot))$ has an equilibrium provided Z is convex and $\{w_i(\cdot)\}$ are continuous for prices at which $\max p \cdot Z$ is well defined. This is Theorem 1.

In this paper we are concerned with particular variable profit assignments economies. These economies, e.g., $\mathscr{E}' = (X_i, \omega_i, \succsim_i, Z, w(\cdot))$, are associated with perfectly competitive private ownership economies, e.g., $\mathscr{E} = (X_i, \omega_i, \succsim_i, Z(\cdot), s_i(\cdot))$, by defining Z as the integral of the correspondence $\beta \rightarrow Z(\beta)$, and for every price p, and every individual i, defining $w_i(p)$ as the integral of i's share of the profits made by each productive unit. In the absence of externalities, when Z is well defined it will be convex

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[2] Rather than m consumers one may postulate m types of consumers, or with little additional difficulty, consider a continuum of consumers.

(by Richter's theorem) and mild conditions on the functions $s_i(\cdot)$ will guarantee that the functions $w_i(\cdot)$ have the continuity property necessary for the application of Theorem 1.

The central point here is that the convexity of Z does not in any way depend on the convexity of the sets $Z(\beta)$. From a descriptive point of view, the convexity of Z is a consequence of the fact that efficient increases in output require a larger number of active productive units, these units "come in" at minimum average cost, and in the order "lower minimum average cost comes in first." Thus, we have the rather classical description of increases in output being achieved by the successive addition of small scale productive units, with the necessity of using on the margin a unit which is less efficient than those which preceded it. If the production possibilities available to each firm are shared by an unbounded measure of similar firms, then constant returns to scale obtains in the aggregate. But, if firms differ greatly in their maximum efficiency, then the boundary of the aggregate production set will exhibit substantial curvature with changes in scale. In the latter case it is typical that economic rents will accrue to the productive sector. The ith agents share of these rents is a function of price, since price determines which firms are profitable. Some price vector may make every one of the firms which an agent owns unprofitable (then his share of economic rents will be zero), while another price vector may result in high profit for several of the firms which he owns. Even in economies with only two firms and two consumers, if the firms are not identical, and if each firm is not held equally by each individual, then the share of aggregate profit which is distributed to each agent will vary with price.

Under rather mild assumption on the correspondence $\beta \to Z(\beta)$, there is a natural correspondence between the equilibria of a private ownership economy $\mathscr{E} = (X_i, \omega_i, \succsim_i, Z(\cdot), s_i(\cdot))$ and the equilibria of the associated variable profit assignments economy $\mathscr{E}' = (X_i, \omega_i, \succsim_i, Z, w_i(\cdot))$. Specifically, $(p, x, z(\cdot))$ is an equilibrium of $\mathscr{E}$ implies $(p, x, \int z(\beta) d\beta)$ is an equilibrium of $\mathscr{E}'$, and with some technical assumptions (p, x, z) for $\mathscr{E}'$ implies there exists an equilibrium of $\mathscr{E}$, $(p, x, z(\cdot))$, such that $\int z(\beta)\, d\beta = z$. Similarly, there is a natural correspondence between the Pareto-efficient allocations of $\mathscr{E}$ and the Pareto-efficient allocations of $\mathscr{E}'$. If the allocation $(x, z(\cdot))$ is Pareto efficient for $\mathscr{E}$, then $(x, \int z(\beta)\, d\beta)$ is Pareto efficient for $\mathscr{E}'$, and if the allocation (x, z) is Pareto efficient for $\mathscr{E}'$ then there exists a Pareto-efficient allocation of $\mathscr{E}$, $(x, z(\cdot))$, such that $z = \int z(\beta)\, d\beta$. With these observations in hand the existence of equilibrium for the perfectly competitive private ownership economy $\mathscr{E}$ is established; this is Theorem 2. The striking feature of the statement of the result is the absence of the assumption that the production sets $Z(\beta)$ are convex. The general argument of the proof is that (a) the production set Z of the associated variable profit assignments economy $\mathscr{E}'$ is convex (Richter's theorem), (b) equilibrium exists for $\mathscr{E}'$

(Theorem 1), and (c) an equilibrium for $\mathscr{E}'$ is naturally associated with an equilibrium for $\mathscr{E}$. Theorem 3 gives conditions under which equilibria of a perfectly competitive private ownership economy $\mathscr{E} = (X_i, \omega_i, \succsim_i, Z(\cdot), s_i(\cdot))$ are Pareto efficient. The idea of the proof is (a) every equilibrium of $\mathscr{E}$ naturally corresponds to an equilibrium of the associated variable profit assignments economy $\mathscr{E}'$, (b) equilibria of $\mathscr{E}'$ are Pareto efficient (standard), and (c) Pareto-efficient allocations of $\mathscr{E}'$ naturally correspond to Pareto-efficient allocations of $\mathscr{E}$.

Finally, Theorem 4 and its corollary establish that every Pareto-efficient allocation of a perfectly competitive private ownership economy $\mathscr{E}$ is an equilibrium subject to a suitable assignment of ownership. Again, the striking feature of the statement is the absence of the assumption that production sets are convex. The general argument of the proof is (a) each Pareto-efficient allocation of $\mathscr{E}$ naturally corresponds to a Pareto-efficient allocation of $\mathscr{E}' = (X_i, \omega_i, \succsim_i, Z, w_i(\cdot))$, (b) the production set Z is convex (Richter's theorem), (c) every Pareto-efficient allocation of $\mathscr{E}'$ is an equilibrium of $\mathscr{E}'$ for a suitable assignment of ownership (standard), and (d) every equilibrium of $\mathscr{E}'$ naturally corresponds to an equilibrium of $\mathscr{E}$. The formal analysis is followed by six remarks which explain some consequences of our formulation; included here is a reinterpretation of the classical theorems of welfare economics. Externalities and taxation are introduced into the analysis, and a potential cause of market failure is identified which does not appear in the Arrow–Debreu theory.

Define $\Delta = \{p \in R_+^l\colon \sum_i p_i = 1\}$; integration is in the sense of Lebesgue. For integrals of correspondences see Hildenbrand [4, Sect. D, II].

THE MODEL

(a) A private ownership perfectly competitive economy $\mathscr{E} = (X_i, \omega_i, \succsim_i, Z(\cdot), s_i(\cdot))$ is

(a1) for each consumer $i = 1, 2,\ldots, m$, a consumption set $X_i \subset R^l$, an initial endowment vector $\omega_i \in R^l$, and a complete preference preordering $\succsim_i \subset X_i \times X_i$,

(a2) for each firm $\beta \in [0, \infty)$, a nonempty production set $Z(\beta) \subset R^l$,

(a3) for each (i, β) a nonnegative number $s_i(\beta)$ which indicates the fraction of firm β owned by individual i. For each β, $\sum_i s_i(\beta) = 1$.

(b) An equilibrium for the private ownership perfectly competitive economy $\mathscr{E} = (X_i, \omega_i, \succsim_i, Z(\cdot), s_i(\cdot))$ is a triple $(p^*, x^*, z^*(\cdot)) \in \Delta \times \Pi X_i \times$ (integrable functions from $[0, \infty)$ to R^l, where $z^*(\beta) \in Z(\beta)$ for all β) satisfying

(b1) $p^* \cdot x_i^* = p^* \cdot \omega_i + \int s_i(\beta)\, p^* \cdot z^*(\beta)\, d\beta$, $i = 1, 2,\ldots, m$, (b2) $\sum_i x_i^* \leqslant \int z^*(\beta)\, d\beta + \sum_i \omega_i$ and $p^* \cdot (\sum_i x_i^* - \int z^*(\beta)\, d\beta - \sum_i \omega_i) = 0$,

(b3) $x_i \succ_i x_i^*$ implies $p^* \cdot x_i > p^* \cdot \omega_i + \int s_i(\beta)\, p^* \cdot z^*(\beta)\, d\beta$, $i = 1, 2,\ldots, m$,

(b4) $p^* \cdot z > p^* \cdot z^*(\beta)$ implies $z \notin z(\beta)$, for a.e. β.

(c) Given the private ownership perfectly competitive economy $\mathscr{E} = (X_i, \omega_i, \succsim_i, Z(\cdot), s_i(\cdot))$, the associated variable profit assignments A–D economy $\mathscr{E}' = (X_i, \omega_i, \succsim_i, Z, w_i(\cdot))$ is defined by

(c1) $Z = \int Z(\beta)\, d\beta$

(c2) for all $p \in \Delta$,

$$\begin{aligned} w_i(p) &= \int s_i(\beta) \sup\{p \cdot z \colon z \in Z(\beta)\}\, d\beta && \text{if it exists and is finite} \\ &= 0 && \text{otherwise.} \end{aligned}$$

(d) An equilibrium for the variable profit assignments A–D economy $\mathscr{E}' = (X_i, \omega_i, \succsim_i, Z, w_i(\cdot))$ is a triple $(p^*, x^*, z^*) \in \Delta \times \Pi X_i \times Z$ satisfying

(d1) $p^* \cdot x_i^* = p^* \cdot \omega_i + w_i(p^*)$, $i = 1, 2,\ldots, m$,

(d2) $\sum_i x_i^* \leqslant z^* + \sum_i \omega_i$ and $p^* \cdot (\sum_i x_i^* - z^* - \sum_i \omega_i) = 0$,

(d3) $x_i \succ_i x_i^*$ implies $p^* \cdot x_i > p^* \cdot \omega_i + w_i(p^*)$, $i = 1, 2,\ldots, m$,

(d4) $p^* \cdot z > p^* \cdot z^*$ implies $z \notin Z$,

(d5) $p^* \cdot z^* = \sum_i w_i(p^*)$.

(e) An allocation for the private ownership perfectly competitive economy $\mathscr{E} = (X_i, \omega_i, \succsim_i, Z(\cdot), s_i(\cdot))$ is a pair $(x, z(\cdot)) \in \Pi X_i \times$ (integrable functions from $[0, \infty)$ to R^l, where $z(\beta) \in Z(\beta)$ for all β) satisfying $\sum_i x_i \leqslant \int z(\beta)\, d\beta + \sum_i \omega_i$. The allocation $(x, z(\cdot))$ is Pareto efficient if there does not exist an allocation $(x', z'(\cdot))$ for $\mathscr{E}$ with the property that $x_i' \succsim_i x_i$ for all i and $x_k' \succ_k x_k$ for some k.

(f) An allocation for the variable profit assignments A–D economy $\mathscr{E}' = (X_i, \omega_i, \succsim_i, Z, w_i(\cdot))$ is a pair $(x, z) \in \Pi X_i \times Z$ satisfying $\sum_i x_i \leqslant z + \sum_i \omega_i$. The allocation (x, z) is Pareto efficient if there does not exist an allocation (x', z') for $\mathscr{E}'$ with the property that $x_i' \succsim_i x_i$ for all i and $x_k' \succ_k x_k$ for some k.

THEOREMS

For reference we state an interchangeability lemma.

LEMMA 1. *If the graph of the correspondence $\beta \to Z(\beta)$ is measurable and $\int Z(\beta)\, d\beta \neq \varnothing$ then for every $p \in R^l$ $\sup\{p \cdot z \colon z \in \int Z(\beta)\, d\beta\} = \int \sup\{p \cdot z \colon z \in Z(\beta)\}\, d\beta$.*

Proof. The proof is similar to that of Proposition D, II, 6 of Hildenbrand [4] with minor modification since the measure is not a probability measure.

LEMMA 2. *Given the private ownership perfectly competitive economy* $\mathscr{E} = (X_i, \omega_i, \succsim_i, Z(\cdot), s_i(\cdot))$ *and the associated variable profit assignments A-D economy* $\mathscr{E} = (X_i, \omega_i, \succsim_i, Z, w_i(\cdot))$, *then*

(i) $(p^*, x^*, z^*(\cdot))$ *is an equilibrium of* $\mathscr{E}$ *implies* $(p^*, x^*, \int z^*(\beta)\, d\beta)$ *is an equilibrium of* $\mathscr{E}''$. *If the correspondence* $\beta \rightarrow Z(\beta)$ *is closed valued, has a measurable graph, and for each i,* $s_i(\cdot)$ *is measurable, then*

(ii) (p^*, x^*, z^*) *is an equilibrium for* $\mathscr{E}''$ *implies there exists an equilibrium of* $\mathscr{E}$, $(p^*, x^*, z^*(\cdot))$, *such that* $\int z^*(\beta)\, d\beta = z^*$.

Proof. Trivial using the interchangeability lemma for part (ii).

LEMMA 3. *Given the private ownership perfectly competitive economy* $\mathscr{E} = (X_i, \omega_i, \succsim_i, Z(\cdot), s_i(\cdot))$ *and the associated variable profit assignments A-D economy* $\mathscr{E}'' = (X_i, \omega_i, \succsim_i, Z, w_i(\cdot))$, *then*

(i) *the allocation* $(x, z(\cdot))$ *is Pareto efficient for* $\mathscr{E}$ *implies* $(x, \int z(\beta)\, d\beta$ *is Pareto efficient for* $\mathscr{E}''$, *and*

(ii) *the allocation* (x, z) *is Pareto efficient for* $\mathscr{E}'$ *implies there exists a Pareto-efficient allocation of* $\mathscr{E}$, $(x, z(\cdot))$, *such that* $z = \int z(\beta)\, d\beta$.

Proof. Trivial.

THEOREM 1. *Given the private ownership perfectly competitive economy* $\mathscr{E} = (X_i, \omega_i, \succsim_i, Z(\cdot), s_i(\cdot))$, *the associated variable profit assignments A-D economy* $\mathscr{E}'' = (X_i, \omega_i, \succsim_i, Z, w_i(\cdot))$ *has an equilibrium if*

(i) *for every i,* X_i *is closed, convex, bounded below,*

$\succsim_i$ *is continuous,*[3] *convex,*[4] *and there is no satiation consumption,* $\omega_i \in \operatorname{int} X_i$;

(ii) Z *is closed,* $Z \cap (-Z) = \{0\}$, *and* $Z \supset (-R_+^l)$: *and*

(iii) *there is a compact cube* $K \subset R^l$ *such that* K^{m+1} *contains all allocations* (x, z) *in its interior*[5] *and for every i,* w_i *is continuous and nonnegative on* $\hat{\Delta} = \{p \in \Delta: \max p \cdot (Z \cap K) = \sup p \cdot Z\}$ *and for every* $p \in \hat{\Delta}$, $\sum_i w_i(p) = \max p \cdot (Z \cap K)$.

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[3] $\succsim_i$ is continuous if, for every $x_i' \in X_i$, the sets $\{x_i \in X_i : x_i \succsim_i x_i'\}$ and $\{x_i \in X_i : x_i' \succsim_i x_i\}$ are closed in X_i.

[4] $\succsim_i$ is convex if $x_i' \succ_i x_i$ implies $\lambda x_i' + (1 - \lambda)x_i \succ_i x_i$ for all $\lambda \in (0, 1)$.

[5] Note that the definition of allocation requires feasibility of (x, z). By a standard argument (i) and (ii) imply that such a K exists.

Proof. By Richter's theorem $Z = \int Z(\beta)\,d\beta$ is convex, so by (ii) there is a $p \in \Delta$ such that $p \cdot Z \leq 0 = p \cdot 0$, and $\hat{\Delta}$ is nonempty. The set $\hat{\Delta}$ is also closed so for each i there is a continuous function $\bar{w}_i\colon \Delta \to R_+$ such that $\bar{w}_i(p) = w_i(p)$ for all $p \in \hat{\Delta}$ and $\sum_i \bar{w}_i(p) = \max p \cdot (Z \cap K)$ for all $p \in \Delta$. By application of a standard technique (the abstract economies approach) due to Arrow and Debreu [1] the variable profit assignments A-D economy $\mathscr{E}'_K = (X_i \cap K, \omega_i, \succsim_i, Z \cap K, \bar{w}_i(\cdot))$ has an equilibrium (p^*, x^*, z^*)which is also an equilibrium of $\mathscr{E}'$.

THEOREM 2. *The perfectly competitive private ownership economy* $\mathscr{E} = (X_i, \omega_i, \succsim_i, Z(\cdot), s_i(\cdot))$ *has an equilibrium if*

(i) *for every i, X_i is closed, convex, bounded below,*

$\succsim_i$ *is continuous, convex, and there is no satiation consumption,* $\omega_i \in \operatorname{int} X_i$;

(ii) *the correspondence* $\beta \to Z(\beta)$ *is closed valued with a measurable graph and* $0 \in Z(\beta)$ *for all* β;

(iii) $\int Z(\beta)\,d\beta = Z$ *is closed,*[6] $Z \cap (-Z) = \{0\}$, *and* $Z \supset (-R_+^l)$; *and*

(iv) *for every i, the function s_i is measurable.*

Proof. Consider the associated variable profit assignments A-D economy $\mathscr{E}' = (X_i, \omega_i, \succsim_i, Z, w_i(\cdot))$. By (i), (iii), and a standard argument there exists a compact cube $K \subset R^l$ such that K^{m+1} contains all allocations (x, z) in its interior. Using this K, assumption (iii) of Theorem 1 holds by (ii) and (iv). By Theorem 1 there is an equilibrium of $\mathscr{E}'$ and by Lemma 2(ii) there is an equilibrium of $\mathscr{E}$.

THEOREM 3. *Suppose that the perfectly competitive private ownership economy* $\mathscr{E} = (X_i, \omega_i, \succsim_i, Z(\cdot), s_i(\cdot))$ *satisfies the condition that X_i contains no point of local satiation for each i. If* $(p^*, x^*, z^*(\cdot))$ *is an equilibrium for* $\mathscr{E}$, *then* $(x^*, z^*(\cdot))$ *is Pareto efficient.*

Proof. Consider the associated variable profit assignments A-D economy $\mathscr{E}' = (X_i, \omega_i, \succsim_i, Z, w_i(\cdot))$. By Lemma 2(i), $(p^*, x^*, \int z^*(\beta)\,d\beta)$ is an equilibrium of $\mathscr{E}'$, so by a standard argument $(x^*, \int z^*(\beta)\,d\beta)$ is Pareto efficient in $\mathscr{E}'$. Thus by Lemma 3(ii), $(x^*, z^*(\cdot))$ (and any allocation $(x^*, z(\cdot))$ such that $\int z(\beta)\,d\beta = z^*$) is Pareto efficient in $\mathscr{E}$.

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[6] The closedness of Z is not implied by (ii). Additional assumptions could be made about the correspondence $\beta \to Z(\beta)$ to ensure that Z is closed. For example, if $Z(\beta)$ is the same for all β and for every nonzero y on the boundary of the least closed convex cone containing $Z(\beta)$ there is a (nonzero) multiple of y in $Z(\beta)$ then Z is closed.

THEOREM 4. *Let the private ownership perfectly competitive economy* $\mathscr{E} = (X_i, \omega_i, \succsim_i, Z(\cdot), s_i(\cdot))$ *be such that*

(i) *for every* i, X_i *is convex and* $\succsim_i$ *is continuous and convex, and*

(ii) *the correspondence* $\beta \to Z(\beta)$ *is closed valued and has a measurable graph.*

Given a Pareto efficient allocation $(x^*, z^*(\cdot))$, *where* x_k^* *is not a satiation consumption for some* k, *there is a nonzero price vector* $p^* \in R^l$ *such that*

(iii) *for each* i, x_i^* *minimizes* $p^* \cdot x_i$ *on* $\{x_i \in X_i : x_i \succsim_i x_i^*\}$,

(iv) *for a.e.* β, $p^* \cdot z > p^* \cdot z^*(\beta)$ *implies* $z \notin Z(\beta)$.

Proof. Consider the associated variable profit assignments A–D economy $\mathscr{E}' = (X_i, \omega_i, \succsim_i, Z, w_i(\cdot))$. The allocation $(x^*, \int z^*(\beta)\, d\beta)$ is Pareto efficient in $\mathscr{E}'$ by Lemma 3(i), and Z is convex by Richter's theorem so by a standard argument there exists a nonzero $p^* \in R^l$ such that (iii) holds and $[p^* \cdot z > p^* \cdot \int z^*(\beta)\, d\beta$ implies $z \notin Z]$. Then (iv) easily follows using (ii) and the interchangeability lemma.

COROLLARY. *If the conditions of Theorem* 4 *hold and* $x_i^* \in \operatorname{int} X_i$ *for each* i *and* $(-R_+^l) \subset Z = \int Z(\beta)\, d\beta$, *then* $(p^*, x^*, z^*(\cdot))$ *is an equilibrium for some assignment of endowments and ownership shares.*

Proof. Trivial.

Remark 1. If for each $j \in \{1,..., n\}$ and $\beta \in [0, \infty)$ we define $Z(j, \beta)$ to be the production set of firm j, β (with ownership shares $s_i(j, \beta)$), then $Z_j = \int Z(j, \beta)\, d\beta$ is convex. If we let $w_i(p, j)$ be consumer i's income from industry j at prices p, and $w_i(p) = \sum_{j=1}^{n} w_i(p, j)$ then the variable profit assignments A–D economy $\mathscr{E}' = (X_i, \omega_i, \succsim_i, Z_1,..., Z_n, w_i(\cdot))$ is quite similar to a standard Arrow–Debreu economy. However, the interpretation is different. Here Z_j is an industry production set, where the industry is made up of a continuum of infinitesimal firms, each of which has no effect on price. Thus the industry acts competitively, and as prices vary the industry output varies both because active firms change their productions and because firms enter and leave the market. This corresponds to the classical intuition about perfect competition.

Novshek and Sonnenschein [7] treated a special case where for each j, $Z(j, \beta)$ is the same nonconvex production set for all β. While each firm had efficient scale bounded away from zero (in the scale of the firm), the industry production set was a convex cone. These are the analogs of U-shaped average cost and horizontal supply in partial equilibrium. They consider a sequence of finite economies $\mathscr{E}_k$ with noninfinitesimal firms. Each economy has the same production sector with a countably infinite number of firms while the consumer sector in $\mathscr{E}_k$ is a k-fold replication of the consumer sector of $\mathscr{E}_1$. In per capita terms, as k becomes large the $\mathscr{E}_k$ converge to the perfectly competitive economy $\mathscr{E}$. In the finite economies firms set quantity in order to maximize profit, recognizing their effect on price.[7] They show that if $\mathscr{E}$ has an equilibrium satisfying a condition called DSD, then whenever the efficient scale of the firms is sufficiently small relative to the market (k is sufficiently large) a Cournot–Nash equilibrium with free entry exists for $\mathscr{E}_k$, and the set of Cournot–Nash equilibria with free entry converges to the set of (perfectly competitive) equilibria of $\mathscr{E}$ which satisfy DSD.[8] These results suggest that in the intuitive model of perfect competition, with entry and exit of many small (infinitesimal) firms producing the changes in output, only those equilibria which satisfy DSD are true equilibria. From this perspective, other equilibria are artifacts of a perfectly competitive specification in which firms are regarded as points rather than infinitesimals. For k large, no equilibria of $\mathscr{E}_k$ can exist near a Walras equilibrium of $\mathscr{E}$ at which DSD fails: entry is always profitable for additional firms.

The DSD condition is a "generalization" of Marshallian stability: at quantities "greater" than equilibrium, demand price is "less" than supply price, and thus prices provide the correct entry and exit signals. If $F(z_1, \ldots, z_n)$ is the function giving the prices that arise when industry productions are $z_1, \ldots, z_n$, and $(p^*, x^*, z_1^*, z_2^*, \ldots, z_n^*)$ is an equilibrium of $\mathscr{E}$, then $p^* = F(z_1^*, \ldots, z_n^*)$ and $p^* \cdot z_j^* = 0$ for all j since Z_j is a cone.

The DSD condition is satisfied at an equilibrium (p^*, x^*, z^*) if for each j, a "small increase in output" to $(1 + \lambda) z_j^*$ for λ small, positive, leads to prices $F(z_1^*, \ldots, (1 + \lambda) z_j^*, \ldots, z_n^*) = p(\lambda)$ such that $p(\lambda) \cdot z_j^* \leqslant 0$ so that entry is unprofitable. The DSD condition is not a pathological condition. It is satisfied at an equilibrium of $\mathscr{E}$ if a weak local version of the weak axiom of revealed preference holds at the equilibrium or if the consumer sector acts as a single consumer near the equilibrium. However, it is also not pathological for a specific equilibrium of $\mathscr{E}$ to fail to satisfy the DSD condition. Failure of DSD can occur in a two-commodity economy in which the aggregate production set is a cone and utility functions are homogeneous.

When firms in an industry do not have identical production sets the DSD condition must be carefully defined. At an equilibrium $(p^*, x^*, z_1^*, \ldots, z_n^*)$ of $\mathscr{E}$, the DSD condition must be applied to those firms which are not yet active and will therefore depend on the industry outputs z_j^*. If there is a positive measure of firms which are inactive in the equilibrium and each of which can "approximately" produce some vector y, then for DSD to be satisfied, a small change in production of λy for λ small, positive must lead to prices $p(\lambda)$ such that $p(\lambda) \cdot y \leqslant 0$ so that entry is unprofitable. If the firms in industry j can be ordered in terms of efficiency (e.g., $Z(j, \beta) \subset Z(j, \beta')$ if $\beta > \beta'$) and only firms $[0, \beta^0]$ are active in equilibrium then DSD must be checked by "entry" of the next most efficient firms, those in $(\beta^0, \beta^0 + \epsilon)$.

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[7] Prices are such that the excess demand of the price-taking consumer sector equals the asserted quantity actions of the firms.

[8] Some technical qualifications are needed; in particular the inverse demand function F must be C^2 in a neighborhood. This of course includes the requirement that F is defined in a neighborhood of the Cournot equilibrium.

Remark 2. Theorem 3 is often referred to as the first theorem of classical welfare economics; it asserts that competitive equilibria are Pareto efficient. Even with DSD added as a requirement for equilibrium, the conclusion will of course still hold. Theorem 4 is the second theorem of classical welfare economics; it asserts that Pareto-efficient allocations are perfectly competitive equilibria subject to a suitable assignment of ownership. Here the result is somewhat different than in the Arrow–Debreu theory. First, we observe that convexity of the set of aggregate production possibilities, which is one of the central hypotheses in standard treatments of the second welfare theorem, is not assumed; here it is a consequence of small efficient scale and the absence of externalities (Richter's theorem). Next we note that our definition of perfectly competitive equilibrium does not require DSD. Even if we succeed (by suitably assigning ownership) to support (in the conventional sense) a given efficient allocation as a perfectly competitive equilibrium, DSD may fail; therefore, prices may give the wrong entry signals, and in this case free entry and exit may drive the economy away from the given efficient allocation However, suppose, for example, that preferences are homothetic. If the efficient allocation is supported as a perfectly competitive equilibrium, and if endowments and ownership are distributed so as to be proportional among individuals, then the consumer sector will act as a single individual and equilibrium will satisfy DSD. This suggests that, rather than convexity, small efficient scale and the possibility of assigning ownership so that prices give the correct entry signals may be the important considerations in formulating the second welfare theorem.

Remark 3. The model can be extended to include externalities. We will indicate how the existence theorem, Theorem 2, can be modified to include production externalities such that each firm's production set depends on the aggregate production of each industry. Using the notation of Remark 1, where z_k is the aggregate production of industry k, let $Z(j, \beta, z_1, \ldots, z_n)$ be the production set of firm j, β when the aggregate productions are $z_1, \ldots, z_n$. Then the aggregate production set (by industry) $\mathscr{Z} = \{(z_1, \ldots, z_n): z_j \in \int Z(j, \beta, z_1, \ldots, z_n)\, d\beta$ for each $j\}$, $\mathscr{Z}_j = \{z_j: (z_1, \ldots, z_n) \in \mathscr{Z}$ for some $z_1, \ldots, z_{j-1}, z_{j+1}, \ldots, z_n\}$, and the aggregate production set $\mathscr{Y} = \{\sum_{j=1}^{n} z_j: (z_1, \ldots, z_n) \in \mathscr{Z}\}$ may not be convex. Let $\hat{\mathscr{Y}}$ (respectively $\hat{\mathscr{Z}}_j$) be the smallest closed convex cone containing $\mathscr{Y}$ (respectively $\mathscr{Z}_j$).

Condition (i) of Theorem 2 is unchanged while conditions (ii), (iii), and (iv) are modified to:

(ii)′ For all $(z_1, \ldots, z_n) \in R^{ln}$, $j \in \{1, \ldots, n\}$, the correspondence $\beta \to Z(j, \beta, z_1, \ldots, z_n)$ is closed valued with a measurable graph and $0 \in Z(j, \beta, z_1, \ldots, z_n)$ for all β.

(iii)′ $\mathscr{L}$ is closed, $\hat{\mathscr{L}} \cap (-\hat{\mathscr{L}}) = \{0\}$, $\hat{\mathscr{L}} \supset (-R_+^l)$, and the $\mathscr{L}_j$ are positively semi-independent.

(iv)′ For every i and j, the function $s_i(j, \cdot)$ is measurable.

We add a condition which is similar to condition (iii):

(v) $\sum_j \int Z(j, \beta, 0, \ldots, 0)\, d\beta \supset (-R_+^l)$ and

$$\Big(\sum_j \int Z(j, \beta, 0, \ldots, 0)\, d\beta\Big) \cap \Big(-\sum_j \int Z(j, \beta, 0, \ldots, 0)\, d\beta\Big) = \{0\}.$$

Finally we add a new condition which requires that the effect of externalities on production is continuous in the aggregate:

(vi) For all i and j, and for any nonempty compact cube $K \subset R^l$ centered at the origin, the correspondences $(z_1, \ldots, z_n) \to K \cap \int Z(j, \beta, z_1, \ldots, z_n)\, d\beta$ and $(z_1, \ldots, z_n) \to K \cap \int s_i(j, \beta) Z(j, \beta, z_1, \ldots, z_n)\, d\beta$ are continuous.

In order to prove the existence of an equilibrium with externalities under these conditions we consider an associated economy with

$$
\begin{aligned}
w_i(p, z_1, \ldots, z_n) &= \sum_{j=1}^{n} \int s_i(j, \beta) \sup\{p \cdot z \colon z \in Z(j, \beta, z_1, \ldots, z_n)\}\, d\beta \\
&\qquad\qquad\qquad\qquad \text{if it exists and is finite,} \\
&= 0 \qquad \text{otherwise.}
\end{aligned}
$$

The existence of an equilibrium for the associated economy follows, as in the proof of Theorem 1, from application of an abstract economy existence theorem, where the industry j production set (constraint set) given $(p, x_1, \ldots, x_m, z_1, \ldots, z_n)$ is $K \cap \int Z(j, \beta, z_1, \ldots, z_n)\, d\beta$. The existence of an equilibrium for $\mathscr{E}$ then follows as in Lemma 2(ii).

Remark 4. Clearly, perfectly competitive equilibria are not efficient in the presence of externalities of the type introduced in Remark 3. Even though with externalities the set of aggregate production possibilities is not convex, is it possible to state and prove a theorem that every efficient allocation is an equilibrium subject to the proper assignment of ownership shares and the appropriate Pigovian taxes?[9] This can be achieved; e.g., by applying the ideas of Shafer and Sonnenschein [8] to the specification of an economy given in Remark 3. One may conclude the convexity of the set of aggregate production possibilities has nothing to do with the second welfare theorem (with externalities and corrective taxes.) Once again, small efficient scale is the appropriate condition. As before, the possibility of not being able to distribute ownership so that DSD is satisfied remains a problem for the result; and with externalities, this may not be possible.

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[9] This is done in a more explicit setting by Chipman [2].

Remark 5. In the Arrow–Debreu theory the convexity of the set of aggregate productions Z plays a key role in establishing the existence of perfectly competitive equilibrium. As we have observed several times, without externalities, the required convexity is a consequence of Richter's theorem, and so does not have to be assumed. But with externalities present, Z may not be convex; nevertheless, the previous remark demonstrated that a general existence theorem still obtains. We assert that the convexity of the set of aggregate production possibilities is not relevant to the problem of the existence of competitive equilibrium. The existence of perfectly competitive equilibrium depends fundamentally on small efficient scale, which in any case is the economically natural condition for perfect competition to apply. Downward sloping demand in the appropriate region, so that prices give the correct entry signals, remains an additional possible requirement. These remarks are very much related to the work of Chipman [2] who makes some similar observations in a more specialized model.

Remark 6. McKenzie [5, 6] introduces a nonmarketed entrepreneurial factor which is private to the firm and owned by the owners of the firm, in order to prove the existence of equilibrium in an economy with convex, noncone production sets via an existence theorem for an economy with constant returns to scale. Application of a similar technique to the economy $\mathscr{E}$ would require the introduction of a continuum of new commodities, one for each firm.[10] On the other hand, application of the technique to the associated economy $\mathscr{E}'$ would yield "ownership shares" $w_i(p)/\sum_k w_k(p)$ which vary (perhaps discontinuously) with price.

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JOURNAL OF ECONOMIC THEORY **22**, 256–278 (1980)

The Limit Points of Monopolistic Competition*

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Received August 1, 1979

1. INTRODUCTION AND SUMMARY

A standard justification for the study of perfect competition is that in large economies, the set of monopolistically competitive equilibria will be approximated by the set of perfectly competitive equilibria. It is the primary purpose of this paper to examine the general validity of this proposition.

In the present context, the distinguishing feature of monopolistic competition is that there exist agents who act in an individually rational way by exploiting the fact that their actions influence prices. However, monopolistic competition eschews the notions of group rationality that are to be found in equilibrium concepts like the core; in particular, monopolistically competitive equilibria will usually be Pareto inefficient. Thus, although the famous limit theorem of Debreu and Scarf [6], showing that core allocations are approximated by perfectly competitive equilibria in large economies, is extremely suggestive, it is not of direct relevance to the present problem.

The possible validity of the proposition rests upon the idea that in large economies, individually rational behavior is "close" to price-taking behavior. In a recent study, Roberts and Postlewaite [14] showed that under seemingly weak assumptions, this was the case. It will become clear from the present study that such a result is not so generally true as Roberts and Postlewaite have suggested.

In keeping with the literature on limit theorems for the core, this paper adopts the replication technique of Debreu and Scarf and, within this framework, investigates the limit points of monopolistically competitive equilibria. This framework of analysis is similar to that of Gabszewicz and Vial [8]. The main difference between the two analyses, and the explanation for the difference in results, is that Gabszewicz and Vial invoke very strong assumptions. In this paper, a general approach is adopted and the assumptions

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* Financial support from the National Science Foundation under Grant SOC78-09084 at the Massachusetts Institute of Technology is gratefully acknowledged. I am indebted to Oliver Hart for both arousing my interest in this topic and for providing comments on an earlier version of the paper. Useful comments were also received from Glenn Loury and the referee.

0022-0531/80/020256-23$02.00/0

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imposed are the standard ones of general equilibrium theory when studied in a differential framework. As one such assumption is the convexity of production sets, monopolistic competition with free-entry as studied by Hart [9], Novshek [12], and Novshek and Sonnenschein [13] is ruled out. Although there seems to be no conceptual difficulty in extending the analysis to cope with free-entry, it will turn out that the structure under investigation admits a sufficiently rich set of possibilities without this complication. It should be stressed that a free-entry assumption has no power in diminishing these possibilities.

As mentioned above, this paper investigates the limit points of monopolistically competitive equilibria. It is shown that if a limit point occurs where the excess demand function given fixed production levels is not at a critical point then the limit point is a perfectly competitive equilibrium. However, if the limit point occurs at a critical point then no such inference can be drawn. In this case, the limiting equilibria are called *critical equilibria*. With critical equilibria, monopolistic elements can remain important even though the economy is large; furthermore, agents may incur large losses by adopting price-taking behavior. Although the occurrence of perfectly competitive equilibria at critical points is pathological (Debreu [3]), the existence of critical equilibria is a robust feature of an economy. For without specific knowledge of production possibilities, it seems that the only way to rule out such equilibria is to impose the restriction that the excess demand function possesses no critical points. As is well known (Arrow and Hahn [2, Chap. 12]), this implies that, for fixed production levels, the economy always has a unique perfectly competitive equilibrium.

If the limit points of monopolistic competition cannot be restricted to perfectly competitive equilibria, one can at least hope that perfectly competitive equilibria will be limit points. Here the results are more satisfactory. For if a perfectly competitive equilibrium occurs at a point where the excess demand function given fixed production levels and the excess demand function including a competitive supply response are not at critical points then the equilibrium will be a limit point of monopolistic competition. Competitive equilibria without these properties are pathological (Debreu [3]) so that competitive equilibria will almost always be limit points. Of course, the existence of a limit point implies that monopolistically competitive equilibria must exist at most stages of replication. In fact, it will be shown that it is almost always the case that, beyond some stage of replication, a monopolistically competitive equilibrium exists (see also Novshek and Sonnenschein [13]).

The general organization of the paper is as follows. The problem to be investigated is formulated in the next section. Section 3 is concerned with showing that, under suitable conditions, a perfectly competitive equilibrium is a limit point and vice versa. Critical equilibria are investigated in Section 4 and their nonpathological nature is demonstrated. Pathological counterexamples are provided in Section 5. These demonstrate why only generic and not universal statements are possible. Then, in Section 6, it is shown that because of their "fragile" nature, critical equilibria, of the form studied in earlier sections, are not robust to the introduction of uncertainty. However, other, very similar, possibilities emerge when uncertainty is introduced. These results suggest that it may be difficult to find a "reasonable" specification of the economy under which it is true that when the economy becomes large, monopolistic competition always approximates perfect competition. Concluding remarks are contained in Section 7.

2. THE MODEL

The structure of the model to be laid out is standard. As we shall be interested in the limit points that arise under monopolistic competition when an economy is replicated, we start by describing an economy E_r at stage r of replication.

There are m types of individuals in the economy and r individuals of each type. Individual $\{i, q\}$ is the q th individual of the ith type. Similarly, there are n types of firms and r firms of each type. Firm $\{j, q\}$ is the qth firm of the jth type. Firms produce and individuals consume goods.[1] There are $l + 1$ goods in the economy, good 1 being the *numeraire* (e.g., leisure). The price of good 1 is taken to be unity and the price vector for all goods is given by $p \in S = \{1\} \times \mathring{\mathbb{R}}_+^l$.[2] An element of $\mathbb{R}^{l+1}$ may be interpreted as a consumption (or production) bundle.

Individual $\{i, q\}$ possesses an initial endowment $\omega_i \in \mathring{\mathbb{R}}_+^{l+1}$ and receives a share $\theta_{ij} \geqslant 0$ $(\sum_i \theta_{ij} = 1)$ of the *average* profits earned by firms of type j; i.e., if π_{jq} is the profit earned by firm $\{j, q\}$ then $\{i, q'\}$ receives

$$\pi_i = \sum_j \theta_{ij} \left(\frac{1}{r} \sum_q \pi_{jq} \right). \tag{1}$$

Thus all individuals of the same type have the same endowment and receive the same level of profits. Individuals of the same type also have identical preferences. $\{i, q\}$ chooses a commodity bundle $x_{iq} \in \mathbb{R}_+^{l+1}$ (the consumption set) to make himself as well-off as possible, subject to the constraint

$$px_{iq} \leqslant \pi_i + p\omega_i, \tag{2}$$

where p is treated as exogenous, i.e., individuals are price takers. As differentiability properties of the economic structure will turn out to be of relevance, it will be assumed that this choice gives rise to a twice continuously differentiable (C^2) demand function

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[1] This is more restrictive than necessary. In fact, individuals are agents who are price takers and firms are agents who recognize the effect of their actions upon prices: competitive producers could be classed as individuals.

[2] $\mathring{\mathbb{R}}_+^l$ is the interior of the positive orthant.

$$f^i: S \times \mathbb{R}_+ \to \mathbb{R}_+^{l+1}. \tag{3}$$

The chosen x_{iq} is given by $f^i(p, \pi_i + p\omega_i)$ and all individuals of the same type make identical choices.

Two further restrictions will be imposed upon demand functions. First, it will be assumed that preferences embody local nonsatiation so that $pf^i(p, \pi_i + p\omega_i) = \pi_i + p\omega_i$. Second, to rule out the possibility that an equilibrium can occur where some prices are zero or infinite, it will be assumed (see Debreu [3]) that, for some type of individual, the sum of demands is unbounded as any price approaches zero or infinity. It may be noted that such an assumption effectively rules out the existence of pure intermediate goods. As the notion of monopolistic competition used in this paper is unlikely to be relevant when pure intermediate goods exist, economies with such goods demand separate attention.

Turning to the production side of the economy, firm $\{j, q\}$ chooses a production plan y_{jq} from a production set $Y_j \subseteq \mathbb{R}^{l+1}$ (negative arguments are inputs, positive arguments are outputs). The following conditions are imposed upon Y_j:

(1) $0 \in Y_j$,

(2) Y_j is closed and bounded above,

(3) Y_j is strictly convex and the boundary of Y_j possesses strict Gaussian curvature.

Apart from the assumption of strict Gaussian curvature, these conditions are standard. But given strict convexity, strict Gaussian curvature is unlikely to be a problem except at zero production. As will become clear, some relaxation of the condition at such points could be admitted without affecting the results.

The profit of firm $\{j, q\}$ is given by

$$\pi_{jq} = p y_{jq}. \tag{4}$$

To analyze the behavior of firms, we start by defining an exchange equilibrium.

DEFINITION. An *exchange equilibrium* in the economy E_r is an array $(p, y) \in S \times \mathbb{R}^{n(l+1)}$, where

$$\sum_i f^i \left(p, p\omega_i + \sum_j \theta_{ij} p y_j\right) - \sum_i \omega_i - \sum_j y_j = 0 \tag{5}$$

and $y \doteq (y_j), j = 1, n$.

Here, y_j is to be interpreted as the *average* production level of type j firms. Given fixed production levels $y = (y_j)$, if (p, y) is an exchange equilibrium then p is the price vector which ensures that all markets clear. With zero production, the exchange equilibrium is the perfectly competitive equilibrium of an exchange economy and this always exists (cf. Debreu [3]). Generically, such equilibria are isolated, finite, and odd in number (Debreu [3]). However, difficulties arise with nonzero production. For then there exists the problem that some individuals may not be able to attain allocations that lie within their consumption sets. This problem is alleviated in the present context because firms, through profit maximization, act to slacken attainability constraints upon individuals.

Firms in the economy have monopolistic power and recognize that a change in production will cause a change in prices which restores equilibrium. Given the possibilities of nonexistence or, more importantly, multiple equilibria of the exchange economy, some selection mechanism must be prescribed. Define $Q(y)$ by

$$Q(y) = \{p' : \{p', y\} \text{ is an exchange equilibrium}\} \subseteq S$$

and let

$$T = \{y : Q(y) \text{ is nonempty}\} \subseteq \mathbb{R}^{n(l+1)}.$$

Consider a mapping

$$P: S \times T \times T \to S \tag{6}$$

which satisfies the property that if $p \in Q(y)$ then

$$P(p, y, y') \in \{p^* : \|p^* - p\| \leq \|p' - p\| \ \forall p^*, p' \in Q(y')\}.$$

It will be assumed that if production levels are initially y and the price vector is p then, with a change in production to y', the new price vector is $P(p, y, y')$ (when equilibrium prices exist). This definition implies that prices change by the smallest amount necessary to restore equilibrium. Essentially, an attempt is being made to choose a well-behaved selection mechanism. However, there exist other mechanisms which are just as appealing. For instance, it could have been assumed that the price moved to the equilibrium that would result under a *tatonnement* price adjustment mechanism. In that case the characterization results would be dependent upon the local stability properties of the equilibria under investigation.

Define $Z(p, y)$ to be the last l arguments of the function

$$\sum_i f^i \Big(p, p\omega_i + \sum_j \theta_{ij} p y_j\Big) - \sum_i \omega_i - \sum_j y_j .$$

By Walras' law, if $Z(p, y) = 0$ then (p, y) is an exchange equilibrium. Z is defined on a neighborhood of (p, y) if individuals' incomes are strictly positive, i.e.,

$$p\omega_i + \sum_j \theta_{ij} p y_j > 0 \qquad \forall_i . \tag{7}$$

Equation (7) will hold if profits are nonnegative ($py_j \geqslant 0$ for all j). Thus if there are nonnegative profits, Z can be differentiated.[3] Let (p, y) be a *regular* (*resp. critical*) *exchange equilibrium* if $Z(p, y) = 0$ and the Jacobian Z_p is nonsingular (resp. singular).

By the Implicit Function Theorem (see, e.g., Dieudonné [7]), if Z_p is nonsingular then in a neighborhood of (p, y) there exists a unique C^2 function $p(\cdot)$ such that $Z(p(y'), y' = 0$. Thus for y' sufficiently close to y, the function P must be the function $p(\cdot)$.

LEMMA 1. *If (p, y) is a regular exchange equilibrium and $py_j \geqslant 0$ for all j then for all y' in some neighborhood of y, $P(p, y, y')$ is a C^2 function of y'.*

The content of Lemma 1 is that under well-defined conditions the selection mechanism will be well behaved. Of course, the conditions required are dependent upon the selection mechanism utilized. Consider again the selection mechanism induced by *tatonnement* adjustment; (p, y) will be a locally stable exchange equilibrium if the eigenvalues of Z_p have strictly negative real parts. As Z is C^2, the equilibria $(p(y'), y')$ thrown up by the Implicit Function Theorem will also be locally stable (see, e.g., Hirsch and Smale [10, Chap. 16]). Thus in this more sophisticated situation, the selection mechanism will be C^2 if (p, y) is a regular *and locally stable* exchange equilibrium with $py_j \geqslant 0$ for all j.

A monopolistically competitive equilibrium is defined as a situation of exchange equilibrium where no firm can increase its profits with the recognition that prices depend upon production levels.[4] Let $y_{-j} = (y_1, ..., y_{j-1}, y_{j+1}, ..., y_n)$.

DEFINITION. A *monopolistically competitive equilibrium* (MCE) in the economy E_r is an array $(p, (y_{jq})) \in S \times \mathbb{R}^{rn(l+1)}$ such that $(p, ((1/r) \sum_q y_{jq}))$ is an exchange equilibrium in E_r,

$$y_{jq} \in Y_j \qquad \forall j, q,$$
$$py_{jq} \geqslant 0 \qquad \forall j, q,$$

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[3] Differentiation of a function with respect to p is taken to mean differentiation with respect to the last l arguments of p; the price of the *numeraire* remains at unity.

[4] The fact that profit-maximization by price makers is generally not *numeraire* free is ignored. See, e.g., Gabszewicz and Vial [8].

and $\nexists\{j, q\}$ and $y'_{jq} \in Y_j$ such that

$$P(p, y, \{y_{\ j}, y_j + (1/r)(y'_{jq} - y_{jq})\})\, y'_{jq} > p y_{jq}. \tag{8}$$

Equation (8) is a Cournot–Nash equilibrium assumption. However, it would be easy to admit conjectural variations, i.e., to endow firms with a belief that other firms will respond to a change in production plan in a well-specified manner. With suitably smooth functions capturing conjectural variations, the analysis of this paper goes through in a straightforward manner.

A MCE changes with replication of the economy and part of this change will be the interest of this paper. However, perfectly competitive equilibria are independent of the stage of replication.

DEFINITION. A *perfectly competitive equilibrium* (PCE) in the economy E_r is an array $(p, ((1/r)\sum_q y_{jq})) \in S \times R^{n(l+1)}$ such that $(p, ((1/r)\sum_q y_{jq}))$ is an exchange equilibrium in E_r,

$$y_{jq} \in Y_j \qquad \forall j, q,$$

and $\nexists\{j, q\}$ and $y'_{jq} \in Y_j$ such that

$$p y'_{jq} > p y_{jq}. \tag{9}$$

As Y_j is strictly convex, it may be noted that $y_{jq} = y_{jq'}$ for all j, q, q': identical firms act identically. Thus, if (p, y) is a PCE then firm $\{j, q\}$ must have y_j as its production plan. It is a simple consequence of the definition of a PCE that if (p, y) is a PCE in E_r then it is a PCE in E_s, where E_s is the economy at any other stage of replication.

At a PCE, all agents are price takers. From the assumptions that have been imposed upon production sets it can be concluded that, given a price vector p, there exists a production plan $y_j^c(p)$ which maximizes profits for type j firms. That y_j^c is a continuous function is a standard result (Arrow and Hahn [2, Chap. 3]). However, by amending the analysis of Debreu [4] to apply to producers instead of consumers (see also Laroque [11]), it may be concluded that, as the boundary of Y_j possesses strict Gaussian curvature, y_j^c will be a continuously differentiable (C^1) function. A PCE will be defined by the relation

$$Z^c(p) = Z(p, y^c(p)) = 0. \tag{10}$$

(p, y) will be a ***regular*** (*resp.* ***critical***) ***PCE*** if $Z_p{}^c$ is nonsingular (resp. singular). As in Debreu [3], Sard's theorem may be invoked to show that, for an open and dense subset of initial endowments, perfectly competitive equilibria will be regular and isolated.[5]

That a PCE exists under the assumptions that have been laid down is well known. However, because of the possibly discontinuous behavior of firms when acting monopolistically (see, e.g., Roberts and Sonnenschein [15]), the existence of a MCE must remain an open question for the present.

Finally, we shall be interested in the properties of monopolistically competitive equilibria when r becomes large. Instead of looking at the production plan of each firm, we shall be interested in the average plan of firms of each type. Define $M(r)$ as follows:

$$M(r) = \left\{\left(p, \left((1/r) \sum_q y_{jq}\right)\right): (p, (y_{jq})) \text{ is a MCE in } E_r\right\}.$$

We shall be interested in the limit points of monopolistic competition.

DEFINITION. A *limiting monopolistically competitive equilibrium* (LMCE) is an array $(\bar{p}, \bar{y}) \in S \times \mathbb{R}^{n(l+1)}$ which is the limit of sequence $\{(p^r, y^r)\}$ such that for all r greater than some r^*, $(p^r, y^r) \in M(r)$.

Our main interest will be the connection that exists between the set of PCEs and the set of LMCEs (note that it makes sense to ask the question whether a PCE is a LMCE and vice versa).

3. PERFECT COMPETITION AND THE LIMITS OF MONOPOLISTIC COMPETITION

The possible equivalence of PCEs with LMCEs depends upon the form of the exchange equilibrium at the point under investigation. By definition, a PCE is an exchange equilibrium. We also have

THEOREM 1. *A LMCE is an exchange equilibrium.*

Proof. Let $(\bar{p}, \bar{y})$ be a LMCE. By definition there exists a sequence $\{(p^r, y^r)\}$, $(p^r, y^r) \in M(r)$ for all r greater than some r^*, such that $p^r \to \bar{p}$ and $y^r \to \bar{y}$. As $(p^r, y^r) \in M(r)$, $r > r^*$, (p^r, y^r) is an exchange equilibrium (from the definitions of $M(r)$ and MCE). Thus $Z(p^r, y^r) = 0\ \forall r > r^*$. Now, as $p^r y_j^r \geqslant 0$ and as the values of endowments are bounded away from zero, the demand functions, and hence Z, are defined at $(\bar{p}, \bar{y})$. Z is C^2 and thus continuous so that $Z(\bar{p}, \bar{y}) = 0$. ∎

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[5] Debreu considered a pure exchange economy with price-taking individuals. However, under the assumptions that have been laid down, price-taking firms are like a special type of individual.

Given Theorem 1, we can separate out those LMCEs which occur at a regular exchange equilibrium from those which occur at a critical exchange equilibrium. In this section, we investigate the regular equilibrium case.

THEOREM 2. *If a LCME is a* regular *exchange equilibrium then it is a PCE.*

Proof. Let $(\bar{p}, \bar{y})$ be a LCME which is also a regular exchange equilibrium. If it is not a PCE then

$$\exists j, y_j^* \in Y_j, \quad \epsilon > 0, \text{ such that } \bar{p}y_j^* - \bar{p}\bar{y}_j + \epsilon \geqslant 0. \tag{11}$$

Here, use has been made of the fact that $\bar{y}_j \in Y_j$, which follows from the closedness and convexity of production sets. Now, as $(\bar{p}, \bar{y})$ is a LMCE, there exists a sequence $\{(p^r, y^r)\}$, $(p^r, y^r) \in M(r)$ $\forall r > r^*$, such that $p^r \to \bar{p}$ and $y^r \to \bar{y}$. If $(p^r, y^r) \in M(r)$ because $(p^r, (y_{jq}^r))$ is a MCE $(y^r = (1/r) \sum_q y_{jq}^r)$ then, from the definition of a MCE

$$\forall q\colon P(p^r, y^r, (y^r_{-j}, y_j^r + (1/r)(y_j^* - y_{jq}^r)))\, y_j^* \leqslant p^r y_{jq}^r \tag{12}$$

if P is defined.

As p^r for all $r > r^*$ and $\bar{p}$ support exchange equilibria, these vectors contain no zero arguments and are bounded above (as production possibilities are bounded). Y_j is bounded above so that y_j^* and y_{jq}^r are drawn from a bounded set, as $\bar{p}y_j^* \geqslant 0$ (from (11)) and $p^r y_{jq}^r \geqslant 0$. Thus $y_j^* - y_{jq}^r$ is bounded and as $p^r \to \bar{p}$ and $y^r \to \bar{y}$, we can choose r^* large enough so that the arguments of $P(\cdot)$ in (11) belong to any small neighborhood of $(\bar{p}, \bar{y})$. Lemma 1 applies, as $(\bar{p}, \bar{y})$ is a regular exchange equilibrium and $\bar{p}\bar{y}_j \geqslant 0$ for all j $(p^r y_j^r \to \bar{p}\bar{y}_j$ and $p^r y_j^r \geqslant 0)$, so that (12) becomes

$$P(p^r, y^r, y^r)\, y_j^* - k/r^* \leqslant p^r y_{jq}^r, \tag{13}$$

where $k \geqslant 0$ is a bounded scalar (as P is continuously differentiable in y_j and relevant production plans are bounded). Averaging (13) over q and noting that $P(p^r, y^r, y^r) = p^r$ gives

$$p^r y_j^* - k/r^* \leqslant p^r y_j^r. \tag{14}$$

Choosing $r^* \geqslant 2k/\epsilon$ and taking limits yields

$$\bar{p}y_j^* - \epsilon/2 \leqslant \bar{p}\bar{y}_j, \tag{15}$$

which contradicts (11). ▌

Remark. It may be noted that Theorem 2 requires and follows from the continuity of P. With the approach of this paper, this result requires only that demand functions be C^1 (so that the Implicit Function Theorem can be applied). An alternative approach is to assume that exchange equilibria are unique (see Gabszewicz and Vial [8]). As the limit of exchange equilibria is an exchange equilibrium (see the proof of Theorem 1), the continuity of P follows directly under the uniqueness assumption. Thus if multiple exchange equilibria are ruled out, a LMCE is always a PCE.

We now prove a partial converse to Theorem 2.

THEOREM 3. *If a* regular *PCE is a* regular *exchange equilibrium then it is a LMCE.*

Remark 1. For an open and dense subset of initial endowments, all perfectly competitive equilibria will be regular and will occur at regular exchange equilibria. Therefore, generically, all PCEs will be LMCEs.

In the present context, however, the most important remark that can be made concerning Theorem 2 is

Remark 2. If a PCE is a LMCE then, beyond some stage of replication, a MCE must exist. As a PCE exists, the existence of a MCE in a large (but finite) economy is generic.[6]

Proof. Let $(\bar{p}, \bar{y})$ be a regular PCE which is also a regular exchange equilibrium. We shall be interested in proving the existence of monopolistically competitive equilibria which are arbitrarily close to $(\bar{p}, \bar{y})$. If the economy is kept within a small neighborhood of $(\bar{p}, \bar{y})$ then Lemma 1 is applicable: given levels of production y', the price vector will be given by a C^2 function $P(y')$. Consider some firm $\{j, q\}$ in the economy E_r. If other types of firms have average production levels y^*_{-j} and other firms of the same type have an average production level y^*_j then $\{j, q\}$ chooses y_{jq} to solve the problem

$$\max \pi(y^*, y_{jq}) = P(y^*_{-j}, y^*_j + (1/r)(y_{jq} - y^*_j))\, y_{jq} \qquad \text{s.t.} \quad y_{jq} \in Y_j \tag{16}$$

(profits may be defined to be zero when there is zero production). As Y_j is bounded above and exchange equilibrium prices are always bounded away from zero and infinity, we can restrict attention to the choice of y_{jq} drawn from a bounded set. Thus, if y^* is sufficiently close to $\bar{y}$, $(y^*_{-j}, y^*_j + (1/r)(y_{jq} - y^*_j))$ will be close to $\bar{y}$ for r large enough and P in (16) will have the desirable C^2 properties alluded to in Lemma 1. The maximand in (16) is therefore C^2 and

$$\pi_{y_{jq}} = P + (1/r)\, P_{y_j} y_j, \tag{17}$$

$$\pi_{y_{jq} y_{jq}} = (2/r)\, P_{y_j} + (1/r^2)\, P_{y_j y_j} y_j. \tag{18}$$

---

[6] Although a similar result is provided by Novshek and Sonnenschein [13], the generic features of the economy that they utilize are not easily interpretable.

For r large enough, P being C^2 ensures that (17) will be nonzero and the curvature of the profit function (see (18)) will be less than the curvature of the boundary of the production set Y_j (as the boundary of Y_j has *strict* Gaussian curvature). Thus the same point will be chosen under a linearization of the maximand; i.e., if $\hat{y}_{jq}$ solves (16) then $\hat{y}_{jq}$ solves

$$\max \pi_{y_{jq}}(y^*, \hat{y}_{jq})\, y_{jq} \qquad \text{s.t.} \quad y_{jq} \in Y_j . \tag{19}$$

However, this is just the problem faced by a perfectly competitive firm when prices are given by $\pi_{y_{jq}}$. $\hat{y}_{jq}$ must therefore satisfy

$$\hat{y}_{jq} = y_j^c(\pi_{y_{jq}}(y^*, \hat{y}_{jq})), \tag{20}$$

where y_j^c is a C^1 function giving the competitive supply function.

Using (20), $(P(y^*), y^*)$ will be a MCE for E_r, r suitably large, if y^* is in a small neighborhood of $\bar{y}$ and

$$\forall j\colon y_j^* = y_j^c(\pi_{y_{jq}}(y^*, y_j^*)). \tag{21}$$

Note that this is a MCE where firms of the same type have identical production plans. Using (17), we seek to show the existence of a y^* close to $\bar{y}$ such that

$$\forall j\colon y_j^* = y_j^c(P(y^*) + \alpha P_{y_j}(y^*)\, y_j^*) \tag{22}$$

and $r = 1/\alpha$. When $\alpha = 0$ in (22), we know that, as $(\bar{p}, \bar{y})$ is a PCE,

$$\forall j\colon \bar{y}_j = y_j^c(P(\bar{y})). \tag{23}$$

As (22) is a continuously differentiable (C^1) function of y^* and α, the Implicit Function Theorem can be applied to show that in the neighborhood of $\bar{y}$ and $\alpha = 0$, there exists a unique C^1 function $y^*(\alpha)$ which solves (22) (feasibility of y^* follows from the fact that it is feasible for competitive producers at some prices). We need only check that the nonsingularity condition holds so that the Implicit Function Theorem can be applied. Stacking the vector functions in (22) and differentiating both sides with respect to y^* gives the result that, for small α, the nonsingularity condition will hold if

$$I - y_p^c(\bar{p})\, P_{y^*} \tag{24}$$

is nonsingular. If (24) is singular then there exists a $v \neq 0$ such that

$$v - y_p^c(\bar{p}) P_{y^*} v = 0.$$

Premultiplying by P_{y^*} yields

$$(I - P_{y^*} y_p^c(\bar{p}))\tilde{v} = 0, \tag{25}$$

where $\tilde{v} = P_{y*}v \neq 0$ (as $v = y_p{}^c(\bar{p})\tilde{v}$). However, invoking (10) and notin that $P_{y*} = -(Z_p)^{-1}Z_y$ gives

$$(I - P_{y*}y_p{}^c(\bar{p}))\tilde{v} = (Z_p)^{-1}Z_p{}^c\tilde{v} \neq 0$$

as $(\bar{p}, \bar{y})$ is a *regular* exchange equilibrium (Z_p nonsingular) and a *regula* PCE ($Z_p{}^c$ nonsingular). This contradicts (25) and proves the nonsingularit of (24). Thus for α small enough, or r greater than some r^*, there exist a MCE and

$$\forall r > r^*: (P(y^*(1/r)), y^*(1/r)) \in M(r). \qquad (26$$

Hence there exists an infinite sequence $\{(p^r, y^r)\}$, $(p^r, y^r) \in M(r)$ for al $r > r^*$, converging along a C^1 manifold to $(\bar{p}, \bar{y})$; i.e., $(\bar{p}, \bar{y})$ is a LMCE. |

Remark. It has been shown that for r large but finite, there exists monopolistically competitive equilibrium. Essentially, existence follow because the nonconcavities in firms' profit functions, as, for instance, analyze by Roberts and Sonnenschein [15], are removed by replication of the economy Note also that it has been shown that monopolistically competitive equilibri exist which are very close to some perfectly competitive equilibrium. No only is this true in the aggregate (similar prices and *average* production levels) but it has also been shown that there exist equilibria where firms of the sam type act identically (not just similarly). Finally, convergence is very wel behaved as it takes place along a C^1 manifold.

4. CRITICAL EQUILIBRIA

The results of the last section give a partial characterization of the connection that exists between the limit points of monopolistic competition an perfectly competitive equilibria. Generically, a set of limit points exists an this set contains the set of perfectly competitive equilibria (Theorem 3) Further, all limit points occur at an exchange equilibrium (Theorem 1) an those which occur at a regular exchange equilibrium are perfectly competitiv (Theorem 2). We now need only investigate limit points which occur at critical exchange equilibrium. Such points will be called critical equilibria

DEFINITION. (p, y) is a *critical equilibrium* if (p, y) is a LMCE and critical exchange equilibrium.

One could hope to show that for an open and dense subset of initial endowments, critical equilibria could not arise. However, despite the usual implications of Sard's theorem concerning the likelihood of critical values (se Debreu [3] for an economic application), we will show that critical equilibri are a robust feature of an economy. Therefore, the set of limit points o monopolistic competition will generally be the union of the set of perfectl competitive equilibria and a set of critical equilibria: if perfect competition is studied because it is a limiting solution then critical equilibria should also be analyzed.

Our results for critical equilibria can be expressed by the following theorem.

THEOREM 4. *The existence of critical equilibria which are not perfectly competitive is nonpathological; i.e., such equilibria may exist and are not necessarily eliminated by perturbations of the excess demand function.*

To prove, it is sufficient to provide an example. First, however, it may be noted that as for almost all y exchange equilibria are regular, monopolistically competitive firms must have an incentive to "seek out" critical exchange equilibria.

It is useful first to explain the rationale of Theorem 4 and this will suggest an example. Consider a situation where firms (which are all identical) produce one good, with price p, using one input (the *numeraire*) supplied by consumers. Fig. 1 gives the excess demand function for the produced good under the assumption that each firm uses the minimum input necessary to produce the output. To give a better understanding of this function, Figs. 2 and 3 give cross-sectional cuts at different output levels and at $Z = 0$, respectively. As is well known (Debreu [5]), if the number of different types of individual m is at least as great as the number of goods $l + 1$ then, at any

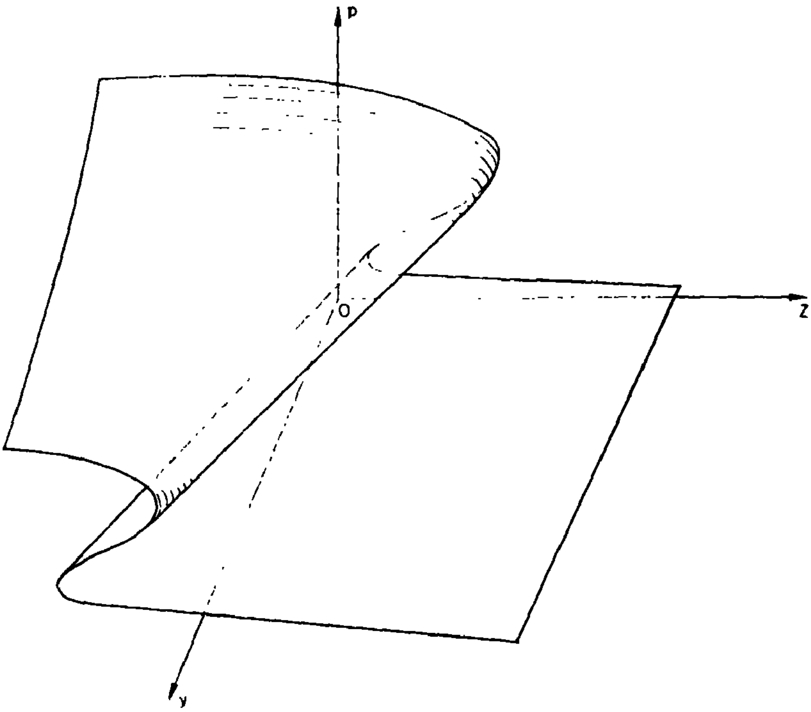

FIGURE 1

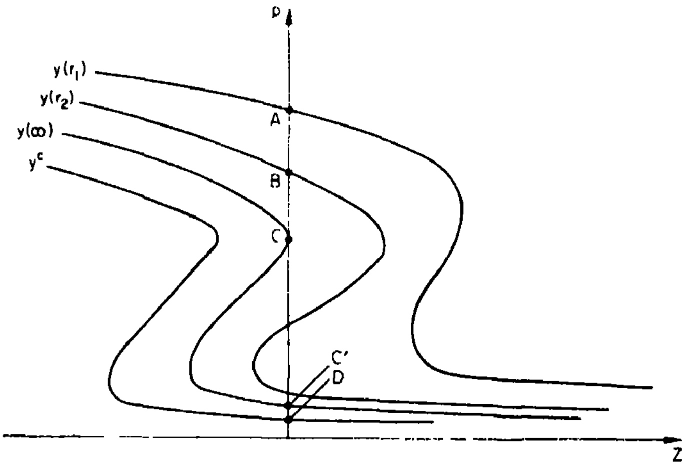

FIGURE 2

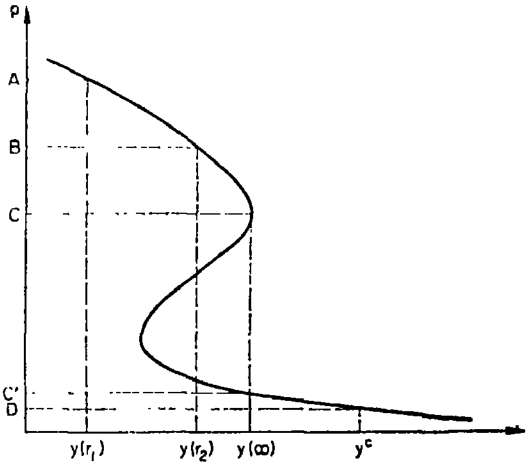

FIGURE 3

articular y, the excess demand function can take any form compatible with Valras' law and continuity.[7] In particular, the excess demand curve passing hrough C and C' in Fig. 2 is possible.

When there is a small number of firms (r_1), monopolistic power is ecognized and a MCE occurs at $p = A$. As the economy is replicated, onopolistic power partially dissipates and a MCE occurs at $p = B$. As ore firms enter, the MCE tends to the point $p = C$ and price becomes ore sensitive to changes in production levels (see Fig. 3). Although there re more firms, monopolistic power is retained. Also C' may be such a low rice that no firm is willing to destroy the equilibrium at C. Thus, as an conomy is replicated, the MCE tends to be "held back" if critical exchange

[7] This holds over any bounded set of strictly positive prices. In fact, we have imposed a striction on the excess demand function when prices tend to zero or infinity.

equilibria are encountered. There is therefore no reason for the economy to tend to a PCE which occurs at a point $p = D$, say, in Fig. 2.

It must now be shown that for r large, monopolistically competitive equilibria can arise at a price close to C. With r large, no firm will contemplate changing the *average* production level by more than a small amount (see the proofs of Theorems 2 and 3). Thus only local behavior around C need be analyzed. If C is a nondegenerate critical point then the Morse lemma can be applied to give the result that the excess demand function is locally quadratic in p. We shall be interested in the existence of equilibria where all firms act identically. Assume that each firm must utilize at least $F(y)$, $F' \geqslant 0$, $F'' > 0$, of the input to produce y (now a scalar). Assume that it is profitable for a firm to be technologically efficient. Then, as the input requirement will depend directly upon the output level, Z can be written as a function of price and the output level and, in a neighborhood of C, excess demand can be approximated by

$$Z = \alpha^2\beta(y^* - y) - \beta(p - C)^2 \tag{27}$$

if, at C, $Z_y < 0$. In (27), α, β, $y^* > 0$ are parameters which change with perturbations in the excess demand function (the critical point occurs at C when average output is y^*). Equating (27) to zero gives the equilibrium price as a function of average output:

$$p = C + \alpha(y^* - y)^{1/2}, \qquad y \leqslant y^*, \tag{28}$$

(the selection mechanism picks out prices which exceed C).

Given this specification, it is not difficult to show that a MCE in E_r exists, where each firm has an output y defined by

$$\alpha(y^* - y) + (C - F'(y))(y^* - y)^{1/2} - \frac{\alpha y}{2r} = 0. \tag{29}$$

This gives the point where first-order conditions of profit maximization are satisfied. Further, it is easy to check that second-order conditions are always satisfied (given only $\alpha > 0$). If $C \geqslant F'(y^*)$, so that price-taking firms would increase production, profit-maximizing output is defined by (29) and tends to y^* as r tends to infinity. From (29), output at stage r (when r is large) is given by

$$y = y^* - \left(\frac{\alpha y^*}{2r(C - F'(y^*))}\right)^2. \tag{30}$$

We have shown that a LMCE occurs at (C, y^*) if firms do not wish to, or cannot, destroy the equilibrium at C and so push the economy to C'. As is clear, if feasible output is bounded at a point just beyond y^* then it will be impossible for any firm to recuperate losses created by the price drop from C to C'. However, (30) implies that C could be destroyed by an increase in *total* production which is proportional to $1/r$. Thus, for large r, each firm *could* destroy the equilibrium at C. In particular, the equilibrium would be destroyed if any firm decided to adopt price-taking behavior. Thus, even in large economies, an agent may lose significantly by adopting price-taking behavior. It is interesting to compare this result with those obtained by Roberts and Postlewaite [14]. They reached the opposite conclusion but, in essence, this is because they ruled out, by assumption, the possibility that critical equilibria could exist.

Finally, a LMCE continues to exist at a point close to C when the excess demand function is subject to C^2 perturbations. For perturbation corresponds to small changes in α, β, and y^*: the existence analysis continues to apply. Nonperfectly competitive critical equilibria have therefore been shown to be nonpathological and Theorem 4 has been proved.

The example given to prove the theorem clearly demonstrates why a critical equilibrium can arise as a LMCE. If the excess demand function has a critical point then it is possible to converge to this point in a way that permits price responses to grow at the rate of replication.

Such a situation can also arise in an economy with many goods. Close to a critical point, firms will recognize that changes in production will significantly affect prices. Consider the form of the price response close to a critical point where Z_p is of rank $l - 1$. Let z_y be the derivative of the excess demand function with respect to a change in a component of some y_j. The price response Δp will be given by

$$Z_p \Delta p + z_y \mathrel{\overset{\sim}{=}} 0. \tag{31}$$

If the last row of Z_p is close to a linear combination of the other rows then (31) may be partitioned to give

$$\left[\begin{array}{c:c} z_p & z_p{}^l \\ \hdashline \lambda z_p + \epsilon t^{-l} & \lambda z^l p + \epsilon t^l \end{array}\right] \left[\begin{array}{c} \Delta p^{-l} \\ \hdashline \Delta p^l \end{array}\right] = \left[\begin{array}{c} -z_y^{-l} \\ \hdashline -z_y{}^l \end{array}\right]. \tag{32}$$

z_p is an $l - 1 \times l - 1$ matrix which, by a suitable reordering of columns, can be taken to be nonsingular (as row rank = column rank = $l - 1$). t is some vector and ϵ is a scalar which tends to zero as the critical point is approached.[8] Equation (32) may be solved to give an approximate result for small ϵ:

$$\epsilon \begin{bmatrix} \Delta p^{-l} \\ \Delta p^l \end{bmatrix} = \begin{bmatrix} -z_p^{-1} z_p{}^l \\ 1 \end{bmatrix} \left(\frac{\lambda z_y^{-l} - z_y{}^l}{t^l - t^{-l} z_p^{-1} z_p{}^l} \right), \tag{33}$$

---

[8] Superscript l denotes the lth argument (column) of a vector (matrix). Superscript $-l$ denotes the vector (matrix) with the lth argument (column) removed.

where it has been assumed that $t^l - t^{-l}z_p^{-1}z_p{}^l \neq 0$ (generically, this will be the case).

Equation (33) shows that if $\epsilon \to 0$ at a rate directly proportional to $1/r$ then each firm in the economy will retain monopolistic power. Note also that, independently of the form of z_p, the direction of price responsiveness will be the same. This is a consequence of the assumption that, at the critical point, Z_p is of rank $l - 1$. If Z_p is of rank $l - k$ then a similar analysis may be used to show that price responsiveness will lie in a k-dimensional space. Thus, although in general all firms will possess monopolistic power when convergence is toward a critical point, the extent of monopolistic power will be inversely related to the rank of Z_p.

5. PATHOLOGICAL POSSIBILITIES

The results that have been derived in this paper depend upon the local differentiability properties of the excess demand function around particular equilibria. This is clearly true of Theorems 2 and 4. Recalling Theorem 3, it was shown that a PCE is almost always a LMCE, nothing being said about when the PCE is critical or occurs at a critical exchange equilibrium. It is the purpose of this section to show that in these pathological cases, the theorem does not necessarily hold: only a generic, and not a universal, statement can be provided.

Three examples will be given. The economy is similar to the one in the example given in Section 4; i.e., there is one type of firm producing one output (at price p) using one input (the *numeraire*). The production technology will be assumed Cobb–Douglas in form, an output of y by a firm requiring an input of y^2.[9] Thus the competitive supply function is given by

$$y^c = p/2. \tag{34}$$

Finally, as we are investigating existence of monopolistically competitive equilibria close to a perfectly competitive equilibrium, we need only look at the economy close to this equilibrium.

EXAMPLE 1. A PCE that is a critical exchange equilibrium.

Let the excess demand function for $p \in [1, 5]$ be

$$Z = (p - 2)^2 + (1 - y). \tag{35}$$

Combining (34) and (35), there are two perfectly competitive equilibria, $(p, y) = (2, 1)$ or $(5/2, 5/4)$. From (35), $(2, 1)$ is a critical exchange equilibrium. Close to (2, 1), there are two possibilities for the price selection mechanism

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[9] To conform with earlier assumptions, output may be bounded at some high level without affecting the analysis.

$$p = 2 + (y - 1)^{1/2} \tag{36a}$$

or

$$p = 2 - (y - 1)^{1/2} \tag{36b}$$

and these are only possible when $y \geqslant 1$. When $y < 1$, assume that the economy moves to some high price ($p > 5$). The behavior of monopolistically competitive firms is easy to analyze by setting up the profit function. Under (36a), output is expanded because price rises with output. In fact, it is easy to show that a MCE cannot exist with average output below 5/4: with (36a), the LMCE is at (5/2, 5/4). Under (36b), price falls with output and each firm has an incentive to reduce output to unity. When all firms have done this there is an incentive for any one firm to destroy the equilibrium and move to an equilibrium with a high price ($p > 5$). Thus under both selection mechanisms, monopolistically competitive firms take the economy away from the PCE at (2, 1).

Example 2. A PCE that is critical.

Around $(p, y) = (2, 1)$, let the excess demand function be given by

$$Z = p - 2y - (y - 1)^2. \tag{37}$$

Combining (34) and (37), there is a PCE at (2, 1) which is critical. Close to this point the price selection mechanism is

$$p = 2y + (y - 1)^2. \tag{38}$$

When output is close to unity, monopolistically competitive firms have an incentive to increase production. In fact, it is trivial to show that every firm wishes to produce more than the average! Thus, prices and output are driven away from (2, 1): it is impossible for (2, 1) to be the limit of monopolistically competitive equilibria.

The final example shows that pathological possibilities do not rule out the possibility that a PCE is a LMCE.

Example 3. A PCE that is a critical PCE and a critical exchange equilibrium.

Around $(p, y) = (2, 1)$, let Z be given by

$$Z = (p - 2)^2 - 4(y - 1)^4. \tag{39}$$

At (2, 1), we have a critical exchange equilibrium and a critical PCE. An admissible price selection mechanism is

$$p = 2 - 2(y - 1)^2. \tag{40}$$

As price rises if output rises when $y < 1$ and falls if output rises when $y > 1$, it is easy to show that, *independently of the stage of replication*, a MCE exists where $(p, y) = (2, 1)$. Thus, the PCE is a LMCE in a very direct sense. The example also shows that it is possible—though unlikely—for perfectly competitive outcomes to be realizable as monopolistically competitive equilibria.

6. UNCERTAINTY AND CRITICAL EQUILIBRIA

Despite the fact that critical equilibria are nonpathological, common sense suggests that they could not arise in "realistic" situations. The reason for such a deep-rooted belief would seem to be that, by their nature, critical equilibria are extremely fragile: if the excess demand function changes, and producers do not respond immediately to the change, then it will often be the case that there will be a jump in prices which leads to a discrete fall in firms' profits. As firms "hold back" at critical equilibria to avoid this discrete drop in profits, it may be conjectured that in situations where firms have incomplete control, the incentives for "holding back" are lost.

To investigate this conjecture, uncertainty will be introduced into the model. In particular, it will be assumed that individuals' endowments are uncertain when firms make their production decisions. Thus equilibrium prices and profits are uncertain. As for behavior toward risk, it will be assumed that firms are risk-neutral.

Consider adding a stochastic component $v_i \in \mathbb{R}$ to the endowment of the *numeraire* good held by individuals of type i. It will be assumed that there is an atomless probability measure ν on $v \in \mathbb{R}^m$ with support

$$V_\epsilon = \{v\colon -\epsilon \leqslant v_i \leqslant \epsilon \forall i\}.$$

Thus, as ϵ becomes small, uncertainty about endowments becomes small.

For individuals, unlike firms, uncertainty is resolved before they take decisions. If firms produce y and some v is realized then the excess demand function Z will be given by the last l arguments of

$$\sum_i f^i \Big(p, p\omega_i + \sum_j \theta_{ij} p y_j + v_i\Big) - \sum_i \omega_i - \sum_j y_i .$$

It will be convenient to write the function Z in the form

$$Z(p, y, v) = X\left(p, \left\{p\omega_i + \sum_j \theta_{ij} p y_j + v_i\right\}_{i=1,m}\right). \tag{41}$$

As firms are risk-neutral, they are only interested in expected prices. If the price selection derived from equating (41) to zero is given by $P(p, y, y'; v)$ (where $Z(p, y, 0) = 0$) then expected price is given by

$$P^*(p, y, y') = \int_{V_\epsilon} P(p, y, y'; v)\, dv. \tag{42}$$

Here, p is used to define the appropriate selection. P^* is defined if and only if P is defined for almost all v.

Economies will be indexed by ϵ. For this purpose it will be assumed that for each ϵ, there exists a unique probability measure on v with support V_ϵ. Corresponding to a MCE and a PCE we have

DEFINITION. A MCE (ϵ) in the ϵ version of the economy E_r is an array $(p, (y_{jq})) \in S \times \mathbb{R}^{rn(l+1)}$ such that $(p, (1/r) \sum_q y_{jq})$ is an exchange equilibrium in the nonstochastic economy; i.e., $Z(p, y, 0) = 0$,

$$y_{jq} \in Y_j \qquad \forall j, q$$

$$P^*(p, y, y)\, y_{jq} \geq 0 \qquad \forall j, q,$$

and $\nexists\{j, q\}$ and $y'_{jq} \in Y_j$ such that

$$P^*(p, y, \{y_{\cdot j}, y_j + (1/r)(y'_{jq} - y_{jq})\})\, y'_{jq} > P^*(p, y, y)\, y_{jq}.$$

DEFINITION. A PCE (ϵ) in the ϵ version of the economy E_r is an array $(p, ((1/r) \sum_q y_{jq})) \in S \times \mathbb{R}^{n(l+1)}$ such that $(p, (1/r) \sum_q y_{jq})$ is an exchange equilibrium in the nonstochastic economy; i.e., $Z(p, y, 0) = 0$,

$$y_{jq} \in Y_j \qquad \forall j, q,$$

and $\nexists\{j, q\}$ and $y'_{jq} \in Y_j$ such that

$$P^*(p, y, y)\, y'_{jq} > P^*(p, y, y)\, y_{jq}.$$

These equilibrium notions are defined relative to expected profit maximization (note that p defines the selection but is not an expected price).

The closeness in the definitions of MCE (ϵ) and PCE (ϵ) to the definitions of MCE and PCE may be noted. If (p, y) is a regular exchange equilibrium then it will be possible to apply the Implicit Function Theorem to show that

$P(p, y, y; v)$ is differentiable in v around $v = 0$. Thus, as $\epsilon \to 0$, $P^*(p, y, y) \to P(p, y, y; 0) = p$. It is then not difficult to show that if (p, y) is the limit point of PCE (ϵ)s as $\epsilon \to 0$ and (p, y) is a regular exchange equilibrium then (p, y) is a PCE (compare with Theorem 2).

Turning to critical equilibria, a LMCE (ϵ) may be defined in the ϵ version of the economy just as LMCE was defined in the standard economy. We wish to know whether a LMCE (ϵ) is a PCE (ϵ).[10] Pursuing an analysis similar to that in Theorem 2, a LMCE (ϵ) will be a PCE (ϵ) if $P^*(p, y, y')$ is continuous in y' at y. There exist two possibilities:

(1) As firms maximize expected profits, and negative profits in some states are not ruled out, there may exist v (of measure zero) under which some individuals are on the boundary of their consumption sets when production is y. When production is changed to y', y' close to y, it is possible that nonattainability will hold for a nonzero measure of states. Thus as P^* will not be defined, the notion of a continuous change will be lost. In this case it is possible for the LMCE (ϵ) to occur at a noncompetitive point where firms "hold back" their production to ensure existence. An example of this is given in Fig. 4 where, for a two-good economy, Z is plotted against p for different values of $y (Z = Z(p, y, 0))$. As the economy is replicated, production expands and price falls from A to B and then to C. At C, no firm contemplates an expansion in output because this leads to the nonexistence of an equilibrium price. However, if all firms expanded then it would be possible to move to D which, it may be assumed, is the perfectly competitive position.

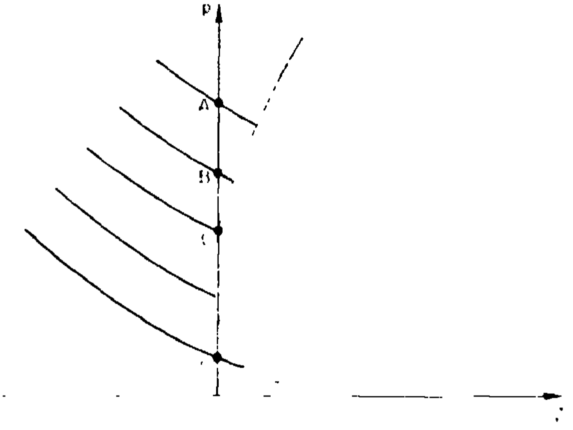

FIGURE 4

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[10] An investigation of the set of PCE (ϵ) as $\epsilon \to 0$ would take us too far astray from the present analysis. It suffices to say that there can exist limit points which are not perfectly competitive for the certain economy.

Although the motivation for this type of equilibrium differs from that of a critical equilibrium, the two concepts are very close. For as expected profits are nonnegative, the nonexistence problem will not usually arise if ϵ is small and profits change continuously with y and v. This is ensured when $Z_p(p, y, 0)$ is nonsingular. However, at critical exchange equilibria, profits can fall discontinuously and the nonexistence problem is more likely to be encountered: limit points close to critical points are not unlikely.

(2) Now assume that P^* is defined for all y' close to y. Looking at (42), P^* will be C^2 in y' if $P(p, y, y'; v)$ is C^2 in y' for almost all v. When $v \to 0$, P will be C^2 if Z_p is nonsingular. As Z is C^2, nonsingularity will hold for all small v so that for small ϵ, P^* will be continuous. If Z_p is singular, so that the determinant, $\det(Z_p)$, is zero, then P will still be C^2 for almost all v, ϵ small, if the differential of $\det(Z_p)$ with respect to v is nonzero. Thus, problems can only arise if there exist p, y such that

$$\begin{aligned} Z(p, y, 0) &= 0, \\ \det(Z_p) &= 0, \\ \frac{\partial}{\partial v} \det(Z_p) &= 0. \end{aligned} \tag{43}$$

Transversality theory (see, e.g., Abraham and Robbin [1]) can be applied to show that if (43) can occur then almost any small perturbation of Z will eliminate the possibility. For, by (41), Z, or alternatively X, is a function of $l + m$ variables whereas (43) is a set of $l + m + 1$ equations.

To sum up, in the ϵ version of the economy with ϵ small, critical equilibria of the form studied in earlier sections are pathological. However, in economies with uncertainty there can exist nonperfectly competitive equilibria as limit points, the incentive for adopting noncompetitive behavior being that changes in production lead to a nonexistence of equilibrium prices. Thus, the introduction of uncertainty is insufficient to ensure that the monopolistically competitive equilibria of large economies approximate a system of perfect competition.

7. CONCLUDING REMARKS

This paper has investigated and characterized the form of the limit points that arise under replication of a monopolistically competitive economy. Of particular interest has been the uncovering of critical equilibria. The nonpathological existence of these equilibria demonstrates a general point. In decentralized systems like perfect competition, there is no possibility for individual agents to exploit irregularities in the economic structure. However, when agents manipulate the system to which they belong, it is possible, and not unlikely, that irregular situations will produce beneficial outcomes. In these circumstances, equilibria cannot be properly understood without the irregularities of the economic structure being understood. In this paper, these irregularities have been critical exchange equilibria. At such equilibria, it has been shown that intuition and standard results are a poor guide to the possibilities that can exist.

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JOURNAL OF ECONOMIC THEORY **22**, 279–312 (1980)

Perfect Competition and Optimal Product Differentiation

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Received February 6, 1979

This paper considers whether a competitive economy will achieve an optimal allocation when firms produce differentiated products. In the model studied, firms market those products which maximize profit given the demand curves facing them. Markets are assumed to be open only for goods actually traded. Competitive conditions are achieved by replicating the consumer sector. It is shown that market equilibrium may be suboptimal if those products not being produced are highly complementary in consumption. Sufficient conditions on preferences and production sets for the market equilibrium to be optimal are given.

1. INTRODUCTION

Product differentiation under imperfectly competitive conditions has been the focus of a number of studies in the last few years (see, for example, Dixit and Stiglitz, 1977; Spence, 1976). Product differentiation under perfect competition has, however, received considerably less attention. One reason for this may be the widely held view that producers of nonhomogeneous goods possess monopoly power and hence that markets where products are differentiated are inherently noncompetitive. In a previous paper (Hart, 1979a), we argued that this is not the case. We showed that what causes a firm to have monopoly power is not that the firm is producing a unique commodity, but rather that the firm is a large part of the aggregate economy or, more precisely, of the potential market for its output. Of course, if a firm is producing a unique commodity, it will in general, even if it is small, earn supernormal profits or rents, i.e., it will have considerable *market* power. This, however, is quite consistent with the fact that it faces a perfectly elastic demand curve, i.e., with the fact that it has no *monopoly* power.

In Hart (1979a), we used the result that a small firm faces a perfectly elastic demand curve to show that a monopolistically competitive equilibrium, in which firms maximize profits subject to correct conjectures concerning the demand for all potential (differentiated) commodities, is approximately Pareto optimal if the economy is large. One assumption which was made in order to establish this result was that consumers' preferences are convex and

0022-0531/80/020279–34$02.00/0

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can be represented by a differentiable utility function. This assumption may seem relatively innocuous at first sight, but on closer examination it turns out to be rather crucial. The purpose of the present paper is to explore the consequences of relaxing this assumption.

Our principal result is that if preferences are not differentiable and convex, a monopolistically competitive equilibrium may not be Pareto optimal even when the economy is large. In particular, the economy can get stuck in a situation where the set of goods being produced is socially inefficient. The reason for this is not that firms possess monopoly power. Rather, it is that, if firms' products are strongly complementary, the gain (both private and social) from several firms simultaneously producing new products may far outweigh the sum of the gains to each firm from producing a new product by itself. To take a very simple example, suppose that neither nuts nor bolts are being produced. The revenue that a bolt manufacturer would obtain from setting up in the absence of a supply of nuts might be expected to be small and hence may not cover costs. The same of course is true of a nut manufacturer who sets up in the absence of a supply of bolts. However, if the two manufacturers set up simultaneously, revenue will in general be much higher and so both manufacturers may be able to make a profit. In an atomistic economy, however, there may be no mechanism which coordinates the actions of the nut manufacturer and the bolt manufacturer. Thus, the equilibrium of the economy may involve neither nuts nor bolts being produced even though such a situation is Pareto suboptimal.

The presence of complementarities of this type captures to some extent the idea that there can be pecuniary externalities in the production process (see, e.g., Scitovsky, 1954). In particular, the profits of one firm depend on the actions of another firm even under competition. We will see that such pecuniary externalities are ruled out if consumers have convex, differentiable utility functions (or if they have quasi-convex utility functions).

While a monopolistically competitive equilibrium may not be Pareto optimal if preferences are not differentiable and convex, it is possible to establish that the equilibrium is optimal in a weaker sense. We will show that, when the economy is large, it is impossible for a central planner to Pareto dominate a monopolistically competitive equilibrium allocation using only the goods which are traded in the monopolistically competitive equilibrium plus at most one new good. That is to say, a monopolistically competitive equilibrium is Pareto optimal subject to the constraint that a planner can open at most one new market.

The paper is organized as follows. The model is presented in Section 2. In Section 3, the properties of monopolistically competitive equilibria in large economies are studied. In Section 4, examples of suboptimal equilibria when preferences are not differentiable and convex are presented. In one of the examples, the nonconvexity of preferences is caused by the presence of transaction costs. Concluding remarks appear in Section 5, and proofs are contained in the Appendix.

2. THE MODEL

We consider an economy in which there are I types of consumers, J types of firms, and $(K+1)$ potential goods, indexed by $i = 1,..., I$, $j = 1,..., J$, and $k = 0,..., K$, respectively. We assume for the sake of simplicity that I, J, K are all finite (the extension to the case where I, J, K are infinite can be carried out along the lines of Hart (1979a)).

Each firm will be assumed to use good 0 as an input to produce goods $1,..., K$ as outputs.[1] In interpreting the model, we will be particularly interested in cases where not all potential goods $k > 0$ are actually produced in equilibrium. In contrast to the usual Arrow–Debreu model, we will assume that if a good $k > 0$ is not produced there is no market for it; that is, no price is quoted for it. In other words, the firms which produce a particular good "make" the market for that good. (However, we assume that there is always a market for good 0, the common input. In addition, it will be assumed that consumers have no endowments of goods $1,..., K$ and hence there is no reason for consumers to trade these goods among themselves.)

Note that our formulation is general enough to include product differentiation in the Chamberlinian or Lancastrian sense. In particular, some of the goods $1,..., K$ may represent different qualities of the same good, or the same good in different locations.

Our objective is to explain which goods are produced, i.e., which markets are open, and to explain the volume of trade in these markets.

We begin with a more precise description of the agents in the economy.

Firms

As indicated above, firms use good 0 to produce goods $1,..., K$. A type j firm $(j = 1,..., J)$ has a production set $Y_j \subset R_- \times R_+^K$, with the interpretation that $y \in Y_j$ means that it is feasible for firm j to produce the vector of outputs $(y_1,..., y_K)$ using $-y_0$ units of good 0.[2] Note that we treat inputs as nonpositive numbers. We assume that for each j:

(A.1) $0 \in Y_j$, and Y_j is closed.

---

[1] We do not consider the case where some goods are intermediate goods. See Makowski (1980) for a discussion of additional coordination problems which can arise when there are intermediate goods.

[2] We use the notation $x \geqslant 0$ if each component of x is nonnegative. Also we define $R_+^m = \{x \in R^m \mid x \geqslant 0\}$, $R_-^m = \{x \in R^m \mid 0 \geqslant x\}$.

(A.2) Given $y \in Y_j$ and $u \in R_-^K$,

$$y + (0, u) \in R_- \times R_+^K \Rightarrow y + (0, u) \in Y_j.$$

(A.3) There exists $\bar{y}_{j_0} < 0$ such that $y \in Y_j$ and $y \neq 0 \Rightarrow y_0 \leq \bar{y}_{j_0}$.

Assumption (A.2) says that there is free disposal of produced goods. Assumption (A.3) says that there is a setup cost for any firm. The existence of setup costs is consistent with the assumption that markets exist only for goods $k > 0$ which are produced (a price will be quoted only if it is profitable for a firm to set up and produce the good in question).

Subject to the payment of a setup cost, there is assumed to be free entry of type j firms, $j = 1,..., J$.

We will not place any convexity restrictions on Y_j (Y_j itself will not be convex, of course, because of (A.3); however, we will also not require $Y_j \backslash \{0\}$ to be convex). In particular, we allow for cases where a firm may be able to produce several goods, but only one at any one time (e.g., if good 1 is a Rolls Royce, and good 2 is a Volkswagen, it may be that $(y_0, 1, 0) \in Y_j$ and $(y_0', 0, 1) \in Y_j$, but there may be no $y \in Y_j$ with y_1 and $y_2 > 0$). We will assume, however, that all goods are divisible.

We will also assume that each good $k = 1,..., K$ can be produced:

(A.4) Given $k = 1,..., K$, there exists j and $y \in Y_j$ such that $y_k > 0$.

Consumers

Each type i consumer ($i = 1,..., I$) is assumed to have a continuous utility function U_i defined on his consumption set, which we will take to be R_+^{K+1}. We will assume that good 0 is desirable and that positive consumption of it is essential to get positive utility:

(A.5) For all $i = 1,..., I$: $U_i(x) > U_i(0) \Rightarrow x_0 > 0$. Furthermore, if $x \in R_+^{K+1}$, then $U_i(x + (u_0, 0,..., 0)) > U_i(x)$ if $u_0 > 0$ and

$$\lim_{u_0 \to \infty} U_i(x + (u_0, 0,..., 0)) = \infty.$$

We also assume that each good $k = 1,..., K$ is desired by some consumer:

(A.6) For each $k = 1,..., K$, there exists i such that: if $x \in R_+^{K+1}$ then $U_i(x + (0,..., 0, u_k, 0,..., 0)) > U_i(x)$ if $u_k > 0$ and

$$\lim_{u_k \to \infty} U_i(x + (0,..., 0, u_k, 0,..., 0)) = \infty.$$

We will not, however, assume that preferences are convex. As will be seen later, the possible nonconvexity of preferences will play an important role in our analysis.[3]

Each type i consumer is assumed to have an initial endowment of goods, $w_i \in R_+^{K+1}$, where $w_{ik} = 0$ for all $k > 0$ and $w_{i0} > 0$, and an initial shareholding in firms. To simplify matters, we will assume that a consumer's shareholding is the same across all firms of the same type, so that we can talk simply about a consumer's shareholding in type j firms. We will also assume that all type i consumers have the same initial shareholdings. We will denote by $\bar{\theta}_{ij}$ the aggregate shareholding of type i consumers in type j firms, where $\bar{\theta}_{ij} \geqslant 0$ and $\sum_{i=1}^{I} \bar{\theta}_{ij} = 1$ for each j.

The Economy E_r

As indicated in the introduction, we are interested in analyzing whether the right mix of products is produced in a perfectly competitive environment. In order to ensure that the environment is indeed perfectly competitive, we make each firm small relative to the aggregate economy by replicating the consumer sector, at the same time keeping the characteristics (i.e., production sets) of the J types of firms $j = 1,..., J$ fixed. (Although the characteristics of the production sector remain fixed, the growth of the consumption sector allows a greater number of firms to operate since setup costs, as a fraction of the economy's aggregate endowment, become smaller.)

This leads us to consider the economy E_r consisting of r consumers of each type $i = 1,..., I$, where each type i consumer has utility function U_i and endowment w_i as indicated above. Since all type i consumers are identical, the shareholding of each type i consumer in type j firms is given by $\bar{\theta}_{ij}/r$.

Exchange Equilibrium in E_r

In order to define an equilibrium for the economy E_r, we proceed in two stages. First, taking firms' production decisions to be already determined, we define an exchange equilibrium relative to these production decisions. Second, we turn to an analysis of how these production decisions are made.

Because of the existence of setup costs, only a finite number of firms will operate in equilibrium in the economy E_r. Suppose that n_j firms of type j choose to operate and label these $(j, 1), (j, 2),..., (j, n_j)$. Let the production decisions of these firms be given by $y_{j1},..., y_{jn_j} \in Y_j$ $(j = 1,..., J)$. Define

$$y_j = (y_{j1} + \cdots + y_{jn_j})/r \qquad (2.1)$$

---

[3] Some of the problems which are caused by nonconvexities in preferences also arise if consumption sets are nonconvex. To simplify matters, we will not consider the complications caused by nonconvex consumption sets in this paper.

to be the aggregate production plan of type j firms deflated by r ($y_j = 0$ if $n_j = 0$, of course), and let $y = (y_1, \dots, y_J)$. Also let $\mathcal{Y}$ denote the array $(y_{jt})_{j=1,\dots,J,\, t=1,\dots,n_j}$.

Consider the resulting exchange equilibrium. Define $R_+^* = R_+ \cup \{\infty\}$, and let $p \in R_+^{*(K+1)}$ be the equilibrium price vector with the understanding that, for $k > 0$, p_k is finite if $\sum_{j=1}^{J} y_{jk} > 0$, i.e., if good k is actually produced, and $p_k = \infty$ if $\sum_{j=1}^{J} y_{jk} = 0$, i.e., if good k is not produced (we will see below that setting $p_k = \infty$ is equivalent to assuming that there is no market for good k).

We will normalize prices so that $p_0 = 1$. Let x_{it} denote the post-trade consumption bundle of the tth consumer of type i. It will be convenient to represent the consumption bundles of type i consumers by a measure. For each i define the probability measure μ_i on the Borel subsets of R_+^{K+1} as follows:

$$\mu_i(B) = \#\{t \mid x_{it} \in B\}/r \tag{2.2}$$

for each Borel subset B of R_+^{K+1}.[4] Here $\mu_i(B)$ is simply the fraction of type i consumers whose consumption bundles lie in B. Let Supp μ_i denote the support of the measure μ_i and let $\| \ \|$ denote the Euclidean norm. Also define $\mu = (\mu_1, \dots, \mu_I)$. Finally, let

$$\begin{aligned} Y_r = \{ y = (y_1, \dots, y_J) \mid & \text{ for each } j,\ y_j = (y_{j1} + \cdots + y_{jn_j})/r \\ & \text{for some } n_j \text{ and some } y_{j1}, \dots, y_{jn_j} \in Y_j \}. \end{aligned} \tag{2.3}$$

We are now ready to define an *approximate exchange equilibrium* in the economy E_r, relative to the deflated production plans y.

DEFINITION. An ϵ-exchange equilibrium, $\epsilon \geq 0$, in the economy E_r is an array (μ, p, y), where $y \in Y_r$ and

$$\begin{aligned} &\mu = (\mu_1, \dots, \mu_I) \text{ and each } \mu_i \text{ is a probability measure} \\ &\qquad \text{defined on the Borel subsets of } R_+^{K+1}; \end{aligned} \tag{2.4}$$

$$p \in R_+^{*(K+1)},\ p_0 = 1, \text{ and, for } k > 0,\ p_k \text{ is finite} \Leftrightarrow \sum_{j=1}^{J} y_{jk} > 0; \tag{2.5}$$

$$p \cdot x_i \leq p \cdot w_i + \sum_j \bar{\theta}_{ij} p \cdot y_j \qquad \text{for all} \quad x_i \in \text{Supp } \mu_i \quad (i = 1, \dots, I); \tag{2.6}$$

For all $x \in$ Supp μ_i,

$$U_i(x') > U_i(x) + \epsilon \Rightarrow p \cdot x' > p \cdot w_i + \sum_j \bar{\theta}_{ij} p \cdot y_j \qquad (i = 1, \dots, I); \tag{2.7}$$

---

[4] $\#C$ denotes the number of elements in the set C.

$r\mu_i(B)$ is integral for all Borel subsets B of R_+^{K+1} $(i = 1, \ldots, I)$; (2.8)

$$\sum_{i=1}^{I} \int x \, d\mu_i - \sum_{j=1}^{J} y_j - \sum_{i=1}^{I} w_i = 0. \tag{2.9}$$

Condition (2.6) says that consumers satisfy their budget constraints. Condition (2.7) says that each consumer's equilibrium bundle provides utility which is within ϵ of maximum utility subject to the budget constraint. Finally, (2.9) says that demand equals supply for each good $0, \ldots, K$. We adopt the convention $\infty \cdot 0 = 0$, $\infty \cdot h = \infty$ if $h > 0$. It then follows that $x_{itk} = 0$ whenever $p_k = \infty$, so that as noted above setting $p_k = \infty$ is equivalent to assuming that there is no trading in good k.

The exchange equilibrium we have defined is an approximate one in the sense of Starr (1969) (it is exact if $\epsilon = 0$). We study approximate rather than exact equilibria because, since preferences are not necessarily convex, an exact exchange equilibrium may not exist. On the other hand, given appropriate restrictions on firms' production plans, the convexifying effects of replication will ensure the existence of an ϵ-exchange equilibrium for large enough r.[5] Note that, given ϵ, an ϵ-exchange equilibrium depends on the particular cardinalization U_i of consumer i's underlying preferences selected. However, the set of all ϵ-exchange equilibria, as we let ϵ vary, is independent of any particular cardinalization.

Note that different type i consumers will in general consume different bundles in an ϵ-exchange equilibrium.

We turn now to the determination of firms' production plans. We will assume that each firm chooses its production plan to maximize profits (measured in units of good 0).[6] We do not assume, however, that firms are price takers (this assumption would in any case be meaningless in the case of goods not being produced since there are no prices to "take"). Instead we assume that each firm, acting under the Cournot-Nash assumption that other firms' production plans are fixed, maximizes profit on the basis of a correct calculation concerning the relationship between the amount of any good it produces and the price which it will receive for this good (this calculation is made both for goods already being produced and for new goods). Such a calculation is of course in general a very complicated one, and the assumption that firms can carry it out is therefore strong. We will make this assumption because we wish to consider the misallocations which result from the inherent characteristics of the economy rather than those which result from the mistakes of the agents.[7]

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[5] An analysis of the existence of various different types of approximate equilibria may be found in Starr (1969).

[6] Conditions under which this objective represents the interests of the firm's owners may be found in Hart (1979b).

There is a problem which must be faced before we can complete the specification of a firm's behavior. In general, there will be many ϵ-exchange equilibria corresponding to a particular choice of production plan for the firm. The firm's decision whether to choose the production plan will depend on which of these equilibria is selected. In order to surmount this difficulty, we follow Novshek and Sonnenschein (1978) in assuming that an exogenous *selection* is made from the set of ϵ-exchange equilibria. Let

$$T_r = \{ y \mid y \in Y_r, \text{ and there exists an } \epsilon\text{-exchange equilibrium } (\mu, p, y) \text{ in } E_r\}. \tag{2.10}$$

Also let G be the set of all probability measures defined on the Borel subsets of R_+^{K+1}. We assume that, for each r, there is a function $F_r\colon T_r \to G^I \times R_+^{*(K+1)}$ satisfying, for all $y \in T_r$,

$$(F_r(y), y) = (\mu, p, y) \text{ is an } \epsilon\text{-exchange equilibrium in the economy } E_r. \tag{2.11}$$

If $F_r(y) = (\mu, p)$, we will use the shorthand $F_{r1}(y) = \mu$, $F_{r2}(y) = p$.

We are now ready to define a monopolistically competitive equilibrium in which firms maximize profits subject to the demand curves facing them. In the following definition, y_{-j} is a shorthand for $(y_1, \ldots, y_{j-1}, y_{j+1}, \ldots, y_J)$.

DEFINITION. An ϵ-monopolistically competitive equilibrium (ϵ-MCE), $\epsilon \geqslant 0$, in E_r is an array $(\mu, p, \mathscr{Y})$, where $\mu = (\mu_i)_{i=1,\ldots,I}$ and $\mathscr{Y} = (y_{jt})_{j=1,\ldots,J,\, t=1,\ldots,n_j}$ such that

$$y_{jt} \in Y_j \text{ for each } t, j; \tag{2.12}$$

$$(\mu, p, y) \text{ is an } \epsilon\text{-exchange equilibrium in } E_r, \text{ where}$$
$$y = (y_1, \ldots, y_J) \quad \text{and} \quad y_j = \left(\sum_{t=1}^{n_j} y_{jt}/r\right) \quad \text{for each } j; \tag{2.13}$$

there does not exist (t, j), $1 \leqslant t \leqslant n_j$, $1 \leqslant j \leqslant J$, and $y'_{jt} \in Y_j$, such that $(y_{-j}, y_j - y_{jt}/r + y'_{jt}/r) \in T_r$ and

$$F_{r2}(y_{-j}, y_j - y_{jt}/r + y'_{jt}/r) \cdot y'_{jt} > p \cdot y_{jt}; \tag{2.14}$$

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[7] To put it another way, whereas in the standard Arrow–Debreu model, a good entrepreneur is simply someone who knows his firm's production set well and who can therefore calculate the profit maximizing production plan, in the present context a good entrepreneur must also be skilled at market research, i.e., at calculating the way the economy's general equilibrium varies with his firm's production plan.

there does not exist j, $1 \leqslant j \leqslant J$, and $y'_j \in Y_j$, such that

$$(y_{-j}, y_j + y'_j/r) \in T_r \quad \text{and} \quad F_{r2}(y_{-j}, y_j + y'_j/r) \cdot y'_j > 0; \tag{2.15}$$

$$p \cdot y_{jt} \geqslant 0 \qquad \text{for all } t, j. \tag{2.16}$$

Condition (2.14) says that no firm which is already operating can increase its profit by moving to a new production plan, while (2.15) says that no new firm can enter and make a positive profit. Finally, (2.16) says that all operating firms break even. (It might seem as though (2.14) already implies (2.16), but this is not the case if $(y_{-j}, y_j - y_{jt}/r) \notin T_r$.)

Finally, it will be useful to define a feasible allocation in the economy E_r.

Definition. A *feasible allocation* in the economy E_r is an array (μ, y), where $y \in Y_r$, satisfying (2.4), (2.8), and (2.9).

3. Properties of Monopolistically Competitive Equilibria in Large Finite Economies

We study now the properties of a monopolistically competitive equilibrium (MCE) in E_r when r is large.

Theorem 1. *Assume* (A.1)–(A.6), *and that Y_j is bounded for all j. Suppose that $\epsilon > 0$ is given. Then we can find a selection function F_r: $T_r \rightarrow G^I \times R_+^{*(K+1)}$ and a number $\bar{r}$ such that, for all $r \geqslant \bar{r}$, there exists an ϵ-MCE in E_r.*

Proof. See Appendix.

Note that Theorem 1 establishes only the existence of an ϵ-MCE for $\epsilon > 0$. Establishing the existence of an exact MCE, i.e., an ϵ-MCE with $\epsilon = 0$, is considerably more complicated and stronger assumptions are needed. See Novshek and Sonnenschein (1978).

Our main interest is to study the welfare properties of an MCE in a large economy. To this end, it is useful to introduce a measure of how far away an equilibrium allocation is from being Pareto optimal. Suppose that we are given a feasible allocation (μ_r, y_r) in E_r which satisfies $\| U_i(x) - U_i(x')\| \leqslant \epsilon$ for all $x, x' \in \operatorname{Supp} \mu_i$ $(i = 1,..., I)$ (this is an approximate equal treatment condition). Let E'_r be the economy which is identical to E_r except that the aggregate endowment of good 0 is $(r \sum_{i=1}^{I} w_{i0}) - rh$ rather than $(r \sum_{i=1}^{I} w_{i0})$, and define

$$\begin{aligned} H_r(\epsilon, \mu_r, y_r) = \sup \{h \mid & \text{ there exists a feasible allocation } (\mu'_r, y'_r) \text{ in } E'_r \\ & \text{such that } U_i(x') \geqslant U_i(x) + \epsilon \text{ for all} \\ & x \in \operatorname{Supp} \mu_{ir}, x' \in \operatorname{Supp} \mu'_{ir} \text{ and } i = 1,..., I\}. \end{aligned} \tag{3.1}$$

The number $H_r(\epsilon, \mu_r, y_r)$ is a measure (in per capita terms) of how far the allocation (μ_r, y_r) is from being a Pareto optimum in E_r, given that we insist that a planner makes everybody better off by at least ϵ. Note that if (μ_r, y_r) is Pareto optimal, then $H_r(\epsilon, \mu_r, y_r) = 0$ for all $\epsilon \geqslant 0$.

As well as being interested in unconstrained Pareto optimality, we will also wish to consider a notion of constrained or restricted Pareto optimality. In particular, we want to study the consequences of allowing a planner to alter firms' production plans and reallocate goods to consumers subject to the constraint that, in the new configuration of firms' production plans, *at most one new good* is being produced relative to the initial allocation (μ, y).

Given the production allocation $y = (y_1, ..., y_J)$, let $M(y) = \{k > 0 \mid \sum_{j=1}^{J} y_{jk} > 0\} \cup \{0\}$ be the set of goods which are produced at y, plus good 0. $M(y)$ represents the set of markets which are open at y. Define

$$\begin{aligned} h_r(\epsilon, \mu, y) = \sup\{h \mid\ & \text{there exists a feasible allocation } (\mu_r', y_r') \text{ in } E_r' \\ & \text{such that } U_i(x') \geqslant U_i(x) + \epsilon \text{ for all} \\ & x \in \operatorname{Supp} \mu_{ir},\ x' \in \operatorname{Supp} \mu_{ir}' \text{ and } i = 1, ..., I, \\ & \text{and } M(y_r') \backslash M(y_r) \text{ contains at most one element}\}. \end{aligned} \tag{3.2}$$

The number $h_r(\epsilon, \mu, y)$ is a measure (again in per capita terms) of how far the allocation (μ_r, y_r) is from being a Pareto optimum in E_r, given that the planner must make everybody better off by at least ϵ, and given that the planner is constrained to produce at most one new good, i.e., to open at most one new market (note that a planner is free to close markets, i.e., $M(y_r')$ can contain fewer elements than $M(y_r)$). Of course, $h_r(\epsilon, \mu_r, y_r) \leqslant H_r(\epsilon, \mu_r, y_r)$.

In Theorems 2 and 3 below, we give conditions under which $h_r(\epsilon, \mu_r, y_r)$ and $H_r(\epsilon, \mu_r, y_r)$ are small when r is large if (μ_r, p_r, y_r) is an ϵ-MCE. In order to prove these theorems, we need to assume that the limit of a sequence of production allocations $y_r \in Y_r$ is a *regular point* with respect to the selection functions F_r.

Let $\hat{Y}_j$ be the smallest closed convex cone with vertex at the origin which contains Y_j. Then $\prod_j \hat{Y}_j$ is the smallest closed set containing all the Y_r, $r = 1, 2, ...$, (see (2.3)). In the following definition, limits of the measure μ_r are taken with respect to the weak convergence topology (see Parthasarathy (1969)) and limits of prices with respect to the standard topology on $R_+^{*(K+1)}$.

DEFINITION. We say that $y \in \prod_{j=1}^{J} \hat{Y}_j$ is a regular point with respect to the selection functions $F_1, F_2, ...$, if:

(1) Given any two sequences (y_r), (y_r') of points in T_r converging to y, and limit points (μ, p), (μ', p') of the sequences $(F_r(y_r))$, $(F_r(y_r'))$, respectively, it is the case that $\mu' = \mu$ and $p_k' = p_k$ for all k such that $k = 0$ or $\sum_{j=1}^{J} y_{jk} > 0$.

(2) Given any sequence (y_r) of points converging to y, where $y_r \in Y_r$ for each r, it is the case that $y_r \in T_r$ for sufficiently large r.

In other words, y is a regular point as long as $F_r(y_r)$ changes continuously as y_r varies in a neighborhood of y. For a discussion of the importance of regularity in cases where firms take into account their effect on prices, see Novshek and Sonnenschein (1978) and Roberts (1980).

Before stating Theorem 2, we need one further assumption:

$$\text{(A.7)} \quad \tilde{Y}_j \cap R_+^{K+1} = \{0\} \qquad (j = 1, \ldots, J).$$

Assumption (A.7) rules out cases where it is possible to get something for nothing in per capita terms in the limit $r \to \infty$ by operating type j firms.

THEOREM 2. *Let* $\epsilon \geqslant 0$ *and the selection functions* $F_1, F_2, \ldots,$ *be given. Suppose that* $(\mu_r, p_r, \mathscr{Y}_r)$ *is a sequence of* ϵ*-MCEs in the economies* E_r, $r = 1, 2, \ldots,$ *and, for each r, define* y_r *as in* (2.1). *Assume* (A.1)–(A.7), *and that each limit point* y *of* (y_r) *is regular. Assume also that for some* $\eta > 0$ *and* r' *the following is true for* $k > 0$: *either* $\sum_{j=1}^{J} y_{jkr} > \eta$ *or* $\sum_{j=1}^{J} y_{jkr} = 0$ *for all* $r \geq r'$. *Define* $h_r(\epsilon, \mu_r, y_r)$ *as in* (3.2). *Then* $\lim_{r\to\infty} h_r(\epsilon, \mu_r, y_r) = 0$.

It should be noted that the condition $\sum_{j=1}^{J} y_{jkr} > \eta$ or $= 0$ for large r rules out situations where the set of markets which are open changes discontinuously in the limit as $y_r \to y$.

Theorem 2 is proved in the Appendix. It is useful, however, to give some intuition behind the result. Suppose to simplify matters that $\mu_r \to \mu$, $p_r \to p$, $y_r \to y$. The proof proceeds by showing that the allocation (μ, y) is a competitive equilibrium allocation relative to the markets $M(y) \cup \{k'\}$ for any $k' = 1, \ldots, K$ in some well-defined limit economy. In particular, in the limit, firms, subject to the constraint that they can market only goods in $M(y) \cup \{k'\}$, maximize profits as price takers at prices p^*, where $p_k^* = p_k$ for $k \in M(y)$ and $p_{k'}^*$ is chosen to be the reservation price for good k', i.e., it is the lowest price at which demand for good k' is zero even though the market for good k' is open. A standard revealed preference argument then shows that $h_r(\epsilon, \mu_r, y_r)$ must be zero in the limit.

To see why firms must maximize profits as price takers in the limit, suppose that they do not. Then this means that there exists $y' \in Y_j$ such that $p^* \cdot y' > 0$. Consider now the consequences of an additional type j firm setting up with the production plan y' in the economy E_r, starting at the ϵ-MCE (μ_r, p_r, y_r). While this action will have some effect on prices, this effect will be very small when r is large since the new prices are given by $F_{r2}(y_{-jr}, y_{jr} + y'/r)$ and this is approximately equal to $F_{r2}(y_{-jr}, y_{jr})$ when r is large. In other words, firms have a negligible effect on prices when r is large since prices depend on per capita production, $(y_{-jr}, y_{jr} + y'/r)$, and an individual firm has a very small influence on this (this argument is only formally correct if y is a regular point).

But since p_r hardly changes and since $p_r \to p^*$ as $r \to \infty$, it follows that the new firm will make positive profits by setting up with the production plan y' in E_r when r is large, which is not consistent with (μ_r, p_r, y_r) being an ϵ-MCE. This completes the proof.

It should be emphasized that Theorem 2 does not tell us that a monopolistically competitive allocation is (approximately) fully Pareto optimal when r is large, but only that it is (approximately) Pareto optimal subject to the constraint that a planner can produce at most one new good, i.e., open at most one new market. In fact, we will see in the examples of the next section that an MCE may be far from optimal in the full sense.

If we are prepared to make further assumptions, however, we can establish the full Pareto optimality of an MCE in the limit $r = \infty$. Consider first how we might try to extend the sketch of the proof of Theorem 2 given above to the case where several new goods can be produced. Define, for each good k' which is not being produced in the limiting allocation (μ, y), $p^*_{k'}$ to be the reservation price for good k'; i.e., $p^*_{k'}$ is the smallest price which is such that demand for good k' is zero if markets for goods $k \in M(y) \cup \{k'\}$ are open. It is not difficult to show that in the limit firms are maximizing profits at these prices. The problem with this approach is that if we now open markets for goods and quote prices p_k for goods k in $M(y)$ and $p^*_{k'}$ for goods k' not in $M(y)$, there is no guarantee that markets will clear! The reason is that $p^*_{k'}$ may be a reservation price for k' when only goods k in $M(y) \cup \{k'\}$ can be purchased, and $p^*_{k''}$ may be a reservation price for k'' when only goods k in $M(y) \cup \{k''\}$ can be purchased, but $p^*_{k'}$ and $p^*_{k''}$ need not be reservation prices when goods k in $M(y) \cup \{k'\} \cup \{k''\}$ can be simultaneously purchased. In particular, if k' and k'' are strongly complementary, then if only one of these goods is available, the willingness of consumers to pay for it may be low, i.e., the reservation price will be low, while if both goods are simultaneously available, the willingness to pay, and hence the corresponding reservation prices, may be much higher.

While the above is in general a problem, there are some situations in which the problem does not arise. Specifically, there are classes of preferences for consumers which have the property that goods are always sufficiently substitutable that the reservation prices of the goods taken one at a time are also reservation prices for all the goods simultaneously.

Given a subset S of $\{1, 2, \ldots, K\}$, define $R^S = \{x \in R^{K+1} \mid x_k = 0 \text{ if } k \notin S\}$.

Definition. We will say that a type i consumer satisfies the *simultaneous reservation price property* (SRPP) if given (1) any subset $\bar{S}$ of $\{0, 1, \ldots, K\}$ containing 0; (2) $p^* \in R_+^{*(K+1)}$ such that p^*_k is finite for all $k \in \bar{S}$; (3) $x \in R_+^{\bar{S}}$

such that $p^* \cdot x > 0$, and for all $S = \tilde{S} \cup \{k'\}$, where $k' = 1,\dots, K$, $x' \in R_+{}^S$ and $U_i(x') > U_i(x) \Rightarrow p^* \cdot x' > p^* \cdot x$; then it is the case that

$$x' \in R_+^{K+1} \quad \text{and} \quad U_i(x') > U_i(x) \Rightarrow p^* \cdot x' > p^* \cdot x. \tag{3.3}$$

In the above definition, we start off with a subset $\tilde{S}$ of marketed goods, prices for these goods, and a feasible consumption bundle x containing only these goods. These prices are then extended to cover all possible goods in such a way that, if we add one new good at a time, although the consumer can now trade in the new good, the price of the good is sufficiently high that he does not wish to do so. The simultaneous reservation price property is satisfied if we may conclude that the consumer will not wish to trade when *all* goods are available.

The next theorem gives conditions for an MCE to be approximately unconstrained Pareto optimal. It applies only to exact MCEs, i.e., to ϵ-MCEs with $\epsilon = 0$.

THEOREM 3. *Let the selection functions* $F_1, F_2, \dots,$ *be given. Suppose that* $(\mu_r, p_r, \mathscr{Y}_r)$ *is a sequence of* ϵ*-MCEs, with* $\epsilon = 0$, *in the economies* E_r, $r = 1, 2, \dots,$ *and, for each r, define* y_r *as in* (2.1). *Assume* (A.1)–(A.7), *and that each limit point* y *of* (y_r) *is regular. Assume also that for some* $\eta > 0$ *and* r' *the following is true for* $k > 0$: *either* $\sum_{j=1}^J y_{jkr} > \eta$ *or* $\sum_{j=1}^J y_{jkr} = 0$ *for all* $r \geqslant r'$. *Define* $H_r(\epsilon, \mu_r, y_r)$ *as in* (3.1). *Then, if SRPP is satisfied by type i consumers for all* $i = 1,\dots, I$, $\lim_{r\to\infty} H_r(0, \mu_r, y_r) = 0$.

Proof. See Appendix.

How strong is SRPP? The following proposition provides two conditions under which SRPP is satisfied.

PROPOSITION 4. *A type i consumer satisfies SRPP if either one of the following conditions holds*:

(1) U_i *is defined on an open subset containing* R_+^{K+1}, *and is differentiable, quasi-concave, and strictly increasing in every argument*;

(2) U_i *is quasi-convex, i.e.,* $U_i(x) = U_i(x') \Rightarrow U_i(\lambda x + (1-\lambda)x') \leqslant U_i(x)$ *if* $0 \leqslant \lambda \leqslant 1$.

Proof. (1) Consider the SRPP definition. Since U_i is quasi-concave and differentiable, (3) of the definition implies by the Kuhn–Tucker necessary conditions that there exists $\lambda \geqslant 0$ such that $\partial U_i/\partial x_k \leqslant \lambda p_k^*$ for all $k \in S$ with equality if $x_k > 0$. Since $p^* \cdot x > 0$, $x \neq 0$ and so

$$\lambda = \max_k \left[\frac{\partial U_i}{\partial x_k} \Big/ p_k^*\right]$$

is independent of k'. Hence, by the Kuhn-Tucker sufficiency conditions, (3.3) holds.

(2) Suppose that (3.3) is violated, i.e., there exists $x' \in R_+^{K+1}$ such that $U_i(x') > U_i(x)$ and $p^* \cdot x' \leq p^* \cdot x$. Assume also that $p_k^* \neq 0$ for all k. For each $k' \notin \tilde{S}$, let $x(k')$ be the bundle such that $x_k(k') = x'_k$ if $k \in \tilde{S}$, $x_k(k') = 0$ if $k \notin \tilde{S} \cup \{k'\}$ and $x_{k'}(k') = (p^* \cdot x' - \sum_{k \in \tilde{S}} p_k^* x'_k)/p_{k'}^*$. Then x' is a convex combination of the $x(k')$, $k' \notin \tilde{S}$, with weights $p_{k'}^* x'_{k'}/(p^* \cdot x' - \sum_{k \in \tilde{S}} p_k^* x'_k)$. But since $p^* \cdot x(k') \leq p^* \cdot x$, we know from (3) of the SRPP definition that $U_i(x) \geq U_i(x(k'))$ for each $k' \notin \tilde{S}$. Therefore, since U_i is quasi-convex, $U_i(x) \geq U_i(x')$, which is a contradiction.

A contradiction can be obtained on similar lines for the case where $p_k^* = 0$ for some k. Q.E.D.

Putting Theorem 3 and Proposition 4(1) together, we obtain a result which was also proved in Hart (1979a): if preferences are representable by a differentiable, quasi-concave utility function, an MCE is approximately Pareto optimal when r is large.

We have indicated previously that what SRPP rules out is the case where goods $k = 1,..., K$ are strongly complementary, i.e., consuming two or more simultaneously is worth much more than consuming any one by itself. We see from Proposition 4 that this sort of complementarity differs substantially from the usual notion of complementarity. For complementarity in the usual sense that compensated cross elasticities of demand are negative is certainly not ruled out by differentiable, quasi-concave utility functions. Yet complementarity in the sense of this paper *is* ruled out under these conditions.

The inefficiencies that arise when the SRPP condition fails can be traced to a lack of coordination of production decisions provided by the market system. If several highly complementary goods are not being produced, then, while it may be socially desirable for these new goods to be produced simultaneously, it may not be in the interest of any single firm to produce one (or some) of these goods alone, since the price that one new good will fetch on the market in the absence of the other goods may be very low.[8] It is important to note that this coordination failure is not caused by the existence of monopoly power or by a divergence between private and social benefits. In fact, the private and social benefits from product innovation are (approximately) equal when r is large. (This follows from the fact that, as shown in the Appendix, a firm can market new goods at their reservation prices. This implies that there is no consumer surplus and hence that consumer surplus plus producer surplus equals profit.) To put it another way, the problem of coordinating the production of new goods must be faced in a planned economy as well as in a market economy.

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[8] *A similar phenomenon has been noted by Spence (1976) in the case where firms are imperfectly competitive.*

Since there is no divergence between private and social benefits, we might expect the coordination problem not to arise if it is technologically feasible for a single firm to produce all the new goods by itself on the same terms as all firms taken together.

THEOREM 5. *Let* $\epsilon \geqslant 0$ *and the selection functions* $F_1, F_2, \ldots,$ *be given. Suppose* $(\mu_r, p_r, \mathscr{Y}_r)$ *is a sequence of* ϵ*-MCEs in the economies* E_r, $r = 1, 2, \ldots,$ *and, for each* r, *define* y_r *as in* (2.1). *Assume* (A.1)–(A.7) *and that each limit point* y *of* (y_r) *is regular. Assume also that for some* $\eta > 0$ *and* r' *the following is true for* $k > 0$: *either* $\sum_{j=1}^{J} y_{jkr} > \eta$ *or* $\sum_{j=1}^{J} y_{jkr} = 0$ *for all* $r \geqslant r'$. *Then the following is a sufficient condition that* $\lim_{r\to\infty} H_r(\epsilon, \mu_r, y_r) = 0$:

$$Y_j = Y \text{ is independent of } j \text{ and } Y + Y \subset Y. \tag{3.4}$$

Proof. See Appendix.

A similar result to Theorem 5 may be found also in Makowski (1980). It should be noted that, in order to get full optimality, it is not sufficient to assume that each firm can produce every good—we require the stronger assumption that each firm can produce these goods on the same terms as the aggregate economy. An example in Section 4 will illustrate the importance of this latter assumption.

We conclude this section with a number of remarks.

(A) The failure of the market mechanism to generate an optimal allocation of resources in the absence of SRPP can be traced to the existence of pecuniary externalities. Pecuniary externalities arise when the profits of one firm depend on the actions of another firm even though perfectly competitive conditions prevail and there are no direct interdependencies through the production sets (see Scitovsky, 1954). We have seen that, if SRPP is violated, then the profits that a firm gets from marketing a new product will depend on whether other firms are marketing products which are complementary to this new product. Hence there are indeed pecuniary externalities according to the above definition. Note that, in our model, pecuniary externalities only arise with respect to the production of new commodities, i.e., the opening of new markets. This is consistent with the interpretation of pecuniary externalities given by Heller and Starrett (1976). Heller and Starrett, however, trace the existence of pecuniary externalities to firms' incorrect expectations about the prices that new products will fetch on the market. In contrast, in our model, pecuniary externalities, and the market failure that they cause, can occur even if all firms have correct expectations, i.e., even if they know the function $F_r(y)$.[9]

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[9] The coordination problems arising in the absence of SRPP bear a resemblance also to those stressed in the economic development literature by Rosenstein-Rodan (1943)

(B) It is important to note that the sufficient condition for SRPP given in Proposition 4(1)—that consumers' utility functions be convex and differentiable—is a condition on *individual* preferences. In particular, replicating the economy does not convexify preferences in such a way that SRPP will hold approximately in a large economy (even if preferences are differentiable). In fact, the size of the economy has no effect at all on whether a vector of prices, each of which is a reservation price for a particular nonproduced good by itself, will serve as reservation prices for all nonproduced goods taken together.

The fact that convex, differentiable preferences prevent coordination failures has been noted also in other contexts. See Marshak and Radner (1972) for examples in the theory of teams and Grossman (1977) for a discussion of planning when there are many planners.

(C) We have noted that, in the absence of SRPP, the profit which firms can make from simultaneously producing new products may outweigh the sum of the profits to each firm from producing a new product by itself. One way around this problem is for an entrepreneur to take over or merge a number of firms of different types and coordinate the production of new products. If such mergers are costless, then any production plan which can be written as $\sum_{j=1}^{J} \sum_{t=1}^{n_j} y_{jt}$ becomes feasible. Letting Y denote the set of all such plans, we see that $Y + Y \subset Y$ and hence Theorem 5 tells us that, under these conditions, the inefficiencies due to coordination failures disappear.

The assumption that mergers can occur costlessly is, however, strong. First, finding out which firms to merge and which combinations of new products to produce will in general require the use of resources. Second, a merged company may be less efficient than a collection of firms operating independently. For example, suppose that ski services and hotel services are complementary goods. Some entrepreneurs may be very good at providing the former, and other entrepreneurs may be very good at providing the latter, but nobody may be very good at providing both services. An entrepreneur who attempts to coordinate production of these two services may find that he faces a production set which is considerably less efficient than the sum of the individual production sets. Finally, note that, if mergers are easy, they may take place not to coordinate the production of new commodities but with the purpose of restricting competition and exploiting monopoly power. In this case, of course, the competitive model studied in this paper loses much of its plausibility.

This is not to deny, of course, that mergers can be important in overcoming coordination problems. In fact, it may be possible empirically to

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and others. The latter are not traced specifically to the production of new products, however, and would seem to have more to do with imperfectly competitive conditions in the production of existing products.

explain some mergers as attempts to internalize pecuniary externalities. See Makowski (1980) on this point.

(D) It should be emphasized that the assumption that there is free entry of firms, subject to the payment of a setup cost, is *not* crucial to our analysis. n particular, it is not this assumption which is responsible for the realization of competitive conditions when r is large, and hence for making Theorems 2, 3, and 5 work. As we have argued in a previous paper (Hart, 1979a), perfect competition is achieved when individual agents are small relative to the aggregate economy; this is true even if an agent is the sole supplier of a particular good. Of course, an agent with unique characteristics will generally earn supernormal profits or rents, but this does not change the fact that the agent will face a flat demand curve. Hence there will be no divergence between marginal revenue and price. The role of free entry is simply to bid away hese rents. In other words, free entry reduces the height of the demand curve but it does not make it any flatter.

While it is therefore possible to generalize Theorems 2, 3, and 5 to the case where there is not free entry along the lines of Hart [1979a], it should be noted that an extra assumption is now required. This is that firms' production sets are bounded. This assumption is important in showing that operating firms are maximizing profits at the prices p in the limit economy. For if an operating firm's production plan, y_r, grows in such a way that y_r/r is bounded away from zero, then by ceasing to operate this firm will have an effect on per capita supplies and hence on prices even in the limit $r \to \infty$, and so competitive conditions may not prevail.

While this can also happen in the free entry case, it does not matter here. For, in this case, in order to show that operating firms are maximizing profit at prices p, it is enough to show that no new firm can make a profit from entering at a fixed scale. Since when r is large, such a firm will have a negligible effect on prices, competitive conditions do prevail in the appropriate sense even when production sets are unbounded. (So, in particular, Theorems 2, 3, and 5 apply when there are increasing returns to scale.)[10]

(E) Theorem 1 establishes the existence of an ϵ-MCE when r is large for he case $\epsilon > 0$. As we have noted, proving the existence of an ϵ-MCE for $\epsilon = 0$ is considerably more difficult and requires stronger assumptions—see Novshek and Sonnenschein (1978). Among other things, it is necessary to assume for exact existence that demand curves are downward sloping in the neighborhood of a competitive equilibrium in the limit economy. While the approximate equilibrium approach studied here therefore yields some simplifications—in the sense that weaker assumptions are needed—the approach should be treated with caution. In particular, why are consumers prepared to put up with a utility level which is ϵ away from the maximal level? It may be possible to justify this assumption by appealing to transaction costs or information costs. In the absence of a rigorous justification along these or other lines, however, the approximate equilibrium approach is somewhat ad hoc in comparison with the exact approach. (Note that while the use of approximate equilibria is important for our existence result, Theorem 1, it is not important for our optimality results, Theorems 2, 3, and 5. The latter apply also to exact MCEs.)

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[10] I am grateful to William Novshek and Hugo Sonnenschein for helpful discussions on the relationship between free entry and perfect competition.

One assumption which we require to prove the existence of even an approximate MCE is that production sets are bounded. If there are everywhere increasing returns to scale, our proof of Theorem 1 fails. It may, however, be possible to prove existence even in the increasing returns case if stronger assumptions are made on the demand side. For the problem that arises in in the usual competitive situation when there are increasing returns to scale —that firms will expand indefinitely—does not apply here. This is because, since firms are not price takers but face demand curves with finite elasticity, such expansion will not generally be profitable. We hope to analyze the question of existence of monopolistically competitive equilibrium under increasing returns to scale in future work.

4. SOME EXAMPLES

We now present some examples which illustrate how Theorem 3 can break down if SRPP does not hold.

Suppose that $K = 2$ and that there is only one type of consumer, whose preferences are represented by the utility function

$$U = x_0^{1/2} + x_1^{1/2} x_2^{1/2}.$$

Let $J = 2$ and assume that $Y_1 = \{(-1, 1, 0)\} \cup \{0\}$, $Y_2 = \{(-1, 0, 1)\} \cup \{0\}$, and that the aggregate per capita endowment of the unreplicated economy $=$ $w_0 = 3$. Consider the following allocation in E_r: no firms set up, $p_0 = 1$, $p_1 = p_2 = \infty$, $x_0 = 3$ for each consumer. It is easy to see that this is an exact MCE for each r. The reason is that if a type 1 or type 2 firm starts to produce good 1 or good 2, it will make losses since

$$\left.\frac{\partial U}{\partial x_1}\right|_{x_2=0} = \left.\frac{\partial U}{\partial x_2}\right|_{x_1=0} = 0.$$

However, the allocation is Pareto dominated by one in which $x_0 = x_1 = x_2 = 1$ and r firms of each type set up in the economy E_r.

Clearly SRPP is violated here since $p_1 = 0$ is a reservation price given the bundle $(w, 0, 0)$ when only goods 0 and 1 can be traded, $p_2 = 0$ is a reservation price when only goods 0 and 2 can be traded, but $p_1 = p_2 = 0$ is not a reservation price vector when all goods can be traded. Preferences are convex in this example. The reason that Proposition 3(1) does not apply is because U is not *differentiable* when $x_1 = x_2 = 0$. (This example corresponds to the case of nuts and bolts considered in the Introduction.)

While it is clear from the above that difficulties can arise even with convex preferences, it might be argued that problems are likely to be rare in this case, since convex preferences are almost everywhere smooth.[11] We turn now to a case where preferences are nonconvex. We will call this the shopping center example.[12]

Consider a region which is served by a single shopping center. The shopping center is assumed to be of unlimited size so that there is no limit to the number of shops which can set up in it. Consumers are located in various parts of the region and must pay a transportation cost to visit the shopping center. We will assume that this transportation cost—which may be regarded as the opportunity cost of the consumers' time—depends on the time of the day when the trip to the shopping center is made. Shops, as well as deciding what goods to sell, must also decide at what times to be open.

It will be assumed that there are n "basic" commodities which can be provided by shops and that the day is split up into a finite number of subperiods, T. Since two commodities which are available at different times of the day are different goods, the total number of possible goods available in the shopping center is Tn. To this we add the good leisure, which consumers are endowed with and which corresponds to the nonproduced good of the model of Sections 2 and 3. Thus $K + 1 = Tn + 1$.

Let us order the goods so that good 0 is leisure, goods $1, \ldots, n$ are the n basic commodities available in subperiod $t = 1$, goods $n + 1, \ldots, 2n$ are the n basic commodities available in subperiod $t = 2$, etc. Also let $e(t, j) \in R^{Tn+1}$ be the vector whose zeroth coordinate is -1, whose $((t - 1)n + j)$th coordinate is $+1$ and whose other coordinates are zero. We will assume that $J = n$ and that $Y_j = \{e(1, j), e(2, j), \ldots, e(T, j)\} \cup \{0\}$, $j = 1, \ldots, n$. In other words, a type j firm, if it sets up, uses one unit of leisure (i.e., labor) to produce one unit of commodity j, and this unit can then be sold in any one subperiod of the day. We assume, however, that it is impossible for a particular shop to be open in more than one subperiod or to produce more than one unit.

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[11] To be more precise, Mas-Colell (1977) has shown that if preferences are convex, almost all consumption bundles have a unique supporting price vector.

[12] In the following example, not only are preferences nonconvex, but they are also discontinuous, and hence nondifferentiable, at some points. It is the nonconvexity of preferences, however, which is responsible for the breakdown of Theorem 3. Note that examples in which preferences are nonconvex but differentiable, and where Theorem 3 breaks down, can easily be constructed.

To simplify matters, we will assume that there is only one type of consumer. The consumer's preferences are represented by the utility function

$$U(x) = V(x_0 - t(x), (x_1 + x_{n+1} + x_{2n+1} + \cdots + x_{(T-1)n+1}), \ldots, \\ (x_n + x_{2n} + \cdots + x_{Tn})), \tag{4.1}$$

where V is some function mapping R_+^{n+1} into R,

$$t(x) = \sum_{t=1}^{T} b_t \delta\left(\sum_{h=(t-1)n+1}^{tn} x_h\right), \tag{4.2}$$

and $\delta(y) = 1$ if $y > 0$, $\delta(y) = 0$ if $y = 0$.

The idea behind is that consumers are interested only in aggregate consumption of each basic commodity and are indifferent about when the commodity is purchased except in so far as transportation costs are affected by this. Transportation costs are paid by consumers themselves out of leisure time. It is assumed that a visit to the shopping center during subperiod t uses up $b_t > 0$ units of leisure, $t = 1, \ldots, T$, but that, once the consumer is at the shopping center, the cost of shopping is zero.

The consumption set of consumers is given by

$$X = \{x \in R_+^{K+1} \mid x_0 - t(x) \geqslant 0\}. \tag{4.3}$$

In contrast to the model of Sections 2 and 3, $X \neq R_+^{K+1}$. In fact, since t is nonconvex, X is not convex. However, this difference between the models is not important; in particular, Theorem 2 can easily be generalized to the shopping center example.

What is important from our point of view is that the nonconvexity of t makes preferences nonconvex. That is, even if V is a quasi-concave (or concave) function, U will not be; in particular, if $b_1 = b_2$, consumers will be indifferent between purchasing one unit of commodity k in subperiod 1 or one unit of commodity k in subperiod 2, but will prefer either of these possibilities to purchasing half a unit in each subperiod.

Not only are preferences nonconvex, but also SRPP is in general violated.[13] Suppose, for example, that $n = 2$ and that

$$V(x_0, x_1, x_2) = x_0 + x_1^{1/2} + x_2^{1/2}. \tag{4.4}$$

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[13] Note that Proposition 4(2) will not generally apply here since utilities may not be quasiconvex over consumption within subperiod t, once the transportation cost b_t has been incurred.

Consider an initial situation in which the consumer consumes only good 0. What is the consumer's willingness to pay for commodity 1 in some subperiod t? Since such a purchase involves a transportation cost of b_t, it is clear that the consumer must receive at least $(b_t)^2$ units of commodity 1 to make the transaction worthwhile. Since V is concave, it follows that the reservation price of commodity 1 in subperiod t—where we normalize so that the price of good 0 is 1—is just

$$p_1^* = \frac{\partial V}{\partial x_1} \Big/ \frac{\partial V}{\partial x_0}, \qquad \text{evaluated at some } x_1 \geq (b_t)^2, \tag{4.5}$$

where

$$-b_t - p_1^* x_1 + x_1^{1/2} = 0. \tag{4.6}$$

At this price the consumer is just indifferent between purchasing commodity 1 and not purchasing it. (We are assuming that the consumer's endowment of leisure is large.)

It is easy to see that (4.5), (4.6) are satisfied by

$$x_1 = 4(b_t)^2, \qquad p_1^* = 1/4b_t.$$

By symmetry, the reservation price of commodity 2 in subperiod t, p_2^*, is also $1/4b_t$.

However, if both commodities 1 and 2 are made available in subperiod t, (p_1^*, p_2^*) are not reservation prices. For if the consumer sets

$$x_1 = x_2 = 2(b_t)^2,$$

then

$$-b_t - p_1^* x_1 - p_2^* x_2 + x_1^{1/2} + x_2^{1/2} > 0, \tag{4.7}$$

and so the consumer is better off purchasing commodities 1 and 2 than not soing so. Hence SRPP is violated.

In fact it is easy to construct cases where Theorem 3 is violated in this example. Choose any subperiod t and consider the economy which is the same as the above except that the set of potential goods consists only of good 0 and commodities 1 and 2 available in subperiod t. Replicate this economy and consider a sequence of ϵ-MCE allocations in which both commodities 1 and 2 are produced. Then if $b_{t'} > \frac{1}{4}$ for all $t' \neq t$, these allocations are also ϵ-MCE allocations for the economy in which the set of potential goods consists of good 0 and commodities 1 and 2 available in all subperiods. For consider the consequences of a shop setting up and supplying one unit of commodity 1 or 2 in subperiod t'. The revenue it obtains from this cannot possibly exceed p_1^* $(= p_2^*)$. But if $b_{t'} > \frac{1}{4}$,

$$p_1^* = p_2^* = 1/4b'_t < 1,$$

and so the shop does not cover its setup cost.

However, if $b'_t < b_t$ for some t', then this sequence of ϵ-MCEs is far from being Pareto optimal since a Pareto improvement could be achieved by getting every shop which is supplying some commodity in subperiod t to supply the same commodity in subperiod t'. In other words, there is a market equilibrium in which all firms are open at the same socially inefficient hours! (There is, however, also another market equilibrium in which all firms are open at the right hours.) The reason that the economy can get stuck with firms being open at the wrong times is that, because of transportation costs, the gain (both private and social) from a single shop switching from subperiod t to t' is small, while the gain from all shops simultaneously switching is large.[14]

It is easy to generalize this example in a number of directions. For example, if we drop the assumption that it is prohibitively expensive for shops to open for more than one subperiod, and if we introduce several types of consumers, then we can construct cases where it is socially desirable for each shop to open during two subperiods, say 9 AM–5 PM and 7 PM–9 PM, but where there are monopolistically competitive equilibria in which shops open only from 9 AM–5 PM. One can also easily generate cases in which there is an inefficient monopolistically competitive equilibrium, in which no shops open at all, i.e., only good 0 is consumed, again because the gain to any single shop from opening in some subperiod is far less than the gain from all shops opening in this subperiod (for a case of this type, see the example below).

It might be thought that the above inefficiencies would disappear if we permitted shops to produce several different commodities simultaneously. The following argument shows that this is not necessarily the case.

Assume that $J = n = 2$, and let the technologies of the two types of shops now be such that, at a cost of one unit of good 0, a type 1 shop can produce the combination (1, C) of commodities 1 and 2, and a type 2 shop can produce the combination (C, 1), where $0 \leqslant C < 1$. The assumption that $C < 1$ means that shop 1 (resp. shop 2) has a comparative advantage in commodity 1 (resp. commodity 2). Assume also that $V(x_1, x_2)$ is as in (4.4).

Consider an initial situation in which no shops operate at all and so only good 0 is consumed. Suppose that a shop sets up and produces the combination (y_1, y_2) in some subperiod t. Then an upper limit to the amount of revenue which the shop can obtain is given by the aggregate willingness of consumers to pay for the bundle (y_1, y_2). To evaluate the latter, assume that the (y_1, y_2) is divided equally among λ consumers (we allow λ to take nonintegral values). Then, the maximum amount which can be extracted from each consumer is given by W where

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[14] Note that the shopping center example can be reinterpreted as a location example. Instead of thinking of goods j, $n + j$, $2n + j$,... as being the same commodity available at different times of the day, we can think of them as being the same commodity available at different locations. The transportation cost is then the cost of traveling to a particular location.

$$-W - b_t + \left(\frac{y_1}{\lambda}\right)^{1/2} + \left(\frac{y_2}{\lambda}\right)^{1/2} = 0. \tag{4.8}$$

The aggregate amount extracted is therefore

$$\lambda W = -\lambda b_t + \lambda^{1/2} y_1^{1/2} + \lambda^{1/2} y_2^{1/2}, \tag{4.9}$$

which is maximized when

$$\frac{d}{d\lambda}(\lambda W) = -b_t + \tfrac{1}{2}\lambda^{-1/2}(y_1^{1/2} + y_2^{1/2}) = 0. \tag{4.10}$$

Substituting (4.10) into (4.9), we obtain

$$R = \frac{(y_1^{1/2} + y_2^{1/2})^2}{4b_t} \tag{4.11}$$

as an upper limit to the shop's revenue.

We see that the shop makes a profit only if

$$\frac{(y_1^{1/2} + y_2^{1/2})^2}{4b_t} \geq 1.$$

Hence no type 1 shop will set up unless

$$(1 + c^{1/2})^2 \geq 4b_t,$$

and if $b_t > \frac{1}{4}$, we can always choose C small enough and positive so that this is not the case. The same argument applies of course to type 2 shops.

Thus even though both types of shops can produce both commodities, neither will find it profitable to enter even when r is large. Furthermore, this is in spite of the fact that it is socially desirable for both types of shops to set up; this can be seen from the fact that total consumer willingness to pay for one unit each of commodities 1 and 2 in subperiod t is given by

$$R = \frac{(y_1^{1/2} + y_2^{1/2})^2}{4b_t} = \frac{(1 + 1)^2}{4b_t} = 1/b_t,$$

which exceeds 2, the cost of setting up one shop of each type, as long as $b_t < \frac{1}{2}$.

In other words, allowing firms (shops) to market several goods does not by itself remove the possibility that the economy will get stuck in a situation where the wrong goods are being produced. In the above example, a profit could be made by a firm which produced one unit each of commodities 1 and 2 at cost less than or equal to 2. However, by assumption, there is no firm capable of doing this. Type 1 firms are able to produce only the combination (1, C) at cost 1, and type 2 firms only the combination (C, 1) at cost 1, where $C < 1$. If the firms could combine their activities, they could produce $(1 + C, 1 + C)$ at cost 2 and so would make a profit (see Theorem 5 and Remark (C) in Section 3). But, acting by themselves, they cannot. We see then that coordination problems arise even if firms can produce several goods simultaneously as long as they cannot produce them in the "right" proportions.

5. CONCLUSIONS

In this paper we have considered whether a competitive economy will provide a Pareto optimal allocation of goods when firms must choose not only what quantities of goods to produce, but also the qualities of these goods. We have shown that in general the answer to this question is no. The reason is that if the products of different firms are strongly complementary (although not in the usual sense), then the private (and social) gain from several firms simultaneously producing new products may far outweigh the sum of the gains to each firm from producing a new product by itself. The market mechanism, however, provides no obvious way for different firms' production choices to be coordinated. Thus, the economy may get stuck in a situation where the wrong (from the Pareto point of view) set of goods is being produced, but each agent is acting optimally given the actions of other agents.

One case where this problem does not arise is when consumers' preferences satisfy what we have called the simultaneous reservation price property. Under these conditions firms' products are sufficiently substitutable that uncoordinated optimizing behavior by individual agents will ensure that the market equilibrium is approximately Pareto optimal in a large economy. A sufficient condition for the simultaneous reservation price property to hold is that consumers' preferences are convex and can be represented by a differentiable utility function. (Note that the convexity condition must hold at the individual level—the existence of large numbers of agents does not convexify in the appropriate sense.) A second case where the coordination problem does not arise is if each firm has access to the production set of the whole economy. This case can occur even if the primitive production sets of individual firms differ, as long as it is possible for firms to merge costlessly so as to take advantage of the sum of their individual production possibilities.

It may be useful to contrast our results with the "first" theorem of welfare economics. This theorem says that in economies with complete markets (i.e., markets for all conceivable commodities), competitive equilibria are Pareto optimal whatever properties preferences have (transaction costs of the type considered in our example of Section 4 cause no problems for this result; see Foley, 1970, or Kurz, 1974). In the model of this paper, however, the properties of consumer preferences play a crucial role. The reason for the difference is that complete market economies provide a much richer set of signals to agents than incomplete market economies.

Consider again the example of the nuts and bolts manufacturers given in the Introduction. We argued that it might be unprofitable for either manufacturer to set up by himself, but profitable for both to set up together. Hence, without coordination, a profitable opportunity may be missed. (Furthermore, this coordination problem can arise even if there is a firm which can manufacture both nuts and bolts; see Section 4.) In a complete market model, this cannot happen. For if it is unprofitable for the bolt manufacturer to set up, it must be because the price of bolts is low and, if it is unprofitable for the nut manufacturer to set up, it must be because the price of nuts is low. But this means that consumers will be able to buy the combination of nuts and bolts at a low price and hence will register large demands for these goods on the two markets. In other words, in a complete market economy consumers can signal their demands for combinations of products which are not being produced, while in an incomplete market model they cannot.

Finally, let us compare our results with those of Drèze and Hagen (1978). Drèze and Hagen also argue that the benefits from several firms simultaneously producing new products may exceed the sum of the benefits from any one firm going it alone, and that, as a result, the market equilibrium may be suboptimal. However, they trace this phenomenon to the fact that if, as they assume, each firm can produce only one good at a time, then the set of feasible allocations of the economy is nonconvex; this in turn implies that the first-order conditions for Pareto optimality are necessary but not sufficient. In our model, however, the nonconvexity emphasized by Drèze and Hagen is unimportant. In particular, as we replicate the economy to obtain competitive conditions, the nonconvexity in the set of feasible allocations disappears (see the Appendix and also Hart (1979a)). In fact, as we have emphasized, in our model any deviations between the benefits from several firms simultaneously producing new products and from one firm going it alone are caused by the nonsatisfaction of the simultaneous reservation price property —which is a condition on preferences—rather than by nonconvexities in the feasible set.

APPENDIX

In this Appendix, we establish Theorems 1, 2, 3, and 5. We begin by defining the limit economy to which the economies E_r converge.

Recall that $\tilde{Y}_j$ is the smallest closed convex cone with vertex at the origin which contains Y_j. Then the limit economy E_∞ is defined as follows: it contains I types of consumers, where each type i consumer has utility function U_i, consumption set R_+^{K+1} and endowment w_i as in Section 2; and there are J "industries" where industry j has a production set given by $\tilde{Y}_j$.

An *allocation* in E_∞ is an array $(\mu, y) = (\mu_1, ..., \mu_I, y_1, ..., y_J)$, where μ_i is a probability measure defined on the Borel subsets of R_+^{K+1} and $y \in R^{(K+1)J}$. As in Section 2, $\mu_i(B)$ denotes the fraction of type i consumers whose consumption bundles lie in B for each Borel subset B of R_+^{K+1}. The allocation is *feasible* if

$$y_j \in \tilde{Y}_j \qquad \text{for each } j; \tag{1}$$

$$\sum_{i=1}^{I} \int x \, d\mu_i - \sum_{j=1}^{J} y_j - \sum_{i=1}^{I} w_i = 0. \tag{2}$$

The difference between E_∞ and E_r then is that in E_∞ we ignore the problems which arise from there having to be an integral number of firms (we replace Y_j by $\tilde{Y}_j$) and from there having to be an integral number of consumers (we drop (2.8)).

It is easy to check that every feasible allocation in E_r is also a feasible allocation in E_∞. One important effect of replacing Y_j by $\tilde{Y}_j$ and of allowing μ_i to be an arbitrary measure is to convexify the economy. In particular, it may easily be checked that the set of feasible allocations in E_∞ is convex.

We now define an approximate competitive equilibrium in E_∞ under the assumption that industries are price takers and that markets are open, i.e., finite prices are quoted, for *all* goods.

DEFINITION. An ϵ-competitive equilibrium in E_∞, $\epsilon \geq 0$, is an array (μ, p, y), where $\mu = (\mu_1, ..., \mu_I)$, $y = (y_1, ..., y_J)$, such that

$$(\mu, y) \text{ is a feasible allocation in } E_\infty; \tag{3}$$

$$p \in R_+^{K+1}; \tag{4}$$

$$p \cdot x_i \leq p \cdot w_i + \sum_{j=1}^{J} \bar{\theta}_{ij} p \cdot y_j \qquad \text{and}$$

$$U_i(x) > U_i(x_i) + \epsilon \Rightarrow p \cdot x > p \cdot w_i + \sum_{j=1}^{J} \bar{\theta}_{ij} p \cdot y_j$$

$$\text{for all} \quad x_i \in \operatorname{Supp} \mu_i \quad (i = 1, ..., I); \tag{5}$$

$$p \cdot y_j \geq p \cdot y_j' \qquad \text{for all} \quad y_j' \in \tilde{Y}_j \quad (j = 1, ..., J). \tag{6}$$

Note that, since $\hat{Y}_j$ is a cone, (6) implies that $p \cdot y_j = 0$ for all j. It is easy to establish

RESULT 1. *Assume* (A.1)–(A.7). *Then an ϵ-competitive equilibrium exists in E_∞ for all $\epsilon \geq 0$.*

Proof. Since E_∞ is a convex economy, standard techniques establish the existence of an exact competitive equilibrium in E_∞. An exact equilibrium is also an ϵ-equilibrium, however, for $\epsilon \geq 0$. Q.E.D.

We will say that (μ, y) is an ϵ-competitive equilibrium allocation in E_∞ if, for some p, (μ, p, y) is an ϵ-competitive equilibrium.

We now turn to the definition of Pareto optimality in E_∞. We will restrict attention to allocations with the approximate equal treatment property, i.e., $|U_i(x) - U_i(x')| \leq \epsilon$ for all $x, x' \in \operatorname{Supp} \mu_i$ $(i = 1, \ldots, I)$.

DEFINITION. Consider a feasible allocation (μ, y) in E_∞ satisfying the approximate equal treatment property. We say that the allocation (μ', y') ϵ-Pareto dominates (μ, y) if $U_i(x') > U_i(x) + \epsilon$ for all $x' \in \operatorname{Supp} \mu_i'$, $x \in \operatorname{Supp} \mu_i$ and all $i = 1, \ldots, I$. We say that (μ, y) is ϵ-Pareto optimal if there is no feasible allocation which ϵ-Pareto dominates it.

The next proposition follows from standard arguments.

RESULT 2. *Let (μ, y) be an ϵ-competitive equilibrium allocation in E_∞. Then (μ, y) is ϵ-Pareto optimal in E_∞.*

So far we have assumed that all markets are open in E_∞, even markets for goods which are not produced in a competitive equilibrium. We now develop a notion of competitive equilibrium which is relevant when some markets are closed.

Recall that, given the production allocation $y = (y_1, \ldots, y_J)$, $M(y) = \{k > 0 \mid \sum_{j=1}^{J} y_{jk} > 0\} \cup \{0\}$ is the set of goods which are produced, plus good 0. Also if S is a subset of $\{0, \ldots, K\}$, $R^S = \{x \in R^{K+1} \mid x_k = 0 \text{ if } k \notin S\}$.

DEFINITION. The feasible allocation (μ, y) in E_∞ is said to be an *ϵ-restricted competitive equilibrium allocation relative to $S \supset M(y)$* (ϵ-RCEA relative to S) if there exists $p \in R_+^{*S}$ such that (μ, p, y) is an ϵ-competitive equilibrium for the economy $\hat{E}_\infty$ in which consumer i's consumption set is $R_+{}^S$ and each $\hat{Y}_j$ is replaced by $\hat{Y}_j = \hat{Y}_j \cap R^S$.

In other words, in a restricted competitive equilibrium, prices are quoted only for a subset of goods and both producers and consumers can supply and démand only these goods (another way of looking at this is that consumers face infinite prices for nonmarketed goods and producers face zero prices for such goods – by the free disposal assumption, producers will never wish to set $y_k > 0$ if $k \notin S$).

Note that we assume that prices are always quoted for good 0 and for all produced goods. Note also that a full ϵ-competitive equilibrium is just an ϵ-RCEA relative to $S = \{0,\dots, K\}$.

We can also generalize the notion of Pareto optimality to cases where not all markets are open.

Definition. The feasible allocation (μ, y) in E_α, satisfying the approximate equal treatment property, is said to be an *ϵ-restricted Pareto optimum relative to $S \supset M(y)$* (ϵ-RPO relative to S) if there is no feasible allocation (μ', y'), satisfying $\sum_{j=1}^{J} y'_{jk} = 0$ for all $k \notin S$, which ϵ-Pareto dominates (μ, y).

In other words, an allocation is ϵ-RPO relative to S if a central planner who can allocate only goods in S cannot achieve a Pareto improvement.

The following is an obvious generalization of Result 2.

Result 3. *Let (μ, y) be an ϵ-*RCEA *relative to $S \supset M(y)$. Then (μ, y) is an ϵ-*RPO *relative to S.*

We return now to a consideration of the finite economies E_r. The following lemma will be useful in the proof of Theorem 1.

Lemma 1. *Suppose that (μ, p, y) is an* exact *competitive equilibrium in E_r. Let (y_r) be a sequence of points converging to y, such that $y_r \in Y_r$ for each r. Then, given $\epsilon > 0$, we can find r' such that for all $r \geq r'$, there exists an ϵ-exchange equilibrium (μ_r, p_r, y_r) in the economy E_r, with $p_r = p$.*

Proof. By the definition of a competitive equilibrium in E_α, each component of p is finite. In addition, $p_k > 0$ for all k by (A.5) and (A.6). For each r sufficienly large, choose an array $(x_{itr})_{i=1,\dots,I,t=1,\dots,r}$ of nonnegative consumption bundles for the r type i consumers satisfying

$$\frac{1}{r}\sum_{i,t} x_{itr} - \sum_{j} y_{jr} - \sum_{i} w_i = 0, \tag{7}$$

and such that, for each i, the distance between Supp μ_i and the set $\{x_{itr} \mid t = 1,\dots, r\}$ tends to zero as r tends to infinity.

Although the x_{itr} satisfy feasibility, they will not generally satisfy the budget constraints

$$p \cdot x_{itr} \leqslant p \cdot w_i + \sum_{j} \theta_{ij} p \cdot y_{jr}. \tag{8}$$

Note, however, that (7) implies that

$$\sum_{i,t} p\cdot x_{itr} = r\left[\sum_i p\cdot w_i + \sum_i\sum_j \bar{\Theta}_{ij}p\cdot y_{jr}\right].$$

Therefore we may define new bundles

$$x'_{itr} = x_{itr} + \tau_{itr}, \tag{9}$$

where the $\tau_{itr}\in R^{K+1}$ are chosen so that (a) for each i, t either $\tau_{itr}\in R^{K+1}_{+}$ or $\tau_{itr}\in R^{K+1}_{-}$; (b) $\sum_{i,t}\tau_{itr} = 0$ and (c)

$$p\cdot\tau_{itr} = p\cdot w_i + \sum_j \bar{\Theta}_{ij}p\cdot y_{jr} - p\cdot x_{itr}. \tag{10}$$

(To construct such τ_{itr}, take goods away from those for whom (8) is violated and give these goods to those for whom (8) holds with strict inequality.)

Let μ_r be the measure corresponding to the bundles (x'_{itr}). We claim that, for large r, (μ_r, p_r, y_r) is an ϵ-exchange equilibrium in E_r. The only condition that remains to be established is (2.7). Suppose (2.7) is violated for large r. Then we can find sequences (x'_{itr}), (x''_{itr}) such that

$$U_i(x''_{itr}) > U_i(x'_{itr}) + \epsilon \qquad \text{and} \qquad p\cdot x''_{itr} \leqslant p\cdot w_i + \sum_j \bar{\Theta}_{ij}p\cdot y_{jr}. \tag{11}$$

Since the components of p are strictly positive, (11) implies that the sequence (x''_{itr}) is bounded. Assume without loss of generality (w.l.o.g.) that $\lim_{r\to\infty} x''_{itr} = x''_i$. Note also that, since $y_{jr}\to y_j$ and since each limit point of the (x_{itr}) lies in Supp μ_i, (10) implies that $\lim_{r\to\infty} p\cdot\tau_{itr} = 0$. Hence since each τ_{itr} has either all nonnegative components or all nonpositive components, it follows that $\lim_{r\to\infty}\tau_{itr} = 0$, and so the (x'_{itr}) are also bounded and each limit point of (x'_{itr}) lies in Supp μ_i. Assume w.l.o.g. that $\lim_{r\to\infty} x'_{itr} = x'_i$. Then (11) implies that

$$U_i(x'') > U_i(x'_i) \qquad \text{and} \qquad p\cdot x'' \leqslant p\cdot w_i + \sum_i \bar{\Theta}_{ij}p\cdot y_j, \tag{12}$$

which contradicts the fact that (μ, p, y) is an exact competitive equilibrium in E_∞. Q.E.D.

COROLLARY. *Suppose that (μ, p, y) is an exact competitive equilibrium in E_∞. Then, given $\epsilon > 0$, we can find $\delta > 0$ and r' such that, for all $r \geqslant r'$: if $y'\in Y_r$ satisfies $\|y'-y\| < \delta$, there exists an ϵ-exchange equilibrium (μ'_r, p, y') in E_r.*

Proof. If not, we can find a sequence (δ_t) tending to zero and a sequence $(y'_{r(t)})$ satisfying $\|y'_{r(t)} - y\| < \delta_t$ and $\lim_{t\to\infty} r(t) = \infty$ for which there is no

ϵ-exchange equilibrium $(\mu'_{r(t)}, p, y'_{r(t)})$ in E_r. Since $y'_{r(t)} \to y$, this contradicts Lemma 1. Q.E.D.

THEOREM 1. *For a statement of Theorem* 1, *see Section* 3.

Proof. Let (μ, p, y) be an exact competitive equilibrium in E_∞. Let $\delta > 0$ and r' be defined to satisfy the above Corollary. Choose the selection function F_r so that, whenever $r \geq r'$, $y' \in Y_r$, and $\| y' - y \| < \delta$, $F_r(y') =$ some (μ'_r, p'_r) with $p'_r = p$. Let F_r be arbitrary elsewhere.

By Carathéodory's theorem (see Rockafeller, 1970), and the assumption that the Y_j are bounded, we may, for each j, write $y_j = \sum_{t=1}^{K+2} \alpha_{jt} y_{jt}$, where $y_{jt} \in Y_j$ for each $t = 1, ..., K + 2$ and the α_{jt} are nonnegative. Choose sequences of integers (n^r_{jt}) such that

$$\lim_{r \to \infty} (n^r_{jt}/r) = \alpha_{jt} \qquad \text{for all } j, t. \tag{13}$$

Let $\mathscr{Y}_r$ represent the array of production plans where n^r_{jt} firms of type j produce y_{jt} for all j, t. For large r, $y_r = (\sum_t (n^r_{1t} y_{1t}/r), ..., \sum_t (n^r_{Jt} y_{Jt}/r))$ satisfies $\| y_r - y \| < \delta$. Therefore, by the Corollary, we can find an ϵ-exchange equilibrium (μ_r, p, y_r) in E_r. We claim that, for large r, $(\mu_r, p, \mathscr{Y}_r)$ is an ϵ-MCE. We have already established (2.12) and (2.13). Condition (2.16) follows from (6) and the fact that, since $\tilde{Y}_j$ is a cone, $p \cdot y_j = 0$. Conditions (2.14) and (2.15) follow from the facts that (a) (6) holds; (b) since the Y_j are bounded, a change in one firm's production plan will, when r is large, have a very small effect on y_r, and hence will keep $\| y'_r - y \| < \delta$ and will not not affect prices. Q.E.D.

THEOREM 2. *For a statement of Theorem* 2, *see Section* 3.

Proof. Assume w.l.o.g. that $p_r \to$ some $p \in \bar{R}_+^{*(K+1)}$. Since $\sum_j y_{jkr}$ is either zero or bounded away from zero for large r, p_k must be finite if $\sum_j y_{jk} \neq 0$. For if not, good 0's price in terms of some other good $\to 0$ as $r \to \infty$, and hence per capita demand for good $0 \to \infty$. In addition, (A.6) implies that $p_k > 0$ for all $k = 1, ..., K$.

Furthermore, (2.9) and (A.7) imply that (y_r) is bounded. It follows, in view of the budget constraints (2.6), that we can find a compact set B such that $\mu_{ir}(B) = 1$ for all i, r. Hence we may assume w.l.o.g. that $(\mu_r, y_r) \to (\mu, y)$. Since (μ_r, y_r) satisfies (1) and (2), and since $\tilde{Y}_j$ is closed, it follows that (μ, y) is a feasible allocation in E_∞.

Let k' be any good $1, ..., K$ which is not in $M(y)$ and let $S = M(y) \cup \{k'\}$. We will prove that (μ, y) is a restricted ϵ-competitive equilibrium relative to S. Theorem 2 will then follow immediately.

In order to do this, we must find prices $p^* \in R_+^{*S}$ which sustain (μ, y). The vector p will not in general do since $p_{k'}$ will be infinite. We therefore define

$p_k^* = p_k$ if $k \in M(y)$, but choose $p_{k'}^*$ as follows. Given $p_{k'}$, let $\bar{p}$ be the vector whose kth component is p_k if $k \in M(y)$ and $p_{k'}$ if $k = k'$. Define

$$p_{k'}^* = \inf \Big\{ p_{k'} \in R_+^* \mid \text{for all } i \text{ and } x_i \in \operatorname{Supp} \mu_i,\ x' \in R_+{}^S \text{ and}$$
$$U_i(x') > U_i(x_i) + \epsilon \Rightarrow \bar{p} \cdot x' > \bar{p} \cdot w_i + \sum_j \bar{\Theta}_{ij} \bar{p} \cdot y_j \Big\}. \quad (14)$$

In other words, $p_{k'}^*$ is the smallest nonnegative price for good k' such that consumers who are able to trade in all goods in S actually end up choosing to consume only goods in $M(y)$; i.e., although good k' is available, the demand for it is zero.

We show now that (μ, y) is a restricted ϵ-competitive equilibrium allocation relative to S at prices p^*. We have shown that (3)–(4) hold. Since any $x \in \operatorname{Supp} \mu_i$ is the limit of a sequence of points $x_r \in \operatorname{Supp} \mu_{ir}$ and since (2.6) holds for each r, we may take limits to establish the first part of (5). The second part of (5) follows from (2.7). It remains to show that (6) is satisfied. Since (2.16) holds for the ϵ-MCE in E_r, we know that

$$p^* \cdot y_j \geq 0 \qquad \text{for each } j. \quad (15)$$

We will therefore have established (6) if we can show that

$$p^* \cdot y \leq 0 \qquad \text{for all} \quad y \in \Upsilon_j, \quad j = 1, \ldots, J.$$

Suppose not. Then there exists some j and $y \in \Upsilon_j$ such that

$$p^* \cdot y > 0.$$

But since Υ_j is the smallest closed convex cone containing Y_j, it follows that

$$p^* \cdot y' > 0 \quad (16)$$

for some $y' \in Y_j$. Otherwise, $\Upsilon_j \cap \{y' \mid p^* \cdot y' \leq 0\}$ would be a smaller closed convex cone containing Y_j.

Consider the consequences of an extra type j firm setting up and choosing the production plan y' in the equilibrium of the economy E_r. We know from the regularity assumption that, for large r, there will be a new ϵ-exchange equilibrium. Hence, by (2.15),

$$F_{r2}(y_{-jr}, y_{jr} + y'/r) \cdot y' \leq 0. \quad (17)$$

Take limits as $r \to \infty$. It follows from the assumption that y is a regular point that

$$\lim_{r \to \infty} F_{r2k}(y_{-jr}, y_{jr} + y'/r) = p_k = p_k^* \qquad \text{if} \quad k \in M(y). \quad (18)$$

Also

$$\liminf_{r\to\infty} F_{r2k'}(y_{-jr}, y_{jr} + y'/r) \geq p^*_{k'}, \tag{19}$$

since otherwise, by definition of $p^*_{k'}$,

$$0 < \lim_{r\to\infty} \sum_i \int x_{k'}\, d\mu_{ir} - \sum_j y_{jk'} + \sum_i w_{ik'} \leq 0,$$

which is a contradiction.

Therefore, putting (17), (18), and (19) together, we get

$$p^* \cdot y' \leq 0, \tag{20}$$

which contradicts (16).

This establishes that (μ, y) is a restricted ϵ-competitive equilibrium allocation relative to S at prices p^*. (Note that (A.4) and (6) imply that $p^*_{k'}$ is finite.)

Suppose now that Theorem 2 is false. Then we can (choosing subsequences if necessary) find a sequence (h_r), bounded away from zero, such that (μ'_r, y'_r) is feasible in E'_r, ϵ'-Pareto dominates (μ_r, y_r) and $M(y'_r)\backslash M(y_r)$ contains at most one element, where $\epsilon' > \epsilon$. Let (μ', y') be a limit point of (μ'_r, y'_r). Then (μ', y') ϵ-Pareto dominates (μ, y), uses strictly less resources in E_∞ than (μ, y), and $M(y')\backslash M(y)$ contains at most one element. This contradicts the fact that, by Result 3, (μ, y) is ϵ-Pareto optimal in E_∞ relative to all sets of the form $M(y) \cup \{k'\}$. Q.E.D.

THEOREM 3. *For a statement of Theorem* 3, *see Section* 3.

Proof. The same argument is used as in the proof of Theorem 2. It is shown that under SRPP (μ, p^*, y) is a competitive equilibrium in E_∞, where $p^*_{k'}$ is defined as in (14) for each $k' \notin M(y)$, and where we set $\epsilon = 0$. Q.E.D.

THEOREM 5. *For a statement of Theorem* 5, *see Section* 3.

Proof. Since all industries are alike, we may take J to equal 1 and drop the subscript j. Suppose that Theorem 5 is false. Then, arguing as in the proof of Theorem 2, we may find a feasible allocation (μ', y') in E_∞ which ϵ-Pareto dominates (μ, y), where $(\mu, p, y) = \lim_{r\to\infty} (\mu_r, p_r, y_r)$.

It follows from (3.4) that $\hat{Y}$, the smallest closed convex cone containing Y, $=\{y \mid$ there exists a sequence (y_r) of points in Y such that $\lim_{r\to\infty} y_r/r = y\}$ (for the latter is a closed convex cone containing Y). Let $y' = \lim_{r\to\infty} y'_r/r$, where $y'_r \in Y$ for each r. Consider now the consequences of a new firm setting up with the production plan $y'_{r^{1/2}}$ in E_r, starting at the ϵ-MCE (μ_r, p_r, y_r).

Since $y'_{r^{1/2}}/r = (y'_{r^{1/2}}/r^{1/2}).\ (1/r^{1/2}) \to 0$, we know from the regularity assumption that, for large r, there will be a new ϵ-exchange equilibrium (μ'_r, p'_r, y'_r). Hence, by (2.15),

$$F_{2r}(y_r + y'_{r^{1/2}}/r) \cdot y'_{r^{1/2}} \leqslant 0. \tag{21}$$

We can assume w.l.o.g. that $\lim_{r\to\infty} F_{2r}(y_r + y'_{r^{1/2}}/r) = p^*$. Moreover, since y is a regular point, $p_k^* = p_k$ for all $k \in M(y)$. Hence, dividing (21) by $r^{1/2}$ and taking limits, we get

$$p^* \cdot y' \leqslant 0. \tag{22}$$

In addition, again since y is regular, we know that $\mu'_r \to \mu$. It follows that

$$x \in R_+^{K+1} \quad \text{and} \quad U_i(x) > U_i(x_i) + \epsilon \Rightarrow p^* \cdot x > p^* \cdot w_i + \bar{\Theta}_i p^* \cdot y$$
$$\text{for all} \quad x_i \in \operatorname{Supp} \mu_i. \tag{23}$$

For otherwise, consumers will not find it optimal to choose consumption plans in the support of μ'_r when r is large. In other words, the p_k^* are reservation prices for the nonproduced goods $k \notin M(y)$ at the allocation (μ, p, y).

However, (23) and the fact that (μ', y') ϵ-Pareto dominates (μ, y) imply, by the usual revealed preference argument, that $p^* \cdot y' > p^* \cdot y$. But since the latter is nonnegative as a consequence of (2.16), this contradicts (22).

Q.E.D.

ACKNOWLEDGMENTS

I have benefited greatly from numerous helpful discussions with Louis Makowski, who has independently obtained results similar to those presented here. See Makowski (1980). I am also very grateful to Andreu Mas-Colell who suggested studying approximate equilibria. Finally, I would like to acknowledge helpful conversations with Hildegard Dierker, Bill Novshek, Joe Ostroy, and Hugo Sonnenschein, and also the comments of an anonymous referee.

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JOURNAL OF ECONOMIC THEORY 22, 313–326 (1980)

Equilibrium in Simple Spatial (or Differentiated Product) Models

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Received August 11, 1979

1. INTRODUCTION

This paper is concerned with equilibrium in models where location (product) and price are strategic variables. Since Hotelling's classic paper [5] a great deal of work on spatial and related differentiated product models has been undertaken, much of it featuring the assumption of zero conjectural variation (ZCV), or zero profits in equilibrium, or symmetry in equilibrium. The difficulties created by these assumptions are sometimes noted, but are typically ignored in the analysis. Here the equilibria of a simple, standard spatial model are investigated without ignoring these difficulties. Equilibrium with free entry when firms are technologically small relative to the market is of particular interest.

It is easy to see why the ZCV assumption, that each firm views the strategies of all other firms as fixed regardless of its own strategy, is inappropriate for spatial models with price and location pairs as strategies when firms have constant marginal cost: no equilibrium at which ZCV is relevant ever exists! Suppose that for some standard spatial model with price and location pairs as strategies, a ZCV equilibrium exists. If all firms are at monopoly solutions (i.e., at locations and prices, and with demands such that if no other firms existed, the same price and location pair would remain an optimal strategy, yielding the same demand) then no firm affects any other firm, and ZCV is irrelevant. On the other hand, if some firm i is not at a monopoly solution, one of its neighbor firms, say j, must be affecting firm i's demand, and therefore firm i must be affecting firm j's demand. Without loss of generality, firm i's profit is not greater than firm j's profit, so that when firm i relocates very near firm j, and slightly undercuts firm j's price, it captures all of firm j's market at a profit arbitrarily near firm j's original profit, plus it retains some of its former market (some of the consumers "in dispute" between firms i and j, who originally purchased from i, but would purchase from j if firm i did not exist). The profit from the retained market is bounded away from zero, so firm i increases its profits by this change of strategy. Thus the only ZCV equilibria when firms have constant marginal cost are those in

0022-0531/80/020313-14$02.00/0

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which all firms are at monopoly solutions.[1,2] Furthermore, if free entry and exit are allowed then each local monopolist must earn zero profit or an entrant, assuming ZCV, will undercut an active firm, anticipating positive profit (or, in the case of negative profit, the monopolist will exit).

In some spatial (or differentiated product) models, firms are assumed to possess a great deal of computational ability or knowledge of the behavior of others. In situations where there are many firms operating, each of which serves only a small fraction of the market, this is not a particularly appealing assumption. On the other hand, when there are only a few firms operating, a typical oligopoly problem arises: what is reasonable firm behavior, and are firms allowed to threaten or collude, etc.? Since there will be many active firms in the main situations of interest here, when firms are technologically small relative to the market and free entry and exit are allowed, only simple conjectural variations will be used.

Here, as in Eaton [1, 2] a modified ZCV is used. Each firm views the strategy of other firms as fixed so long as its own strategy does not cause its delivered price to match or undercut any other firm's price at that other firm's own location. Each firm believes that no other firm will allow itself to be undersold at its own location and that other firms will reduce price if they are undercut at their own locations. This modification effectively rules out the problem with ZCV.[3]

Modified ZCV is a minimal assumption in the sense that any assumption which will allow existence of equilibrium must effectively rule out undercutting and modified ZCV does this with a minimum change from ZCV. It may seem more reasonable to assume that firms respond whenever their profits become negative. This type of alternative conjectural variation does not affect the validity of the main results of the paper contained in Proposition 2. In a typical equilibrium of Proposition 2 all firms earn strictly positive profit and no firm is (locally) constrained by modified ZCV. For any conjectural variation in which other firms do not respond to sufficiently small changes from this equilibrium configuration each active firm will have a local profit maximum at the modified ZCV equilibrium. If the alternative conjectural variations are "well behaved" globally, the modified ZCV equilibrium will also be an equilibrium with the alternative conjectural variations. Thus the results for modified ZCV are more general than is immediately apparent.

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[1] If firms have average cost curves which are very sharply U shaped, then this type of strategy change may not improve profit, since demand may substantially change.

[2] Lerner and Singer [6] discuss this type of strategy change in a duopoly context.

[3] If $(p(i), x(i))_{i=1}^{n}$ is a modified ZCV equilibrium in price-location strategies then (except for monopoly solutions) it is not a Nash equilibrium in price-location strategies. However, $(p(i), x(i), 0)_{i=1}^{n}$ is a Nash equilibrium for a corresponding game in which strategies are triples of price, location, and response price. If firm i's price is undercut so that it faces zero demand then it uses the response price (here zero) instead of $p(i)$.

(Remark 3 discusses some cases in which firms conjecture that other firms will respond to any change in price.)

The problem with the assumption of zero profits in equilibrium is that it usually leads to the generic nonexistence of equilibrium in bounded models, because the number of active firms must be an integer.[4] Free entry is usually used as the rationale for this assumption, and though the conjectural variation assumption is not always explicitly clear, the implicit assumption often appears to be ZCV, which also leads to nonexistence, even in unbounded models.[5]

There are two types of symmetry of equilibrium assumptions which cause problems. The first is the assumption that all firms will charge the same price, or will locate symmetrically. The firms are then allowed to maximize profit only with respect to one variable, without explanation as to why the equilibrium should be assumed symmetric. The second symmetry "assumption" allows firms to maximize with respect to both price and location, but considers only those equilibria which are symmetric. While this is a legitimate method for showing existence of equilibrium, the symmetric equilibria of different markets are often compared. Unless it can be shown that nonsymmetric equilibria do not exist, these comparisons are not valid comparative statics.

Here the equilibria of a simple, standard spatial model are analyzed. The market is one dimensional, bounded, and without a boundary: a circle with circumference L. All firms have identical cost functions consisting of a fixed cost F plus a constant marginal cost times output (by measuring all prices as net of marginal cost, without loss of generality the marginal cost is zero), and choose location and price. Modified ZCV is assumed, so firms believe the strategies of other firms are fixed unless a strategy change for the firm undercuts some other active firm in which case the affected firm will respond by lowering its price. A continuum of consumers is uniformly distributed on the circle with density A, and each consumer only purchases from firms with the lowest delivered price, with consumers paying transportation costs. The delivered price at x' from a firm located at x and charging price p is $p + c \mid x - x' \mid$. All consumers possess identical linear demand functions $h(p) = (a - bp)/A$ which are functions of delivered price. The demand function can be generated in the usual manner from a utility function for $l + 1$ standard commodities one of which is available from several firms at various delivered prices.

The equilibria for this model are characterized both for the case where the number of firms is exogenously fixed, and for the case of free entry and exit. Only integral numbers of firms are allowed and true free entry and exit, rather than a zero profit condition, is used. Except for some monopoly solutions all equilibria are symmetric. When fixed cost is small relative to the market then equilibrium with free entry and exit exists and is approximately competitive (i.e., all prices are approximately equal to marginal cost). Here (approximately) competitive results arise endogenously when technologies are small with respect to the market and there is free entry and exit, rather than exogenously as assumptions.

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[4] Salop [8] mentions this problem.

[5] Eaton [2] shows that symmetric positive profit equilibria often exist in a model using modified ZCV and free entry.

The paper is organized as follows. Section 2 contains results for linear demand and constant marginal cost. Section 3 contains remarks which generalize the results to include nonlinear demand, U-shaped average cost, and differentiated products. The Appendix contains the proofs, which are mainly computational.

2. EXISTENCE AND PROPERTIES OF EQUILIBRIUM

Let the price and location for firm i be denoted by $p(i)$ and $x(i)$ respectively. All equilibria which differ only in the "names" of firms or consumers (e.g., equilibria in which two firms interchange prices and locations, or in which all firms move clockwise by an equal amount) are identified so let $t(i) = cx(i) - cx(i-1)$. The results for exogenous n, linear demand and constant marginal cost are summarized in Proposition 1. (By measuring all prices as net of marginal cost, without loss of generality the marginal cost is zero.) For exogenous n the only n firm equilibria which exist are ones in which all active firms charge the same price. Whenever n is large enough so that all individual firms cannot be at monopoly solutions, then a unique n firm equilibrium exist, and it is symmetric in location as well as price.

PROPOSITION 1. *With linear demand, constant (zero) marginal cost, and parameters* (a, b, c, A, F, L), *for all positive integers n there is an n firm equilibrium (without free entry and exit).*

(a) *All firms are local monopolists if and only if* $n \leqslant \max\{1, 3bcL/4a\}$:

(i) *for* $n = 1 \geqslant 3bcL/4a$ *the equilibrium is unique with price* $p^* = (a/2b) - (cL/8)$ *and profit* $\pi^*(F) = bL[(a/2b) - (cL/8)]^2 - F$;

(ii) *for* $1 \leqslant n \leqslant 3bcL/4a$ $(\neq 1)$ *equilibrium is not unique unless* $n = 3bcL/4a$. *For all firms* $t(i) \geqslant 4a/3b$, $p^* = a/3b$ *and* $\pi^*(F) = (4a^3/27b^2c) - F$.

(b) *There is a nonmonopoly equilibrium if and only if* $n > \max\{1, 3bcL/4a\}$. *The equilibrium is unique, and symmetric with, for all firms,* $t(n) = cL/n$, *price* $p^*(n) = (a/2b) + (3cL/4n) - (1/2) \cdot \{[(a/b) - (cL/2n)]^2 + (3c^2L^2/n^2)\}^{1/2}$ *and profit* $\pi^*(n, F) = (Lb/n)[(acL/8bn) - (25c^2L^2/16n^2) + (7cL/8n) \times \{[(a/b) - (cL/2n)]^2 + (3c^2L^2/n^2)\}^{1/2}] - F$.

The equilibria for $1 < n < 3cbL/4a$ are not unique and need not be symmetric. The possible lack of symmetry does not affect the welfare properties of equilibrium, since the nonsymmetric gaps between the neighboring monopolists' markets consist of consumers who do not purchase any of the product, and the measure of consumers in this situation is independent of the arrangement of the n monopolists. However, this lack of symmetry does affect the potential for entry, with symmetric equilibria offering the smallest maximum gap and thus the lowest profit potential for entrants.

There are two properties that are important for the proof. First, in an equilibrium no active firm is (locally) constrained by modified ZCV (if any firm is locally constrained some firm is not at an optimum location). The second important property is boundary price equalization. Whenever there is a uniform density of consumers, each with identical downward sloping demand, firms maximize demand at each price they charge by locating in such a manner that the farthest consumers in each direction that purchase from the firm face the same delivered price. As the firm moves toward the center of its market, even if the measure of consumers purchasing from the firm remains constant, demand increases because the delivered prices faced by consumers in the firm's market area become lower in a statistical sense. For each price p the measure of consumers purchasing from the firm at delivered prices less than or equal to p increases (weakly for all p and strictly for some p) as the firm moves toward the center of its market. The remainder of the proof is mainly computational.

Proof of Proposition 1. See Appendix.

If we introduce free entry and exit, given parameters (a, b, c, A, F, L) the set of equilibria with free entry and exit is the union over all n of the n firm equilibria of Proposition 1 at which active firms earn nonnegative profit, and the best a potential entrant could do by entering is to earn a nonpositive profit.[6]

The following example shows that equilibrium with free entry and exit does not always exist. Let $a = b = c = 1$ and $L = \epsilon$. The monopoly price is $(4 - \epsilon)/8$ with profit $[\epsilon(4 - \epsilon)^2/64] - F$. For all n greater than one, the only possible n firm equilibrium has prices on the order of ϵ/n and firm profits which exceed $-F$ by a term on the order of $(\epsilon/n)^2$. An entrant into the monopoly market charging price $(4 - \epsilon)/8$ anticipates profit of $[\epsilon(4 - \epsilon)/32] - F$. Thus for all small ϵ there is a value of F such that the monopoly profit and the profit for an entrant given $n = 1$ are both positive, but for $n > 1$ profits are negative in every n firm equilibrium. Hence for small ϵ and appropriate F no equilibrium with free entry and exit exists.

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[6] For $n = 0$ there is an n-firm equilibrium with free entry and exit if the monopoly profit in Proposition 1 is nonpositive.

Proposition 2 shows that whenever fixed costs are sufficiently small relative to the other parameters, equilibrium with free entry and exit exists and all equilibria are approximately competitive (i.e., all consumers are able to purchase the commodity at delivered prices which are approximately equal to the marginal cost-of production). Let $\mathscr{E}(a, b, c, A, F, L)$ denote the set of equilibria with free entry and exit when the parameters are (a, b, c, A, F, L).

Proposition 2. *With linear demand, constant (zero) marginal cost and parameters* (a, b, c, A, L) fixed,

(a) *there exists an* $F^* > 0$ *such that for all* $F \in (0, F^*]$ $\mathscr{E}(a, b, c, A, F, L) \neq \emptyset$,

(b) *the number of equilibria is* $|\mathscr{E}(a, b, c, A, F, L)| = \mathcal{O}(F^{-1/2})$ *and*

(c) *if* $(F(k))_{k=1}^{\infty}$ *is a sequence of fixed costs such that* $F(k) \to 0$, *and for each* k, $m(k)$ *is the maximum delivered price paid by any consumer in any equilibrium in* $\mathscr{E}(a, b, c, F(k), L)$ *then* $m(k) \to 0$.

Proof. See Appendix.

3. REMARKS

(1) With constant marginal cost and linear individual consumer demand functions Proposition 1 shows that essentially all equilibria are symmetric and Proposition 2 shows that when technologies (fixed costs) are small relative to the market then equilibrium with free entry and exit exists and is approximately competitive. The results of Proposition 2 can be generalized under a variety of assumptions: if technologies are small relative to the market then equilibrium with free entry and exit exists, is symmetric, and is approximately competitive. The important requirements are that all firms have identical technologies with initial increasing returns and that there be a uniform density of consumers with identical demand functions which are downward sloping at the competitive (minimum average cost) price.

If $C(y)$ is a basic cost function with initial increasing returns to scale, for $\alpha \in (0, 1)$ let $C_\alpha(y) := \alpha C(y/\alpha)$. Then α is a measure of the technology: for the technologies of Section 2, if C has fixed cost F then C_α has fixed cost αF; if C attains minimum average cost at finite output y^* then C_α attains (the same) minimum average cost at αy^*. When $1/\alpha$ is an integer the rescaled cost function C_α can be interpreted as measuring the original cost function in "per capita" or per replication terms after a $(1/\alpha)$-fold replication of the consumer sector.

The proof technique sketched here can be applied to nonlinear demand and U-shaped average cost. The first step is to note that the boundary price equalization property discussed in Section 2 still applies. Let $p(q, \alpha)$ be the optimal price charged by a firm with cost function C_α (exit not allowed) which is located between firms at $x \pm (q/c)$ which charge a price equal to the minimum (infinum) of average cost. For each small α, $p(q, \alpha)$ and $(p(q, \alpha) + q)/2$ are invertible as functions of q for appropriate small q. Then, without allowing entry or exit, there is a *symmetric n firm equilibrium* if there exists a q such that $(n/c)(q - p(q, \alpha)) = L$. An entrant into this situation has optimal price $p([p(q, \alpha) + q]/2, \alpha)$. The last step is to show that for small α there exists an appropriate large n such that an n firm equilibrium exists at which active firms earn positive profit and entrants anticipate negative profit from entry ($[p(q, \alpha) + q]/2q$ is bounded away from one for appropriate (q, α) pairs so the initial increasing returns yields a situation similar to that of the proof of Proposition 2 where $[\pi_e(n, F) + F]/[\pi^*(n, F) + F] \to [9/16]$). For small α all equilibria are approximately competitive because of free entry and all equilibria are symmetric because of the invertibility of $p(q, \alpha)$ and $[p(q, \alpha) + q]/2$.

It is possible to modify the technique just sketched to apply to nonlinear transportation costs or differentiated products. For the differentiated product case there are no transportation costs but preferences vary over the differentiated products. For example suppose t is a taste parameter of consumers (distributed uniformly on $[0, L]$) and g: $[0, L/2] \to [0, 1]$ is a strictly decreasing function with $g(0) = 1$. Suppose all consumers view the differentiated products as perfect substitutes in the appropriate ratios[7] (given by g). For a consumer with parameter t let products be "named" by their distance counterclockwise (+) or clockwise (−) from t. For a consumption vector z of l standard commodities and consumption $y(x)$ for each differentiated commodity $x \in [-L/2, L/2]$, a consumer of taste t has utility $u(z, \int_{-L/2}^{L/2} g(|x|)y(x)\,dx)$, where u is a standard utility function for $l + 1$ commodities. Consumers only purchase those commodities at the lowest "effective price" $p(x)/g(|x|)$. This is similar to a situation with one characteristic where consumers differ in their subjective beliefs about the amount of the characteristic contained in each good.

(2) The results of Section 2 and Remark 1 show that with linear demand, or with downward-sloping demand, technology small with respect to the market, and free entry and exit it is legitimate to perform comparative statics using only the symmetric equilibria since no other equilibria exist (and with an appropriate dynamic the symmetric equilibria are stable). With linear demand essentially all equilibria are symmetric but with general downward-sloping demand nonsymmetric equilibria may exist when the technology is large with respect to the market. This is illustrated in Example 1.

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[7] See [7] for a related discussion of this substitutability.

EXAMPLE 1. Let $L = 32/21$, $F = 3/4$, and $c = A = 1$, let there be zero marginal cost, and let consumer demand functions be

$$\begin{aligned} h(p) &= 2 - p/r && \text{if } p \in [0, 1] \\ &= 1 - p/r && \text{if } p \in (1, r] \\ &= 0 && \text{if } p \in (r, \infty). \end{aligned}$$

For every $\epsilon < 1/21$, if $r \geqslant 2[(1/21) - \epsilon]^{-1}$ and $\delta \leqslant \epsilon$ there is a two-firm equilibrium with free entry and exit which has symmetric locations and nonsymmetric prices $(13/21) - \delta$ and $(13/21) + \delta$.

In terms of the literature on spatial models the results for perfectly inelastic demand cause more of a problem. Models in which each consumer purchases one unit of the spatial commodity regardless of price (or for all prices below some reservation price) are commonly treated (see, for example, [3, 8]). With inelastic demand the boundary price equalization argument of Section 2 breaks down. With constant marginal cost, if fixed cost is small relative to the market then equilibrium with free entry and exit exists and is approximately competitive, but almost all equilibria are nonsymmetric. For each n-firm symmetric equilibrium there is an $(n - 1)$-dimensional set of nonsymmetric equilibria "near" it. Starting at an equilibrium with symmetric locations there will be an equilibrium for each small perturbation of the locations: each firm's location does not affect its own profit and for each perturbed set of locations there is a price equilibrium.

(3) The result of Section 2, proving the existence of equilibrium with free entry and exit when fixed costs are small, can be extended to a class of conjectural variation functions all of whose members include the modification to ZCV which effectively eliminated price undercutting. A simple subclass of conjectural variation functions to which the result can be extended is the class of conjectural variation functions that agree with the modified ZCV except for the belief that a neighboring firm will change its price in proportion to the change in the given firm's price, i.e., firm i believes $\Delta p(i \pm 1) = (1 - k)\Delta p(i)$, where $k \in (0, 1]$. For sufficiently small F, there is an n-firm equilibrium with free entry and exit in which all firms charge price approximately cL/kn, when

$$[(3/4) + \epsilon]L(ac/kF)^{1/2} < n < (1 - \epsilon)L(ac/kF)^{1/2}.$$

The above result also holds when k is replaced by a function of the original distances between the firms, $|\bar{x}(i \pm 1) - \bar{x}(i)| = d$, of the form kd^s for $s \in [0, 1]$. Then firm i believes that $\Delta p(i \pm 1) = (1 - kd^s)\Delta p(i)$ (this applies only when firms are sufficiently close that $kd^s \in (0, 1]$), and for sufficiently small fixed costs there is an equilibrium with free entry and exit.

(4) One instance in which unmodified ZCV may be reasonable is that in which firms choose location and quantities rather than prices. The difficulty in working with quantity choice models in a spatial (or differentiated products) framework arises in determining prices which correspond to a given set of location and quantity choices by firms. In the linear demand model of Section 2, if n firms are positioned symmetrically and all charge the same price, then at the corresponding quantities, q, the matrix $[\partial q/\partial p']$ has a negative dominant diagonal, and is therefore nonsingular. Thus the inverse demand function, for quantity changes with locations fixed, can be found for quantity vectors q near the original symmetric q^*. If only one firm, i, changes its quantity action, $q(i)$, then $\partial p(i)/\partial q(i)$ and $\partial p(i \pm 1)/\partial q(i)$ can be found, and, with some computation, it can be shown that

$$\frac{\partial p(i \pm 1)}{\partial q(i)} \Big/ \frac{\partial p(i)}{\partial q(i)}$$

is (locally) approximately $1 - kn^{-1/2}$, where n is the number of active firms and k is a constant determined by the parameters of the model. Quantity-setting behavior with ZCV can be thought of as price-setting behavior with a conjectural variation about neighbors' price changes which is exactly the price change (along with price changes for more distant firms) which maintains quantity levels for other firms. Locally, near a symmetric price location $2n$-vector (p, x), the necessary conjectural variation is approximately $\partial p(i \pm 1)/\partial p(i) = 1 - kd^s$, where d is the distance between firms and $s = \frac{1}{2}$. Recall that $\partial p(i \pm 1)/\partial p(i) = 1 - kd^{1/2}$ is one of the conjectural variation functions for which equilibrium with free entry and exit exists when fixed costs are sufficiently small.

(5) In a general equilibrium, differentiated products model where firms choose quantities, Hart [4] proves the convergence of equilibrium with free entry and exit to competitive equilibrium as firms become technologically small relative to the economy. However, he does not consider the question of the existence of equilibrium. The existence result of Section 2 could be easily extended to general equilibrium if the distribution of ownership shares was symmetric (i.e., each set of consumers of measure μ would own μ/AL of each firm). Substantial problems arise in both partial and general equilibrium when the symmetry assumptions of the model (uniform consumer density, identical consumer demand functions, identical ownership shares, identical cost functions) are weakened in any significant way. It is then no longer possible to treat a representative firm: each firm must be considered separately. As the technologies become small relative to the economy the endogenously determined number of firms necessarily becomes large and increases the complexity of the problem.

4. CONCLUSION

In spatial (or differentiated products) models in which location (product) and price are strategic variables, ZCV must be modified in order for the results to be nonvacuous. With modified ZCV, constant marginal cost production, and linear demand, essentially all equilibria are symmetric. Though equilibrium with free entry and exit does not always exist, if fixed costs are sufficiently small relative to the other market parameters, then an equilibrium with free entry and exit does exist, and all equilibria with free entry and exit are approximately competitive. The results can be generalized to cover a variety of assumptions including nonlinear demand and U-shaped average cost. In these cases, when the technology is small relative to the market, equilibrium with free entry and exit exists and is approximately competitive.

APPENDIX

Proof of Proposition 1

A monopolist charging price $p < a/b$ faces demand $2\int_0^r A \cdot h(p + cs)\,ds$, where $r = \min\{[(a/b) - p]/c,\ L/2\}$, so

$$\begin{aligned} D(p) &= 0 && \text{if } p \geqslant a/b \\ &= (a - bp)^2/bc && \text{if } a/b > p \geqslant (a/b) - (cL/2) \\ &= (a - bp)L - (bcL^2)/4 && \text{if } (a/b) - (cL/2) \geqslant p. \end{aligned}$$

Maximizing profit, $\pi(p, F) = pD(p) - F$, the monopolist's optimal price is $a/3b$ if $3cbL/4a > 1$ and is $(a/2b) - (cL/8)$ if $3cbL/4a \leqslant 1$. When $3cbL/4a \leqslant 1$, the single monopolist serves the entire market, and earns profit $bL((a/2b) - (cL/8))^2 - F$. When $3cbL/4a > 1$, the single monopolist serves a market of length $4a/3bc < L$, and earns profit $(4a^3/27b^2c) - F$. When $3cbL/4a > 1$, for any $n \leqslant 3cbL/4a$, n firms can operate as local monopolists, each charging price $a/3b$, as long as the distance between any two firms is at least $4a/3bc$. These equilibria (with $1 < n \leqslant 3cbL/4a$) need not be symmetric in location since location does not affect the monopolist's profit (as long as the distance $4a/3bc$ is maintained between monopolists).

Now consider an n-firm equilibrium in which some firm is not at a monopoly solution. The demand faced by firm i, from consumers located between firms i and $i + 1$ is the integral of individual consumer demands (which are functions of the delivered price from firm i) for consumers between i and $i + 1$ whose lowest delivered price is from i,

$$\int_0^r A \cdot h(p(i) + cs)\,ds,$$

where

$$r = \min\{[p(i+1) - p(i) + t(i+1)]/2c,\ [(a/b) - p(i)]/c\}$$

or $(1/8c)(p(i+1) - p(i) + t(i+1))(4a - b[3p(i) + p(i+1) + t(i+1)])$ if $p(i+1) + p(i) + t(i+1) \leqslant 2a/b$, and $(a - bp(i))^2/2bc$ otherwise, while the demand from consumers located between firms $i-1$ and i is $(1/8c)(p(i-1) - p(i) + t(i))(4a - b[3p(i) + p(i-1) + t(i)])$ if $p(i-1) + p(i) + t(i) \leqslant 2a/b$, and $(a - bp(i))^2/2bc$ otherwise.

With firm i's price fixed, demand (and hence profit) is maximized by locating in such a manner that the farthest consumers in each direction that purchase from firm i face the same delivered price.

Therefore in a nonmonopoly equilibrium, for all i,

$$p(i) + p(i+1) + t(i+1) = p(i-1) + p(i) + t(i)$$

and every firm is at an interior (unconstrained) profit-maximizing solution, so the first- and second-order conditions for profit maximization can be used to characterize equilibrium. For the following analysis, constrain firm i to remain between firms $i-1$ and $i+1$, so a "global" solution is only global in that restricted domain.

The unique global profit-maximizing solution for location, $x(i)$, satisfies $p(i+1) - p(i-1) + cx(i+1) + cx(i-1) - 2cx(i) = 0$ (this is the location that equalizes the boundary-delivered prices). Taking the partial derivative of profit with respect to price, $p(i)$, after the optimal value of $x(i)$ has been substituted into the profit function, the first-order condition requires that $(b/16c)\{36p^2(i) - 8p(i)(s + 4a/b) + (8sa/b) - s^2\} = 0$, where $s = p(i+1) + p(i-1) + cx(i+1) - cx(i-1)$. Note that the second-order condition is only satisfied at the smaller of the two solutions to this equation. Substitute $t(i) = cx(i) - cx(i-1)$ into the two systems of first-order conditions. The n first-order conditions for locations yield only $n-1$ independent equations, but $1^{*\prime}t = cL$,[8] so, solving for t as a function of p,

$$\begin{bmatrix} 0 & 1 & 0 & \cdots & 0 & -1 \\ -1 & 0 & 1 & 0 & \cdots & 0 \\ & \ddots & \ddots & \ddots & & \vdots \\ 0 & & \ddots & \ddots & \ddots & 0 \\ \vdots & \ddots & -1 & & 0 & 1 \\ 0 & \cdots & 0 & & 0 & 0 \end{bmatrix} p + \begin{bmatrix} -1 & 1 & 0 & \cdots & \cdots & 0 \\ 0 & -1 & 1 & 0 & \cdots & 0 \\ \vdots & \ddots & \ddots & \ddots & & \vdots \\ \vdots & & \ddots & \ddots & \ddots & 0 \\ 0 & \cdots & 0 & -1 & & 1 \\ 1 & 1 & \cdots & 1 & & 1 \end{bmatrix} t = \begin{pmatrix} 0 \\ \vdots \\ \vdots \\ \vdots \\ 0 \\ cL \end{pmatrix}$$

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[8] 1^* is the n vector of ones and $'$ denotes transpose. The vector t has ith component $t(i)$ for all i.

and

$$t = (cL/n)\,1^* + (2/n)\,1^*1^{*\prime}p - \{I + D\}p, \tag{1}$$

where D is the $n \times n$ matrix with all zero entries except for ones just below the diagonal ($i-1, i$ entries) and in the upper right corner ($1, n$ entry). Substituting this expression into the n equations corresponding to the first-order conditions for prices, and simplifying,

$$48\begin{bmatrix} p(1) & 0 & \cdots & 0 \\ & & \ddots & \vdots \\ 0 & p(2) & & 0 \\ \vdots & & \ddots & \\ 0 & \cdots & 0 & p(n) \end{bmatrix} p - 8((6a/b) + k)p + ((16a/b)k - 4k^2)\,1^* = 0, \tag{2}$$

where $k = (cL + 2(p'1^*))/n$. Thus every $p(i)$ must satisfy the same quadratic equation, and a computation shows that the larger solution fails to satisfy the second-order condition for profit maximization. Thus all prices are the same, and letting $p^*(n)$ be this common price, substituting $p = p^*(n)\,1^*$ into (2) yields further simplification to

$$p^{*2}(n) - ((a/b) + (3cL/2n))\,p^*(n) + (acL/bn) - (c^2L^2/4n^2) = 0$$

with the correct solution being

$$p^*(n) = (a/2b) + (3cL/4n) - \tfrac{1}{2}\{[(a/b) - (cL/2n)]^2 + (3c^2L^2/n^2)\}^{1/2}. \tag{3}$$

For large values of n, this is approximately $(cL/n) - (3c^2L^2b/4n^2a)$, or approximately cL/n. Since all the prices are equal, from (1), $t = (cL/n)\,1^*$.

This result was based upon the assumption that in the equilibrium, some firm was not at a monopoly solution. Now check to see which values of n satisfy that condition by finding those values of n for which the borderline consumer between neighboring firms faces a delivered price less than a/b; i.e., $p^*(n) + (cL/2n) < a/b$. Substituting (3) into this inequality, this condition holds if and only if $n > 3cLb/4a$. Thus for $n \leqslant \max\{3bcL/4a, 1\}$, the only possible equilibria are the previously discussed monopoly equilibria, and for $n > \max\{3bcL/4a, 1\}$, all equilibria are symmetric, with prices and locations equal to the just-computed values, and firm demands and profits

$$D^*(n) = (Lb/n)[(a/2b) - (cL/n) + \tfrac{1}{2}\{[(a/b) - (cL/2n)]^2 + (3c^2L^2/n^2)\}^{1/2}]$$

and

$$\pi^*(n, F) = (Lb/n)[(acL/8bn) - (25c^2L^2/16n^2) + (7cL/8n) \times \{[(a/b) - (cL/2n)]^2 + (3c^2L^2/n^2)\}^{1/2}] - F,$$

respectively.

It remains to show that each firm is at a true global profit-maximizing solution. This is obvious for monopoly solutions. By the symmetry of all nonmonopoly equilibria no firm can improve profit by jumping between two firms separated by a gap of L/n (as compared to its present position in a gap of $2L/n$). It is always most profitable to retain location $x(i)$ because of the boundary price equalization property and modified ZCV. At location $x(i)$ it is not possible to undercut any other firm since $p^*(n) - (cL/n) < 0$. Q.E.D.

Proof of Proposition 2

Since in every equilibrium, all firms charge the same price, the best location for a potential entrant is exactly between two firms which are at least as far apart as all other neighboring pairs of firms. An optimal location for the entrant is $(x(i) + x(i + 1))/2$, where $x(i + 1) - x(i) \geqslant x(j + 1) - x(j)$ for all j (if $n = 1$, $x(1) + (L/2)$ is the optimal location). If $3cbL/4a > 1$ and $x(i + 1) - x(i) \geqslant 8a/3bc$, then the entrant can be at a monopoly solution, just as all other firms. When $n = 1$ and $L \leqslant (4a/69bc) \times [(73)^{1/2} - 2]$ then the optimal price for the entrant, $[(a/2b) - (5cL/8)]^+$, "almost undercuts" the monopolist and the entrant faces demand $bL[(a/2b) + (3cL/8)]^-$ and earns profit $\{(bL/4)[(a^2/b^2) - (acL/2b) - (15c^2L^2/16)] - F\}^-$. In all other cases, using the first- and second-order conditions for profit maximization, the optimal price, demand, and profit for the entrant are

$$(s/9) + (4a/9b) - (1/18)[(64a^2/b^2) - (40sa/b) + 13s^2]^{1/2},$$

$$(b/216c)\{(56sa/b) - 11s^2 - (32a^2/b^2) \\ + [s + (4a/b)] \cdot [(64a^2/b^2) - (40sa/b) + 13s^2]^{1/2}\},$$

and

$$(b/3888c)\{-35s^3 + (12s^2a/b) + (480sa^2/b) - (512a^3/b^3) \\ + [(64a^2/b^2) - (40sa/b) + 13s^2]^{3/2}\} - F,$$

respectively, where $s = 2p^*(n) + cx(i + 1) - cx(i)$.

(a) For active firms $F + \pi^*(n, F) = (acL^2/n^2) + o(n^{-2})$ and for entrants $F + \pi_e(n, F) = (9acL^2/16n^2) + o(n^{-2})$. Thus there exists an integer $N > \max\{1, 3bcL/4a\}$ such that for all $n \geq N$, $F + \pi^*(n, F) > (0.81)(acL^2/n^2)$ and $F + \pi_e(n, F) < (0.64)(acL^2/n^2)$. Let $F^* = \min\{(acL^2/100), \frac{1}{2}\min_{n \leqslant N} \{\pi_e(n, F) + F\}\}$. To show that $\mathscr{E}(a, b, c, F, L) \neq \emptyset$ for all $F \in (0, F^*]$, fix $F \in (0, F^*]$. Then by the choice of F^*, $(0.64)(acL^2/N^2) - F > \pi_e(N, F) \geqslant 2F^* > F$. Let n be the smallest integer greater than N such that $(0.64)(acL^2/n^2) - F \leqslant 0$. By the choice of F^*, $(0.64)[acL^2/(n-1)^2] - F > 0$ implies $(0.81)(acL^2/n^2) - F > 0$ so $\pi^*(n, F) > (0.81)(acL^2/n^2) - F > 0 \geq (0.64)(acL^2/n^2) - F > \pi_e(n, F)$ and there is an n-firm equilibrium with free entry and exit.

(b) It is clear from the proof of part (a) that for any $\epsilon > 0$, for each sufficiently small F there will be an n-firm equilibrium with free entry and exit for each integer $n \in [(\frac{3}{4} + \epsilon)L(ac/F)^{1/2}, (1 - \epsilon)L(ac/F)^{1/2}]$, and there will be no n-firm equilibrium with free entry and exit for each integer $n \in [0, (\frac{3}{4} - \epsilon)L(ac/F)^{1/2}]$ or $n \in [(1 + \epsilon)L(ac/F)^{1/2}, \infty)$.

(c) For all n, $F + \pi_e(n, F) > 0$ so there exists an $F_n > 0$ such that $\pi_e(n, F) > 0$ for $F < F_n$ $(F_n = (9acL^2/16n^2) + o(n^{-2}))$. Thus as $F(k)$ converges to zero, $n(k)$, the smallest number of active firms in any equilibrium in $\mathscr{E}(a, b, c, F(k), L)$, becomes arbitrarily large. Hence for large k the maximum delivered price $m(k) = p^*(n(k)) + (cL/2n(k)) < (3cL/2n(k))$ and $m(k)$ converges to zero. Q.E.D.

ACKNOWLEDGMENTS

Comments by Hugo Sonnenschein and a referee have improved the presentation of this paper. All errors are of course my own.

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JOURNAL OF ECONOMIC THEORY 22, 327–338 (1980)

Entry (and Exit) in a Differentiated Industry*

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Received August 6, 1979

The entry process in an industry embodying more or less close substitutes is considered. One examines whether the increase in the number of substitutes induces pure competition when prices are chosen noncooperatively. It is shown that there exists an upper bound on the number of firms which can compete in the market: when this upperbound is reached, any further entry entails the exit of an existing firm. In spite of this fact, new entries imply the decrease of prices to the competitive ones.

Since Cournot [3], there has been a long-standing tradition according to which entry into a homogeneous market, where oligopolists use quantity strategies, restores pure competition (see, for instance, [5] or [7]). Intuitively, when the number of firms increases, the ability of each oligopolist to alter the value of the inverse demand function through his own strategic choice must necessarily diminish, and vanishes at the limit. By contrast, if the firms use price strategies on the same market, it has been known since Bertrand [1] that pure competition obtains already with two firms. Clearly the loss induced by undercutting the competitor's price is broadly compensated by capturing the whole demand. With Hotelling [4] and Chamberlin [2], the idea was developed that firms operate through product differentiation in order to avoid price competition "à la Bertrand." Nevertheless, within such a context, the problem remains open whether, by analogy with the homogeneous case, the increase in the number of substitutes in the industry induces pure competition when prices are chosen noncooperatively.

In order to deal with this problem, the approach employed for the homogeneous case suggests starting out with an entry process where the entrants arrive in the industry with products which are more or less close substitutes for the existing ones, and then studying the asymptotic behavior of noncooperative prices when the number of entrants tends to infinity. It is the purpose of this article to show through an example that this procedure

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* The major part of this work was done while the first author was visiting Bonn University. Financial support from the Sonderforschungsbereich 21 is gratefully acknowledged.

0022-0531/80/020327-12$02.00/0

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cannot be transposed as such to the case of a differentiated industry with price strategies.

Our example suggests that *the number of firms which can coexist in a differentiated industry cannot exceed a finite value* n^* (in our example, the number n^* is determined by a set of parameters which describe our simple economy, inspired by a previous work of the authors [6]). Surprisingly, if more than n^* firms try to remain on the market, they will necessarily jostle each other, and this struggle will provoke the exit of one of them. Still more surprising is that *this upper bound on the number of firms does not preclude that entry reinforces the tendency toward pure competition.*

In fact, the entry process decomposes into three successive phases. The first one corresponds to the situation where the number of firms is such that the whole market is not supplied at the equilibrium prices. The second obtains when the whole market is served, but where room is left for the entry of some additional firms. In the third, and last, phase, the number of potential firms is larger than n^*. It will be shown that in both the first and second phases, new entries entail decreases in prices of the products already sold on the market. As for the third phase, a new entry is now necessarily accompanied by the exit of another firm. But, in spite of the fixed number of firms still allowed on the market, equilibrium prices must necessarily decrease to the competitive ones when the number of entrants increases.

Finally, it must be noted that competition can also be restored with means other than the number of firms, because product differentiation adds a new dimension to the rivalry among firms. Our example suggests that, more than from the number of firms, perfect competition could emerge from the close substitutability among the products, thus confirming the "objection péremptoire" of Bertrand against Cournot.

The authors have recently proposed a model for dealing with a situation of differentiated duopoly [6]. Its extension to an arbitrary number of firms can provide a natural framework for settling an example in which the above questions can be discussed.

Imagine an industry constituted by *n firms*, indexed by k, $k = 1,..., n$; firm k sells, at no cost, product k; all these products are more or less close substitutes for each other. Let $T = [0, 1]$ be the set of *consumers*, which are assumed to be ranked in T by order of increasing income, and let the income $R(t)$ of consumer $t \in T$ be given by

$$R(t) = R_1 + R_2 t, \qquad R_1 > 0, R_2 \geqslant 0.$$

Consumers are also assumed to make indivisible and mutually exclusive purchases. Thus if consumer t decides to buy one of the products, k, he buys that product only, and a single unit of it.

Let us denote by $u(k, R)$ the utility of having one unit of product k and an income R, and by $u(0, R)$ the utility of having no unit of any product and an income R. In our example we take the further specification[1]

$$u(k, R) = u_k \cdot R = u_1 \cdot [1 + \alpha(k - 1)] \cdot R, \qquad \alpha \geqslant 0, \tag{1}$$

and

$$u(0, R) = u_0 \cdot R \tag{2}$$

with $u_1 > u_0 > 0$.

In this specification, u_k is a "utility index" which ranks the quality of the products: if $k > h$, product k is more desired than product h. The parameter α is a measure of the substitutability between products k; thus, for $\alpha = 0$ all products are perfect substitutes, and substitutability decreases when α increases.

Let p_k be the price quoted by oligopolist k; product k would be bought by customer t rather than product $j, j \neq k$, if and only if

$$\begin{aligned} u(k, R(t) - p_k) &= u_1 \cdot [1 + \alpha(k - 1)](R_1 + R_2 t - p_k) \\ &> u_1 \cdot [1 + \alpha(j - 1)](R_1 + R_2 t - p_j) = u(j, R(t) - p_j). \end{aligned} \tag{3}$$

To be sure that t buys product k, we must further have

$$\begin{aligned} u(k, R(t) - p_k) &= u_1 \cdot [1 + \alpha(k - 1)](R_1 + R_2 t - p_k) \\ &\geqslant u_0 \cdot (R_1 + R_2 t) = u(0, R(t)). \end{aligned} \tag{4}$$

(The consumer must prefer to buy product k rather than nothing.)

In what follows we want to characterize a noncooperative price equilibrium for the above framework. By this we mean an n-tuple of prices such that no firm can increase its profit by any unilateral deviation and such that each of the n products obtains a positive market share. To this end, let us derive the contingent demand function for each oligopolist, under the assumption that each of the n firms has a positive market share. Let $M_k(\bar{p}_1, \ldots, p_k, \ldots, \bar{p}_n)$ be the *market share* of firm k defined by

$$M_k(\bar{p}_1, \ldots, p_k, \ldots, \bar{p}_n) = \{t \mid t \text{ buys product } k \text{ at prices } (\bar{p}_1, \ldots, p_k, \ldots, \bar{p}_n)\},$$

and let (t, τ) be a pair of consumers such that $t < \tau$ (which implies that t is poorer than τ). Using (1), it is easily seen that if t chooses to buy product k, then τ will not choose to buy a product $k - j$ of lower quality, with $j \in$

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[1] Note that this formulation does not imply any restriction with respect to our analysis in [6], when $n = 2$.

$\{1,\dots, k-1\}$. Similarly, again using (1), if τ chooses to buy product k, then t will not choose to buy a product $k+j$ of higher quality, with $j \in \{1,\dots, n-k\}$. It follows that $M_k(\bar{p}_1,\dots, p_k,\dots, \bar{p}_n)$ is defined by an interval I_k of $[0, 1]$ and that the intervals I_k are ranked from the left to the right of $[0, 1]$ in an increasing order. Given our hypothesis that each consumer of brand k buys only a single unit of that product, the contingent demand function of oligopolist k is equal to the length of the interval I_k. To determine that length, it is sufficient to identify both extremities of I_k. First, let k be a product such that $1 < k < n$. The lower extremity of I_k is given by the consumer who is just indifferent between buying product k or product $k-1$, since by assumption all the firms are on the market. Similarly the upper extremity is defined by the consumer who is just indifferent between purchasing product k or product $k+1$. According to (3), the lower and upper extremities of I_k are consequently given by

$$t_k(\bar{p}_1,\dots, p_k,\dots, \bar{p}_n) \underset{\text{def}}{=} \frac{u_k p_k - u_{k-1} p_{k-1}}{(u_k - u_{k-1}) R_2} - \frac{R_1}{R_2} \tag{5}$$

and

$$t_{k+1}(\bar{p}_1,\dots, p_k,\dots, \bar{p}_n) \underset{\text{def}}{=} \frac{u_{k+1} p_{k+1} - u_k p_k}{(u_{k+1} - u_k) R_2} - \frac{R_1}{R_2}, \tag{6}$$

so that the demand function $\mu[M_k(\bar{p}_1,\dots, p_k,\dots, \bar{p}_n)]$ is equal to

$$\mu[M_k(\bar{p}_1,\dots, p_k,\dots, \bar{p}_n)] = t_{k+1}(\bar{p}_1,\dots, p_k,\dots, \bar{p}_n) - t_k(\bar{p}_1,\dots, \bar{p}_n). \tag{7}$$

Let us now consider the case of product n. Since no brand of higher quality exists, the upper extremity of I_n is provided by the richest customer $t = 1$, so that the demand function of oligopolist n is given by

$$\begin{aligned}\mu[M_n(\bar{p}_1,\dots, \bar{p}_k,\dots, p_n)] &= 1 - t_n(\bar{p}_1,\dots, \bar{p}_k,\dots, p_n)\\ &= 1 - \frac{u_n p_n - u_{n-1} p_{n-1}}{(u_n - u_{n-1}) R_2} + \frac{R_1}{R_2}.\end{aligned} \tag{8}$$

Finally let us envision the case of product 1. Since no brand of lower quality is offered and if the whole market is not served, the lower extremity of I_1 is provided by the consumer who is just indifferent between buying product 1 or buying nothing, i.e., using (4) with $k = 1$,

$$t_1(p_1,\dots, \bar{p}_k,\dots, \bar{p}_n) \underset{\text{def}}{=} \frac{u_1 p_1}{(u_1 - u_0) R_2} - \frac{R_1}{R_2}; \tag{9}$$

the demand function of oligoplist 1 is then given by

$$\mu[M_1(p_1,\dots, \bar{p}_k,\dots, \bar{p}_n)] = t_2(p_1,\dots, \bar{p}_k,\dots, \bar{p}_n) - t_1(p_1,\dots, \bar{p}_k,\dots, \bar{p}_n). \tag{10}$$

On the other hand, if the whole market is served, the lower extremity of I_1 is defined by the poorest customer $t = 0$, so that the demand function of oligopolist 1 is then equal to

$$\mu[M_1(p_1, \ldots, \bar{p}_k, \ldots, \bar{p}_n)] = t_2(p_1, \ldots, \bar{p}_k, \ldots, \bar{p}_n). \tag{11}$$

Denote by $P_k(\bar{p}_1, \ldots, p_k, \ldots, \bar{p}_n) = p_k \cdot \mu[M_k(\bar{p}_1, \ldots, p_k, \ldots, \bar{p}_n)]$ the profit function of firm k. A *noncooperative price equilibrium* is defined as an n-tuple of prices $(p_1^*, \ldots, p_k^*, \ldots, p_n^*)$ such that no firm k can increase its profit by any unilateral deviation from p_k^* when other firms j stick to prices $p_j^*, j \neq k$, and $\mu[M_k(p_1^*, \ldots, p_k^*, \ldots, p_n^*)] > 0, \forall\, k = 1, \ldots, n$.[2]

Let us now determine the equilibrium prices, successively, when the whole market is not served $(\sum_{k=1}^{n} \mu[M_k(p_1^*, \ldots, p_n^*)] < 1)$ and when it is entirely served $(\sum_{k=1}^{n} \mu[M_k(p_1^*, \ldots, p_n^*)] = 1)$. In the first case, the demand functions are described by (7), (8), and (10). Assuming that there exists a noncooperative price equilibrium, the following first-order conditions must be satisfied:

$$\left.\frac{\partial P_1}{\partial p_1}\right|_{(p_1^*, \ldots, p_n^*)} = \frac{(1+\alpha)\, u_1 p_2^* - 2u_1 p_1^*}{\alpha u_1 R_2} - \frac{2u_1 p_1^*}{(u_1 - u_0)\, R_2} = 0,$$

$$\left.\frac{\partial P_k}{\partial p_k}\right|_{(p_1^*, \ldots, p_n^*)} = \frac{(1+\alpha k)\, u_1 p_{k+1}^* - 2[1+\alpha(k-1)]\, u_1 p_k^*}{\alpha u_1 R_2} - \frac{2[1+\alpha(k-1)]\, u_1 p_k^* - [1+\alpha(k-2)]\, u_1 p_{k-1}^*}{\alpha u_1 R_2} = 0, \quad k = 2, \ldots, n-1,$$

$$\left.\frac{\partial P_n}{\partial p_n}\right|_{(p_1^*, \ldots, p_n^*)} = 1 - \frac{2[1+\alpha(n-1)]\, u_1 p_n^* - [1+\alpha(n-2)]\, u_1 p_{n-1}^*}{\alpha u_1 R_2} + \frac{R_1}{R_2} = 0,$$

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[2] The contingent demand functions we have derived above are only valid in the restricted domains of price strategies for which all the n oligopolists obtain a positive market share. Without this assumption the possibility should indeed be recognized that, for any group of products, n-tuples of prices can be found which would cancel the demand for any product in that group. Nevertheless, it can be shown that the equilibrium prices calculated in the restricted domains are in fact equilibrium prices over the whole domains. The reason is that the profit function of each oligopolist is quasi-concave over the whole domain of its strategies, so that a local best reply is also a global one. This subject will be dealt in a forthcoming paper.

which reduce to the system of difference equations:

$$\begin{gathered}(1 + \alpha)(u_1 - u_0)\, p_2^* - 2[(u_1 - u_0) + \alpha u_1]\, p_1^* = 0,\\ (1 + \alpha k)\, p_{k+1}^* - 4[1 + \alpha(k - 1)]\, p_k^* + [1 + \alpha(k - 2)]\, p_{k-1}^* = 0,\\ k = 2,\dots, n - 1,\\ -2[1 + \alpha(n - 1)]\, p_n^* + [1 + \alpha(n - 2)]\, p_{n-1}^* = -\alpha(R_1 + R_2).\end{gathered}$$

Using the change of variables defined by

$$x_k = [1 + \alpha(k - 1)]\, p_k^* \tag{12}$$

the above system can be rewritten as

$$\begin{gathered}(u_1 - u_0)\, x_2 - 2[(u_1 - u_0) + \alpha u_1]\, x_1 = 0,\\ x_{k+1} - 4x_k + x_{k-1} = 0, \qquad k = 2,\dots, n - 1,\\ -2x_n + x_{n-1} = -\alpha(R_1 + R_2),\end{gathered} \qquad \{1\}$$

the solution of which is

$$x_k = A_n(2 + 3^{1/2})^k + B_n(2 - 3^{1/2})^k$$

with

$$A_n = \frac{\alpha(2 - 3^{1/2})[3^{1/2}(u_1 - u_0) + 2\alpha u_1](R_1 + R_2)}{\left(\begin{array}{l}(3u_1 - 3u_0 + 2\cdot 3^{1/2}\alpha u_1)(2 + 3^{1/2})^{n-1}\\ \qquad - (3u_1 - 3u_0 - 2\cdot 3^{1/2}\alpha u_1)(2 - 3^{1/2})^{n-1}\end{array}\right)}$$

and

$$B_n = \frac{\alpha(2 + 3^{1/2})[3^{1/2}(u_1 - u_0) - 2\alpha u_1](R_1 + R_2)}{\left(\begin{array}{l}(3u_1 - 3u_0 + 2\cdot 3^{1/2}\alpha u_1)(2 + 3^{1/2})^{n-1}\\ \qquad - (3u_1 - 3u_0 - 2\cdot 3^{1/2}\alpha u_1)(2 - 3^{1/2})^{n-1}\end{array}\right)}.$$

Given (12), we finally obtain

$$p_k^* = \frac{1}{[1 + \alpha(k - 1)]}\, [A_n(2 + 3^{1/2})^k + B_n(2 - 3^{1/2})^k]. \tag{13}$$

These prices are only valid when the whole market is not served. In order to guarantee that this condition is satisfied, we must have, in particular, that the equilibrium price p_1^* for oligopolist 1 obtained from (13) is larger than $[(u_1 - u_0)/u_1]\, R_1$ (by (4) with $k = 1$), i.e.,

$$\frac{R_1}{R_2} < \frac{2u_1\alpha\, 3^{1/2}}{\left(\begin{array}{l}3(u_1 - u_0)[(2 + 3^{1/2})^{n-1} - (2 - 3^{1/2})^{n-1}]\\ \quad + 2u_1\alpha\, 3^{1/2}[(2 + 3^{1/2})^{n-1} + (2 - 3^{1/2})^{n-1} - 1]\end{array}\right)}. \tag{14}$$

In the case where the whole market is served, the demand functions are defined by (7), (8), and (11). Hence an interior noncooperative price equilibrium must verify the first-order conditions

$$\left.\frac{\partial P_1}{\partial p_1}\right|_{(p_1^*,\dots,p_n^*)} = \frac{(1+\alpha)\,u_1p_2^* - 2u_1p_1^*}{\alpha u_1R_2} - \frac{R_1}{R_2} = 0,$$

$$\left.\frac{\partial P_k}{\partial p_k}\right|_{(p_1^*,\dots,p_n^*)} = \frac{(1+\alpha k)\,u_1p_{k+1}^* - 2[1+\alpha(k-1)]\,u_1p_k^*}{\alpha u_1R_1}$$

$$- \frac{2[1+\alpha(k-1)]\,u_1p_k^* - [1+\alpha(k-2)]\,u_1p_{k-1}^*}{\alpha u_1R_2} = 0,$$

$$k = 2,\dots, n,$$

$$\left.\frac{\partial P_n}{\partial p_n}\right|_{(p_1^*,\dots,p_n^*)} = 1 - \frac{2[1+\alpha(n-1)]\,u_1p_n^* - [1+\alpha(n-2)]\,u_1p_{n-1}^*}{\alpha u_1R_2}$$

$$+ \frac{R_1}{R_2} = 0,$$

which, for the change of variables (12), are equivalent to the system of difference equations

$$\begin{aligned} x_2 - 2x_1 &= \alpha R_1, \\ x_{k+1} - 4x_k + x_{k-1} &= 0, \qquad k = 2,\dots, n-1, \\ -2x_n + x_{n-1} &= -\alpha(R_1+R_2), \end{aligned} \qquad \{2\}$$

whose solution is given by

$$x_k = A_n'(2+3^{1/2})^k + B_n'(2-3^{1/2})^k$$

with

$$A_n' = \frac{(3-2\cdot 3^{1/2})[\alpha(2-3^{1/2})^{n-1}R_1 - \alpha(R_1+R_2)]}{3[(2+3^{1/2})^{n-1} - (2-3^{1/2})^{n-1}]}$$

and

$$B_n' = \frac{(3+2\cdot 3^{1/2})[\alpha(2+3^{1/2})^{n-1}R_1 - \alpha(R_1+R_2)]}{3[(2+3^{1/2})^{n-1} - (2-3^{1/2})^{n-1}]}$$

so that p_k^* is defined by

$$p_k^* = \frac{1}{[1+\alpha(k-1)]}\,[A_n'(2+3^{1/2})^k + B_n'(2-3^{1/2})^k]. \qquad (15)$$

These prices are only valid when the market is entirely served and when each oligopolist gets a positive market share. To this effect, the equilibrium price p_1^* of oligopolist 1 given by (15) must be smaller than or equal to $[(u_1 - u_0)/u_1]\, R_1$, i.e.,

$$\frac{2u_1\alpha\, 3^{1/2}}{\left(\begin{array}{l}3(u_1 - u_0)[(2 + 3^{1/2})^{n-1} - (2 - 3^{1/2})^{n-1}] \\ \qquad + u_1\alpha\, 3^{1/2}[(2 + 3^{1/2})^{n-1} + (2 - 3^{1/2})^{n-1} - 2]\end{array}\right)} \leqslant \frac{R_1}{R_2} \tag{16}$$

and $\mu[M_1(p_1^*,\ldots, p_k^*,\ldots, p_n^*)]$ must be positive, i.e.,

$$\frac{R_1}{R_2} < \frac{2}{(2 + 3^{1/2})^{n-1} + (2 - 3^{1/2})^{n-1} - 2}\,. \tag{17}$$

So far, we have characterized equilibrium prices in an industry embodying n firms selling products whose respective utility indices satisfy relationship (1). Equipped with this framework, we may now study the change in prices and market shares when the number of firms increases. To proceed in that direction, we shall assume that new firms always enter the market with higher-quality products, namely, that the $(n + 1)$st firm enters the market with a utility index u_{n+1} equal to $(1 + \alpha n)\, u_1$.[3]

To begin with, let us assume that the starting number n of firms is such that the whole market is not served. In this case, we know that condition (14) must hold and that equilibrium prices are given by (13). Defining $\bar{n}$ as the largest integer for which condition (14) is verified, it means that $n \leqslant \bar{n}$. If a new firm enters, then either $n + 1$ is still smaller than or equal to $\bar{n}$, or $n + 1$ is greater than $\bar{n}$. In the first alternative, it is easily verified that *the "after-entry" equilibrium prices for existing firms*, still given by (13), *are smaller than the "pre-entry" equilibrium prices.* Consequently, as long as the whole market is not served, equilibrium prices form a decreasing sequence of the number of the new entrants. In the second alternative, the whole market is served. Assuming that an interior noncooperative price equilibrium is observed with $(\bar{n} + 1)$ firms (which is fulfilled if condition (16) holds for $\bar{n} + 1$), the equilibrium prices are then given by (15). Computing $\partial p_k^*/\partial n$ from (15), we obtain the result that, as soon as the whole market is served, equilibrium prices again form a decreasing sequence when the number of entrants increases.

Our major finding is that this entry process cannot allow a continuously increasing number of firms with a positive market share. Indeed, for all the firms already on the market to maintain a positive share, we know that condition (17), among others, must be satisfied. As the right-hand side of (17) is a decreasing function of n, there exists a maximal number, say n^*, of firms for which (17) can still hold. For $n > n^*$, the converse of (17) must be verified: income disparities, as expressed in our model by the value of R_2, are no longer sufficient to sustain an industry embodying a larger number of firms. In other words, *the income distribution determines endogenously the maximal number of products which defines the industry.* It is our belief that the number n^* can be viewed as a kind of long-run equilibrium number of firms in the sense that, when this number is reached, no room is left for a larger number of products.

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[3] Although the use of this particular entry process entails some loss of generality, that loss is comparable with the loss of generality which follows from assuming that all firms are identical, a hypothesis usually made in the theory of entry with homogeneous products. Moreover, analogous processes are considered in the recent literature in location theory (see, for instance, [8]).

Does it mean that no other firm with a higher-quality product can enter when the long-run equilibrium number n^* is reached? The answer is no. Indeed the entry of a new firm can take place, provided, however, that it is accompanied by the exit of another. Assume that a firm selling a product with utility index $u_{n^*+1} = (1 + \alpha n^*) u_1$ decides to enter. It then follows from the definition of n^* that at least one other firm must necessarily obtain a null market share at the after-entry equilibrium. In fact, it can be shown that only one firm must exit at the after-entry equilibrium and that it can be only firm 1 (see the proposition given in the Appendix). Consequently, after entry of firm $n^* + 1$, the industry again embodies n^* firms, but now with indices $\{2, \ldots, n^* + 1\}$. More generally, the entry of firm $n^* + m$, with utility index $u_{n^*+m} = [1 + \alpha(n^* + m - 1)] u_1$, would similarly lead to an industry profile defined by the firms $\{m + 1, \ldots, n^* + m\}$. Interestingly, *the after-entry equilibrium prices form a sequence decreasing to the competitive prices as the number m of new entrants increases.* Indeed, the equilibrium price of firm k is given by

$$p_k^* = \frac{1}{1 + \alpha(k - 1)} [A'_{n^*}(2 + 3^{1/2})^{k-m} + B'_{n^*}(2 - 3^{1/2})^{k-m}],$$

with $k = m + 1, \ldots, m + n^*$ (again see the proposition of the Appendix). Among other things, this result implies that we may observe low prices in an industry embodying a fixed, and possibly small, number of firms provided m is large enough.

Let us illustrate the whole process we have just described when the long-run equilibrium number n^* is equal to 2. Let firm 1 initially be a monopolist in the industry and sell a product with utility index u_1. By the choice rule (4) with $k = 1$, one easily checks that the market share $M_1(p_1)$ is defined by the interval I_1 whose lower and upper extremities are respectively given by

$$t_1(p_1) = \text{Max} \left\{0, \frac{u_1 p_1}{(u_1 - u_0) R_2} - \frac{R_1}{R_2}\right\}$$

and 1. The demand function $\mu[M_1(p_1)]$ faced by firm 1 is then

$$\text{Min} \left\{1, 1 - \frac{u_1 p_1}{(u_1 - u_0) R_2} + \frac{R_1}{R_2}\right\}.$$

Assuming $R_1/R_2 < 1$, a simple calculation shows that the monopoly price is equal to $\frac{1}{2}[(u_1 - u_0)/u_1](R_1 + R_2)$. Since, at this price, all the consumers are not served, room is left for entry of firm 2 with utility index $u_2 = (1 + \alpha) u_1$. Assuming also that condition (16) holds for $n = 2$, namely,

$$\frac{\alpha}{3(u_1 - u_0) + \alpha u_1} \leqslant \frac{R_1}{R_2},$$

we deduce from the above that the whole market is supplied at the equilibrium prices

$$p_1^* = \frac{\alpha(R_2 - R_1)}{3} \quad \text{and} \quad p_2^* = \frac{\alpha(2R_2 + R_1)}{3(1 + \alpha)};$$

consequently, $\bar{n} = 1$.[4] Suppose further that condition (17) is not fulfilled for $n = 3$, namely, that $\frac{1}{6} \leqslant R_1/R_2$; accordingly, $n^* = 2$. Hence under the hypothesis

$$\max \left\{\frac{\alpha}{3(u_1 - u_0) + \alpha u_1}, \frac{1}{6}\right\} \leqslant \frac{R_1}{R_2} < 1,$$

two firms, and only two firms, may remain in the industry forever. If firm 3 with utility index $u_3 = (1 + 2\alpha) u_1$ would in turn enter the market, then a new equilibrium would emerge at which firm 1 has been enforced to exit, with prices

$$p_2^* = \frac{\alpha(R_2 - R_1)}{3(1 + \alpha)} \quad \text{and} \quad p_3^* = \frac{\alpha(2R_2 + R_1)}{3(1 + 2\alpha)}.^5$$

It remains to study the role of substitutability among the products at the equilibrium. First note that, whatever the fixed number of products, a value of α sufficiently small exists for condition (16) to be satisfied. In this case, equilibrium prices are then given by (15). Second, these equilibrium prices form a decreasing sequence which converges to zero as the substitutability between products increases, that is, as α tends to zero. As stated above, pure competition is the limit of a process where products become more and more homogeneous. At the limit this is nothing else but the "objection péremptoire" of Bertrand against Cournot.

APPENDIX

PROPOSITION. *Let q and m be two arbitrary integers such that $q \leqslant m + 1$ and let the firms defined by the set of indices $\{q, \ldots, n^*, \ldots, n^* + m\}$. A non-*

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[4] A more extensive discussion of the market solution for $n = 2$ is contained in [6].

[5] This illustration shows that the long-run equilibrium number is not necessarily very large. A priori, to any value of n, there corresponds an income distribution which authorizes a number of firms at most equal to this value. Possibly, with a high degree of income dispersion, a large number of products will be observed.

cooperative price equilibrium involves exactly n^* *firms given by the set of indices* $\{m + 1, \ldots, n^* + m\}$. *Furthermore, the equilibrium price of firm* k, *with* $k = m + 1, \ldots, n^* + m$ *is given by*

$$p_k^* = \frac{1}{1 + \alpha(k - 1)} \cdot [A'_{n^*}(2 + 3^{1/2})^{k-m} + B'_{n^*}(2 - 3^{1/2})^{k-m}]. \tag{18}$$

Proof. Assume that the firms defined by $\{q, \ldots, n^* + m\}$ have a positive market share at the equilibrium and that the firms defined by $\{1, \ldots, q - 1\}$ have been put out of business. In this case, the equilibrium prices must verify the following first-order conditions:

$$(1 + \alpha q)\, p_{q+1}^* - 2[1 + \alpha(q - 1)]\, p_q^* - \alpha R_1 = 0,$$

$$(1 + \alpha k)\, p_{k+1}^* - 4[1 + \alpha(k - 1)]\, p_k^* + [1 + \alpha(k - 2)]\, p_{k-1}^* = 0,$$
$$k = q + 1 \cdots n^* + m - 1,$$

$$\alpha(R_1 + R_2) - 2[1 + \alpha(n^* + m - 1)]\, p_{n^*+m}^*$$
$$+ [1 + \alpha(n^* + m - 2)]\, p_{n^*+m-1}^* = 0.$$

Using the change of variables defined by (12), we obtain

$$\begin{aligned} x_{q+1} - 2x_q - \alpha R_1 &= 0, \\ x_{k+1} - 4x_k + x_{k-1} &= 0, \qquad k = q + 1, \ldots, n^* + m - 1, \\ \alpha(R_1 + R_2) - 2x_{n^*+m} + x_{n^*+m-1} &= 0. \end{aligned} \tag*{\{3\}}$$

Furthermore, setting $j = k - q + 1$ yields the system

$$\begin{aligned} x_2 - 2x_1 - \alpha R_1 &= 0, \\ x_{j+1} - 4x_j + x_{j-1} &= 0, \qquad j = 2, \ldots, n^* + m - q, \\ \alpha(R_1 + R_2) - 2x_{n^*+m-q+1} + x_{n^*+m-q} &= 0. \end{aligned} \tag*{\{4\}}$$

This system is identical to system {2}, with $n = n^* + m - q + 1$, so that its solution is given by

$$x_j = A'_{n^*+m-q+1}(2 + 3^{1/2})^j + B'_{n^*+m-q+1}(2 - 3^{1/2})^j,$$
$$j = 1, \ldots, n^* + m - q + 1,$$

i.e.,

$$x_k = A'_{n^*+m-q+1}(2 + 3^{1/2})^{k-q+1} + B'_{n^*+m-q+1}(2 - 3^{1/2})^{k-q+1},$$
$$k = q, \ldots, n^* + m,$$

which leads to

$$p_k^* = \frac{1}{1+\gamma(k-1)} \cdot [A'_{n^*+m-q+1}(2+3^{1/2})^{k-q+1} + B'_{n^*+m-q+1}(2-3^{1/2})^{k-q+1}]. \quad (19)$$

By definition of a noncooperative price equilibrium, each firm must have a positive market share. In particular, this must be true for firm q, so that

$$u_{q+1}p_{q+1}^* - u_q p_q^* > \alpha R_1 .$$

The latter condition is verified if and only if condition (17) holds for $n^* + m - q + 1$, which is impossible as long as $q < m + 1$. Consequently, firm q cannot afford a positive market share. As a similar argument applies to firms $k = q + 1,..., k = m$, *exactly* n^* firms, namely, the firms defined by $\{m + 1,..., n^* + m\}$, remain on the market and the corresponding equilibrium prices are provided by (19), where $q = m + 1$, i.e., by (18). Q.E.D.

ACKNOWLEDGMENTS

We thank J. Drèze and an anonymous referee for their helpful comments.

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JOURNAL OF ECONOMIC THEORY **22**, 339–362 (1980)

Efficiency Properties of Strategic Market Games: An Axiomatic Approach*

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Received July 23, 1979

The paper investigates the conditions under which an abstractly given market game will have the property that if there is a continuum of traders then every noncooperative equilibrium is Walrasian. In other words, we look for a general axiomatization of Cournot's well-known result. Besides some convexity, continuity, and nondegeneracy hypotheses, the crucial axioms are: anonymity (i.e., the names of traders are irrelevant to the market) and aggregation (i.e., the net trade received by a trader depends only on his own action and the mean action of all traders). It is also shown that the same axioms do not guarantee efficiency if there is only a finite number of traders. Some examples are discussed and a notion of strict noncooperative equilibrium for anonymous games is introduced.

1. INTRODUCTION

In this paper we shall attempt an axiomatization of a central theme of economic theory, namely, the idea that the noncooperative equilibria of

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* This paper brings together and supersedes Dubey and Shubik (1978) and Mas-Colell (1978). We are indebted for discussion, suggestions, and amendations to K. Arrow, W. Heller, L. Hurwicz, Ch. Kahn, E. Maskin, A. Postlewaite, and W. Thomson. The usual caveat applies. A. Mas-Colell's research was supported by NSF Grant SOC76-19700A01 and SOC77-06000. The latter grant was at the Institute for Mathematical Studies in the Social Sciences, Stanford University. The work of Dubey and Shubik relates to Department of the Navy Contract N00014-77-C-0518, issued by the Office of Naval Research under contract authority NR047-006.

0022-0531/80/020339-24$02.00/0

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market-like economies with many relatively small traders are Walrasian (hence efficient) or, in other words, that price-taking behavior is, in a mass market, the natural consequence of "message taking" behavior. The classical partial equilibrium reference is, of course, Cournot's (1838). Two modern general-equilibrium version of Cournot's model are represented by Shubik (1973) [see also Shapley (1976), Shapley and Shubik (1977), Jaynes, Okuno, and Schmeidler (1978)] and by Gabszewicz-Vial (1972) [see also Hart (1979) and Novshek–Sonnenschein (1978)]. The work reported here develops from the Shubik line.

We take as our conceptual starting point the notion of Strategic Market Game, to be understood as a complete specification of trading rules and transaction constraints with respect to which a noncooperative equilibria is reached. This is to be contrasted with the theory of the core of a market, [which originated in Edgeworth (1881); see Hildenbrand (1974) and his references] where final outcomes are completely unconstrained by institutional mechanisms of trade.

The game being played in real economic life is quite complex. It is clear that there is not a unique, somehow given, way to formalize it as a strategic market game. So it is sound research strategy to build a gallery of models emphasizing different important features, their fine structure reflecting the host of institutional features of the economy not usually taken into account by the economic theorist, but of importance in the understanding of marketing and transaction technologies. Nevertheless, experience with different models indicates that a number of facts are quite robust to changes of specification. Among them: the noncooperative equilibria of markets with a finite number of traders tend (this is a suitably vague verb) to be inefficient, and the noncooperative equilibria of markets with many relatively small participants tend to be (almost) efficient. Thus, it seems justified in order to bring out the robustness of these phenomena to engage in axiomatics.

We proceed by sidestepping the modeling problem and assuming (admittedly, this is quite a lot to assume) that a strategic market game is given to us in an abstract manner. We then try to identify general principles yielding the efficiency results. Our axiomatization will be fairly transparent and, from the mathematical point of view, trivial. The basic principles turn out to be:

(i) convexity: traders have available a convex set of strategies.

(ii) anonymity: from the point of view of the market, only the message sent by the trader matters.

(iii) continuity: of outcomes with respect to strategies.

(iv) aggregation: the trading possibilities of any player are influenced by the messages of the other players only through the *mean* of those messages (and not, say, through the variance).

(v) nondegeneracy: it must be possible for individual players to influence to a substantial extent their trading possibilities in the market.

With the given axioms we first prove some inefficiency results for the finite number of traders case and then establish efficiency for the large number of small traders case. As in Aumann (1964) for "large number of small traders," we shall adopt the idealization of the continuum. We do this merely for convenience and in order to sharpen the essential facts. We will, however, devote one section to the asymptotic, in contrast to limit, theory. It is fortunate that the consideration of a large number of small traders (in the limit a continuum) has simplifying implications for noncooperative analysis. Indeed, under general conditions, if individual players are, in whichever is the relevant sense, very small relative to the size of the game, the only sensible expectations (at "equilibrium" or at "disequilibrium" plays) of individual players is that the rest of the players will not react (or react very little) to changes of their actions. Thus, in contrast to games among few, simple Cournot–Nash equilibrium emerges as the natural noncooperative solution. Our treatment of Cournot–Nash equilibrium in the continuum builds on a paper of Schmeidler (1973) which deserves to be better known.

There are clear similarities and connections between our formal structure and the extensive literature growing out of Hurwicz's (1960) seminal contribution on the designing of resource allocation mechanism. But it is important to keep in mind that the spirit of our work is very different. In particular, we do not attach normative significance to our results. For us, the market mechanism is given, and we merely analyze its properties.

The paper is organized as follows: Sections 2–5 present the basic model. Section 6 enumerates the axioms. Section 7 looks at the continuum as a limit of the finite but large situation. Section 8 discusses a nondegeneracy issue for equilibria. Sections 9 and 10 present efficiency results for the finite and continuum cases, respectively. Section 11 looks at examples; and, finally, Section 12 defines and discusses a type of noncooperative equilibria of special relevance to market situations.

2. COMMODITIES, AGENTS, ASSIGNMENTS

The commodity space is R^l.

We let I, the set of agents names, be either a finite set equipped with the counting measure or the interval [0, 1] with Lebesgue measure (denoted λ). This will allow us to treat simultaneously the finite and the continuum cases.

An *assignment* is an integrable map $\mathbf{x}\colon I \to R^l$ such that $\int \mathbf{x}(t)\, dt = 0$. Note that we are concerned only with net trades.

3. ECONOMICS

We now look at the agents in I as the traders of an economy.

A *trader* is characterized by:

(i) a closed, convex, nonempty set $X \subset R^l$ of net trades, interpreted as the set of contracts the trader can honor, and

(ii) a complete and transitive preferences relation $\gtrsim \subset X \times X$. Every X is assumed to satisfy *free disposability*, i.e., if $x \in X$, then $x + R_+^l \subset X$. Preferences $\gtrsim$ are taken to be continuous, convex, and *monotone*. The set of characteristics is $\mathscr{A} = \{(X, \gtrsim)\}$. As in Hildenbrand (1974), one can view $\mathscr{A}$ as a measurable space by endowing it with the σ-field derived from the closed convergence topology on the closed subsets of R^l.

We define an economy à la Aumann (1963), i.e., as a (measurable) map $\mathscr{E}: I \to \mathscr{A}$. We denote $\mathscr{E}(t)$ by $(X_t, \gtrsim_t)$.

An assignment $\mathbf{x}: I \to R^l$ is *feasible* if $x(t) \in X_t$ for a.e. $t \in I$.

The existence of two types of trade constraints is a basic distinction in our treatment. We have the *public constraints* embodied in the fact that the sum of net trades must be zero (i.e., markets should clear) and the *private constraints* requiring that individual net trades belong to the individual trading sets. As a matter of interpretation we take that the private constraints are not publicly observable.

A feasible assignment $\mathbf{x}$ is efficient if there is no other feasible assignment $\mathbf{x}'$ such that $\mathbf{x}'(t) \gtrsim_t x(t)$ for a.e. $t \in I$ and $x'(t) \succ_t x(t)$ for a set of t's of positive measure.

A feasible assignment $\mathbf{x}$ is *Walrasian* if there is $p \in R^l$ such that for a.e. $t \in I$, $\mathbf{x}(t)$ is $\gtrsim_t$ maximal on $\{x \in X_t : p \cdot x \leqslant 0\}$. It is, of course, well known that, with the hypothesis made, a Walrasian assignment is efficient.

Remark 1. Note that (i) individual production (i.e., firms with a single owner) is encompassed by the model, (ii) consumption or technological externalities are not permitted.

4. STRATEGIC MARKETS GAMES

We now look at the set of agents I as a set of players.

There is given a nonempty set S called the message, action, or strategy space. We assume that S is a *subset of a separable Banach space*. It will be conceptually simpler if the reader thinks of S as lying in a finite-dimensional space, i.e., a subset of some R^m.

A *play* is an integrable function $\mathbf{s}: I \to S$.

A *strategie market game* is specified by the *set of players I*, the *strategy space S* and an *outcome rule Φ*, which associates with every play $\mathbf{s}$ an assignment $\Phi(\mathbf{s})\colon I \to R^l$. If Φ is understood, we write $\mathbf{x}_\mathbf{s}$ for $\Phi(\mathbf{s})$.

For the measure theoretic treatment to be sensible, we should require that the outcome rule satisfies the following property: *if* $[\mathbf{s}] = [\mathbf{s}']$, *then* $[\mathbf{x}_\mathbf{s}] = [\mathbf{x}_{\mathbf{s}'}]$. The symbol [ ] denotes the equivalence class of the function (i.e., all functions which are a.e. equal to the given function). Informally speaking, this property, which is nonvacuous only for the case $I = [0, 1]$, has the implication that (again for $I = [0, 1]$) individual actions do not have a macroscopic influence on market outcomes (i.e., pecuniary externalities are absent), which is a well-known heuristic requirement for a competitive process to yield efficient outcomes.

Let us illustrate with some examples. They will be discussed with more detail in Section 11, from where, especially for Example 1, a clearer understanding of the underlying economics will be gained.

EXAMPLE 1. [Shubik (1973)]. There are two commodities. The strategy space is R_+^2. A message (s_1, s_2) is to be interpreted as follows: s_1 (resp. s_2) is the amount of commodity 1 (resp. 2) a trader proposes to deliver to the market. Let $\mathbf{s}$ be a play. Suppose first that $\int \mathbf{s}_1, \int \mathbf{s}_2 > 0$. Then the outcome is given by

$$\mathbf{x}_\mathbf{s}(t) = \left(\frac{\int \mathbf{s}_1}{\int \mathbf{s}_2}\, \mathbf{s}_2(t) - \mathbf{s}_1(t), \frac{\int \mathbf{s}_2}{\int \mathbf{s}_1}\, \mathbf{s}_1(t) - \mathbf{s}_2(t)\right).$$

If either $\int \mathbf{s}_1 = 0$ or $\int \mathbf{s}_2 = 0$, we put $\mathbf{x}_\mathbf{s} = 0$.

In the next two examples the strategies are whole demand-supply functions.

EXAMPLE 2. Again $l = 2$. The strategy set is $S = \{f \in C([0, r]) : f(0) \geqslant 0, f(r) \leqslant 0\}$, where r is some large number and for $p \in [0, r]$ one interprets $f(p)$ as a proposal to buy ("sell" if negative) $f(p)$ units of the second commodity in exchange for $pf(p)$ units of the first commodity. Let $\mathbf{s}$ be a play. Define

$$p = \max \left\{p \in [0, r] \colon \left(\int \mathbf{s}\right)(p) = 0\right\}.$$

Then the outcome is given by

$$\mathbf{x}_\mathbf{s}(t) = (-p\, \mathbf{s}(t)(p), \mathbf{s}(t)(p)).$$

This example shows that there is substance in the implicit hypothesis that to every play $\mathbf{s}$ there corresponds a unique assignment. The rule for choosing one among the possibly many price equilibria is completely arbitrary. In the next example, the equilibrium is naturally unique.

EXAMPLE 3. Identical to Example 2 except that now $S = \{f \in C([0, r]) : f(0) \geqslant 0, f(r) \leqslant 0$ and f is decreasing$\}$.

Remark 2. In the case where S is infinite dimensional (which shall not be emphasized), "integrable" is taken to mean Pettis integrable [see, for example, Yosida (1971)].

Remark 3. We could in the definition of a play require simply that $\mathbf{s}$ be measurable, thus allowing for nonintegrable plays. But not much would be gained, since in fact we will have a need to integrate.

Remark 4. In the present approach the outcomes of a play are not necessarily feasible assignments. This is in keeping with the modelling of market institutions as blind to private characteristics. Any player is free to send any message to the market although the outcome of a play can be realized only if almost every trader can honor the proposed contract without going bankrupt. Thus, our strategies are more in the nature of proposals than actual actions, and to that extent the use of the term "game" may be a little abusive. A modeling via Debreu's Generalized Games (1952) would have been possible. The present situation can be contrasted with the well-known price tatonnement. Here public constraints are always satisfied, but private ones need not be. There private constraints do hold, but not necessarily the public ones (i.e., at given arbitrary prices, there may be excess demand or supply).

Remark 5. The concept of strategic market games introduced in this section is formally related to the concept of allocation mechanism growing out of Hurwicz's (1960) approach to the *design* of resource allocation methods. [See, for example, Hurwicz (1979) and Schmeidler (1978)].

5. NONCOOPERATIVE EQUILIBRIA

For fixed I let an economy $\mathscr{E} : I \to \mathscr{A}$ and a strategic market game Φ with message space S be given.

DEFINITION. A play $\mathbf{s}$ is a Cournot–Nash (CN) equilibrium if $\mathbf{x}_\mathbf{s}$ is feasible and for almost every $t \in I$, $\mathbf{x}_\mathbf{s}(t)$ is $\succsim_t$ maximal on $X_t \cap \{\mathbf{x}_{\mathbf{s}'}(t) : \mathbf{s}'(t') = \mathbf{s}(t')$ for all $t' \neq t\}$.

The assignment $\mathbf{x}_\mathbf{s}$ corresponding to a CN play will be called a CN assignment.

Since we will very quickly specialize our model, we shall not dwell on a discussion of this definition.

Remark 6. For the continuum, i.e., $I = [0, 1]$, the definition of CN equilibrium has taken inspiration in Schmeidler's (1973) work on the noncooperative theory of games with a continuum of players.

6. AXIOMS

Let the strategic market game Φ on a set of players I be given.

We shall restrict our consideration to strategic market games whose outcomes depend only on the strategies played and not on the names of the players. More precisely: (i) if two traders choose the same action, they get assigned the same net trade, and (ii) for a given fixed action of a trader, the net trade assigned depends only on the distribution of the strategies of all traders. Those are particular conditions, but they embody the anonymity principles of well-functioning markets: the trading possibilities of any particular economic agent are limited *only* by their availability of commodities to trade.

Our wish to impose an anonymity axiom explains two aspects of our treatment of strategic market games: the strategy sets of different players were assumed equal and the outcome of a play was not required to satisfy the private feasibility constraints.

Let $\mathcal{M}$ be the set of probability measures on S with finite mean. To every $\mathbf{s} : I \to S$ there corresponds an $\nu_{\mathbf{s}} = \lambda \circ \mathbf{s}^{-1} \in \mathcal{M}$.

Anonymity Axiom. There is a function $G : S \times \mathcal{M} \to R^l$ such that, for all plays $\mathbf{s}$, $x_{\mathbf{s}}(t) = G(\mathbf{s}(t), \nu_{\mathbf{s}})$ for all $t \in I$.

Clearly, for I finite, the function G is not unique, while for $I = [0, 1]$, it is trivially so.

Under the Anonymity Axiom, a play $\mathbf{s}$: $I \to S$ is a CN equilibrium if $G(\cdot, \nu_{\mathbf{s}}) : I \to R^l$ is a feasible assignment and, for a.e. $t \in I$, $G(\mathbf{s}(t), \nu_{\mathbf{s}})$ is $\gtrsim_t$ maximal on $X_t \cap \{G(s, \nu_{\mathbf{s}}) : s \in S\}$.

A second, important axiom is:

Continuity Axiom. The function G given by the Anonymity Axiom is jointly continuous when $\mathcal{M}$ is endowed with the topology of the weak convergence [see (Hildenbrand, 1974, p. 48), for a definition].

Under the continuity axiom the uniqueness of G in the continuum case (i.e., $I = [0, 1]$) can be sharpened to: if for all $\nu \in \mathcal{M}$, G and G' are continuous and coincide on supp(ν), then $G = G'$.

A third axiom will be:

Convexity Axiom. The strategy set S is convex.

Given the previous axioms, our key hypothesis is:

Aggregation Axiom. For any $\nu, \nu' \in \mathcal{M}$ if $\int i d\nu = \int i d\nu'$, then $G(\cdot, \nu) = G(\cdot, \nu')$. The symbol i denotes the identity map.

In words: the net trade proposed to a player by the strategic market game does only depend on the message sent by this player and the mean message of all players. Particular as this condition is, we would argue that it reflects well the aggregate nature of the impact of demand and supply in organized markets. To be sure, it is quite possible that an economic trading situation may be most succinctly and naturally described in a way which does not satisfy the Aggregation Axiom. The point is, however, whether or not it can be described in some way for which it does, even if it perhaps involves a much expanded, but still convex, strategy space. For example, a typical competitive problem among firms may involve as individual strategies, price and quality (suppose there are only m possible qualities). The concept of mean quality is then irrelevant to the equilibrium problem. It may not even be defined. So, the Aggregation Axiom is not satisfied. On the other hand, if we view as the set of messages the much larger set of supply (and demand) functions for the m quality goods (with arguments the m vector of prices), then the Aggregation Axiom will hold.

We will postpone a more detailed discussion of the role of the different axioms until after the statement of the efficiency results in Section 10.

If the four axioms of this section are satisfied, it will be simpler to describe the strategic market game Φ by the continuous function $F: S \times S \to R^l$ such that $G(s, \nu) = F(s, \int i d\nu)$.

7. THE CONTINUUM AS A MODEL FOR A LARGE NUMBER OF PARTICIPANTS

Suppose we have a continuum of traders $I = [0, 1]$, an economy $\mathscr{E}: I \to \mathscr{A}$ and a strategic market game Φ, with player set I, which satisfies the Anonymity, Continuity, Convexity, and Aggregation Axioms of the previous section, i.e., the strategic market game can simply be defined by a continuous function $F: S \times S \to R^l$. In which sense does this amount to a model for an economy and strategic market game with a large number of participants? We address this question now. Given the limited nature of the strategic games we deal with (anonymous, aggregable), this is not a difficult problem. Even so we shall not attempt to obtain the most general possible results.

Suppose we are given a sequence of trader sets I_n, $\# I_n < \infty$, economies $\mathscr{E}_n : I_n \to \mathscr{A}$, and strategic market games Φ_n with player set I_n and a common convex strategy space S. We assume that $\# I_n \to \infty$.

As in Hildenbrand (1974), we say that $\mathscr{E}_n \to \mathscr{E}$ if the distribution of characteristics induced by $\mathscr{E}_n$ converges weakly to the distribution of characteristics of $\mathscr{E}$.

For the sequence of strategic market games we assume: (i) there is a function $H : S \times \hat{S} \to R^l$, where $\hat{S}$ is the cone spanned by S, such that for all n, plays $\mathbf{s} : I_n \to S$ and $t \in I_n$, $\Phi_n(\mathbf{s})(t) = H(\mathbf{s}(t), \sum_{t \in I_n} \mathbf{s}(t))$ and (ii) the function H is homogeneous of degree zero in its second arguments, i.e., $H(s, \lambda b) = H(s, b)$ for all $\lambda > 0$. This could be called the Homogeneity Axiom.

We say that $\mathscr{E}_n$, Φ_n *converges to* $\mathscr{E}$, F if (i) $\mathscr{E}_n \to \mathscr{E}$ and

(ii) $F = H \mid S \times S$. Since H, F remain fixed, we identify $(\mathscr{E}_n, \Phi_n)$ with $\mathscr{E}_n$.

A comment to this definition may be useful. The requirement that H be common to all terms of the sequence is in order since we want to formalize the idea that the number of participants in some *fixed* market institutions increases. For the same reason, i.e., the market institutions do not depend on the number of participants, the function H is given in an unnormalized way. Thus the homogeneity axiom emerges as fundamental since it is this axiom which allows us to replace sums by averages and therefore to define a meaningful continuum model embodying the notion of the negligibility of individual participants. The sequences generated as in Examples 1–3 satisfy the axiom.

If assertions about CN equilibria in the continuum are to be relevant, then it should at least be true that the "limits" of CN equilibria for $\mathscr{E}$ are CN equilibria for the limit $\mathscr{E}$. But what is a limit of CN equilibria? We borrow the notion of equilibrium distribution introduced by Hart, Hildenbrand, and Kohlberg (1974). Given a (Borel) distribution μ on $\mathscr{A} \times S$, let $\mu_{\mathscr{A}}$, μ_S be the respective marginals. We say that μ is a CN distribution for $\mathscr{E}_n$ (resp. $\mathscr{E}$) if (i) $\mu_{\mathscr{A}}$ is the characteristic distribution of $\mathscr{E}_n$ (resp. $\mathscr{E}$) and (ii) for all $(X, \succsim, s) \in \operatorname{supp}(\mu)$, $F(s, \int i\mu_S)$ is $\succsim$-maximal on $X \cap \{F(s', \int id\mu'_S)$: all $s' \in S\}$, where i is the identity map on S and μ' is the distribution obtained by transferring a total mass of $1/n$ from $(X, \succsim, s)$ to $(X, \succsim, s')$ (resp. $\mu = \mu'$). Note that the concept of CN distribution captures everything essential in the definition of CN equilibria in the sense that if two CN equilibria have the same distribution, then they are identical up to a relabeling of the traders' names.

Then we have:

PROPOSITION 1. *Assume that for all n and $t \in I_n$, $X_t = R^l$. Let μ_n be a CN distribution for $\mathscr{E}_n$ and $\mu_n \to \mu$ weakly. Then μ is a CN distribution for $\mathscr{E}$.*

Proof. If $\mu_n \to \mu$, then $\mu_{n\mathscr{A}} \to \mu_{\mathscr{A}}$ and $\mu_{nS} \to \mu_S$. So the first condition in the definition of CN distribution is satisfied for μ. Now let $(\succsim, s) \in \operatorname{supp}(\mu)$ (we can forget about X as $X = R^l$). Take $(\succsim_n, s_n) \to (\succsim, s)$, $(\succsim_n, s_n) \in$ supp. (μ_n). Call $x = F(s, \int id\mu_S)$ and for arbitrary $s' \in S$ let $y = F(s', \int id\mu_S)$. By the continuity of F, $x_n \to x$ and $y_n \to y$, where $x_n = F(s_n, \int id\mu_{nS})$ and

$y_n = F(s', \int i\mu'_{ns})$. Hence $x \gtrsim y$ as $x_n \gtrsim_n y_n$ for all n. So, the second condition is also satisfied. ▌

Proposition 1 is a closed graph property for the CN equilibrium correspondence (defined from distribution on $\mathscr{A}$ to distribution on $\mathscr{A} \times S$). It is well known that in order to prove results of this sort it is essential that constraint sets correspondences be continuous. To guarantee this we take $X_t = R^l$ in Proposition 1. Of course, more general conditions could be given.

A stronger result than Proposition 1 would be the upper hemicontinuity of the equilibrium correspondence. Only then can we assert that the equilibria of large economies are near the equilibria of the continuum. Upper hemicontinuity obtains under the hypothesis of Proposition 1 and, for example, the very strong assumption that S be compact (this is trivial to prove). If upper hemicontinuity fails (as it may well be the case in the context, for example, of an Example 3), then the behavior of the equilibria in arbitrarily large economies may differ markedly from the continuum situation. See Thompson (1978) for examples where precisely this happens. This is an important point which does surely deserve closer investigation.

8. Proper and Full Cournot–Nash Equilibria

Our aim is to establish efficiency properties for CN equilibria. It is clear that, informally speaking, a necessary condition for efficiency is that individual net trades be responsive enough to individual actions. Indeed, if all markets are closed, every play is a CN equilibrium. It will be useful to distinguish two types of not totally unrelated local rigidities. The first would arise if in some markets there were quantitative limits on trades and some traders had reached them. The second would arise if some markets were effectively closed at the CN equilibrium. This motivates the following definitions.

Let the agents set I, economy $\mathscr{E}$ and strategic market game Φ be given. Suppose that $\mathbf{s}$ is a CN equilibrium. For every t let $B_t = \{\Phi(\mathbf{s}')(t) : \mathbf{s}'(t') = \mathbf{s}(t') \text{ for all } t' \neq t\}$.

We say that the CN equilibrium $\mathbf{s}$ is *proper* if for some $n \leqslant l$ and a.e. $t \in I$, a neighborhood of $\mathbf{x}_\mathbf{s}(t)$ on B_t is homeomorphic to R^n. If we can take $n \geqslant l - 1$, we say that the equilibrium is *full.*

Note that the notion of proper equilibrium is compatible with traders being at the boundary of their private trading sets.

We shall see in Section 11 that it is quite possible, and even typical, that for the same economy a strategic market game exhibit both full and nonfull CN equilibria. Improper equilibria would appear to be more pathological in "flexible prices" models, but they would be typical in rationing models.

9. On the Efficiency of the CN Net Trade Allocations in the Finite Number of Traders Case

Let $F: S \times S \to R^l$ represent a strategic market game satisfying the four axioms of Section 7. It is fairly clear from the usual examples that if F acts on an economy with a finite number of traders, there is no reason to expect the CN equilibrium allocations to be efficient. We shall here illustrate this point by proving, under a number of simplifying but reasonable hypotheses, that given F and $n \geqslant 2$, there are economies with n traders exhibiting nonefficient full CN equilibria (which, further, cannot be perturbed away).

Let I be a traders index set with $\#I = n$. Given a play $\mathbf{s}: I \to S$ and $t \in I$, it is convenient to define: $\bar{s}_t = 1/(n-1) \sum_{t' \neq t} \mathbf{s}(t')$. Let also $H: S \times S \to R^l$ be given by $H(s, s') = F(s, (1/n)((n-1)s' + s))$. Then $B_{s'} = H(S, s')$ is the public budget of any $t \in I$ at any play $\mathbf{s}$ with $s' = 1/(n-1) \sum_{t' \in I\, t' \neq t} \mathbf{s}(t')$.

We shall make the following hypothesis:

(I) S is open and H is C^1.

(II) For all s, s', rank $D_s H(s, s') = l - 1$ and Range $D_s H(s, s') \cap R_+^{\,l} = \{0\}$.

(III) For some s and s' rank $D_{s,s'} H(s, s') = l$.

These are very reasonable hypotheses. Assumption (II) asserts that it is not possible by an infinitesimal displacement to get something from nothing, but that, subject to this fact, rank $D_s H(s, s')$ is maximal, i.e., $l - 1$. This is a clear requirement if there is to be any hope that the allocations induced by the CN equilibria of the strategic market game be efficient. Assumption (III) asserts that at some combination of actions the strategic market game allows, if cooperation prevails, for (local) unrestricted variation of an individual allocation (i.e., cooperation makes it possible for some individual to get something from nothing). It is satisfied if, for example, $H(S \times S)$ contains an open set.

Proposition 2. *If H satisfies* (I), (II), *and* (III), *there is an economy* $\mathscr{E}: I \to \mathscr{A}$ *and a play* $\mathbf{s}: I \to S$ *such that*:

(i) $\mathbf{s}$ *is a CN equilibrium,*

(ii) *the net trade allocation of* $\mathbf{s}$ *is not efficient.*

Proof. Let s and s' be as in Assumption (III). With $I = \{1, \ldots, n\}$, we consider the play $\mathbf{s}(1) = s$, $\mathbf{s}(j) = s'$ for $2 \leqslant j \leqslant n$. We let $s_i = \mathbf{s}(i)$, $s_i = \bar{s}_i$. By (II), for every i there is a unique $p_i \geqslant 0$, $\|p\| = 1$, such that $p_i \cdot D_s H(s_i, s_i') = 0$. Let $\mathbf{x}: I \to R^l$ be the net trade induced by $\mathbf{s}$. If $\mathbf{x}$ is to be efficient for any economy which satisfies our hypothesis and makes **s** into a CN equilibrium, then p_i must be the same for all i. Indeed, for every i, we can take $X_i = R^l$ and choose monotone preferences $\gtrsim_i$ representable by a C^1 utility function and such that **x**(i) is $\gtrsim_i$ maximal on $B_{s'_i}$. By Assumption (II) this can be done (it is easy to verify that at any s' and $x \in B_{s'}$, $B_{s'} \cap (\{x\} + R_+^l) = \{x\}$. See Fig. 1), and it makes **s** into a CN equilibrium. By

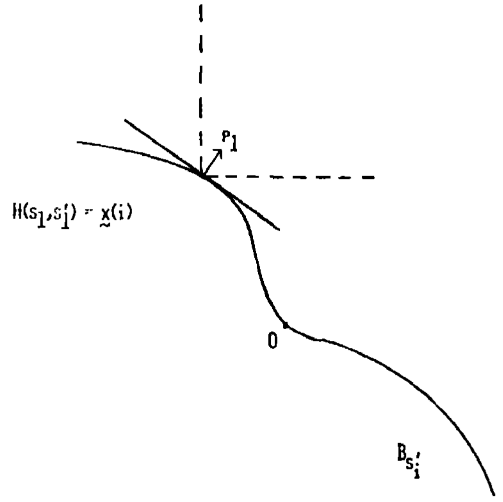

FIGURE 1

the usual efficiency conditions, if **x** is efficient, then the normalized gradients of utility functions, which equal the p_i's must be the same.

So, let $p_1 = \cdots = p_n = p$. Differentiating $\sum_{i=1}^n H(s_i, s_i')$ with respect to s_j, we get $D_s H(s_j, s_j') + \sum_{i \neq j} D_{s'} H(s_i, s_i') = 0$ for all j. Therefore, $\sum_{i \neq j} p \cdot D_{s'} H(s_i, s_i') = 0$ for all j. As $p \cdot D_{s'} H(s_j, s_j') - p \cdot D_{s'} H(s_{j'}, s_{j'}') = \sum_{i \neq j} p \cdot D_{s'} H(s_i, s_i') - \sum_{i \neq j'} p \cdot D_{s'} H(s_i, s_i') = 0$ for all j, j', we conclude that $p \cdot D_{s'} H(s_i, s_i') = 0$ for all i. In particular, $p \cdot D_{s'} H(s_1, s_1') = p \cdot D_{s'} H(s, s') = 0$. But then $p \cdot D_{s,s'} H(s, s') = 0$, which implies rank $D_{s,s'} H(s, s') \leqslant l - 1$ and contradicts Assumption (III). Therefore not all p_i's are the same, and we are done. ▌

Remark 7. Proposition 2 is a weak result as it asserts only the existence of an economy exhibiting a nonefficient CN equilibrium. We would surmise that the general picture is as follows:

Postulate Hypothesis (III) for all s, s'. Then restricting ourselves to the class of smooth economies, it would be the case that, but for some exceptional economies (i.e., "generically"), the set of efficient net allocations is a codimension one smooth manifold in the space of net trades allocations and the set of CN allocations induced by the strategic market game is also a smooth manifold. Further both manifolds intersect transversally. This means that if a particular noncooperative theory of market allocation is deterministic in spirit (i.e., generically the set of CN equilibria is finite), then but for coincidental cases, no CN allocation will be efficient. This we have verified for the case dim $S = l - 1$ [see Dubey (1979a), where a particular model is thoroughly analyzed]. More generally, the efficient CN allocations will form a lower-dimensional submanifold of the CN manifold. The investigation of this topic, which does require the methods of differential topology, we have not pursued at much length although we think its clarification is of importance. For a study of the generic properties of CN equilibria in a purely game theoretic context, see Dubey (1978).

Remark 8. Without the Continuity Axiom the conclusions of Proposition 2 may well fail [see, for example, Dubey (1979b) and Schmeidler (1978)] The same is true for the Aggregation Axiom [see Hurwicz (1979) model, where the CN equilibria are efficient, but outcomes depend on the squares of the messages].

10. On the Efficiency of the CN Net Trade Allocations in the Continuum of Traders Case

We now take up an examination of efficiency of CN allocations in the continuum of traders-players situation. We will be led to qualitatively different results from those in the finite case.

From now on $I = [0, 1]$, and we have given a strategic market game satisfying the Anonymity, Continuity, Convexity, and Aggregation Axioms. So, the mechanism is summarized by function $F: S \times S \rightarrow R^l$. After stating our main results, we will discuss the role of each of the axioms.

The first obvious consequence of being in the continuum of agents is that all traders face the same public budget set, as no single trader can affect the average action and anonymity holds. For every $s' \in S$ let $B_{s'} = F(S, s')$. Then given a play $\mathbf{s}: I \rightarrow S$, the common public budget set is $B_{\bar{\mathbf{s}}}$ where $\bar{\mathbf{s}} = \int \mathbf{s}$. Of course, a play $\mathbf{s}$ is a CN equilibrium if for a.e. $t \in I$ $F(\mathbf{s}(t), \bar{\mathbf{s}})$ is $\succsim_t$ maximal on $B_{\bar{\mathbf{s}}} \cap X_t$.

The second and fairly obvious consequence of being in the continuum of agents is that for every $s' \in S$, $B_{s'}$ is in fact a convex set. Aggregation is the crucial axiom here.

Proposition 3. *Under the Anonymity, Continuity, Convexity, and Aggregation Axioms, $B_{s'}$ is a convex set for every s'.*

Proof. Let $s' = c$ be given. By the continuity axiom, we can assume without loss of generality that c belongs to the relative interior of S. We proceed to show that $F(\cdot, c): S \rightarrow R^l$ is then linear. Let $f_c: S \rightarrow R$ be any of the coordinate functions of $F(\cdot, c)$. We know that if $\mathbf{s}$ is a play with $\int \mathbf{s} = c$, then $\int (f_c \circ \mathbf{s})\, dt = 0$. If f_c, which is continuous, is not linear, then an easy argument shows the existence of $s_1, s_2 \in S$ and $0 \leqslant \alpha \leqslant 1$ such that

$$\alpha f_c(s_1) + (1-\alpha) f_c(s_2) \neq f_c(\alpha s_1 + (1-\alpha)\, s_2).$$

Let $\mathbf{c} : I \to S$ be a play with $\int \mathbf{c} = c$ and such $\lambda(\mathbf{c}^{-1}(s_1)) > 0$, $\lambda(\mathbf{c}^{-1}(s_2)) > 0$. Such a $\mathbf{c}$ exists because c belongs to the relative interior of S. Now let $\mathbf{c}'$ be as $\mathbf{c}$ except that for a small $\epsilon > 0$ an amount of mass equal to $\alpha\epsilon$ (resp. $(1-\alpha)\epsilon$) is transferred from $\{s_1\}$ (resp. $\{s_2\}$) to $\alpha s_1 + (1-\alpha) s_2$. We still have $\int \mathbf{c}' = c$. Therefore, $\int (f_c \circ \mathbf{c})\, dt \neq \int (f_c \circ \mathbf{c}') dt$. But this is impossible because $\int (f_c \circ \mathbf{c}) dt = 0 = \int (f_c \circ \mathbf{c}) dt$.

Hence, $F(\circ, c)$ is linear and since S is convex, $B_c = F(S, c)$ is convex. ∎

For all s' we also have $0 \in B_{s'}$, since the constant play $\mathbf{s}'(t) = s'$ should yield, by anonymity and aggregation, the null set trade. Let $L_{s'}$ be the linear subspace spanned by $B_{s'}$. Then we have:

PROPOSITION 4. *Under the Anonymity, Continuity, Convexity, and Aggregation Axioms, if* $\mathbf{s} : I \to S$ *is a proper CN play, then for a.e.* t. $F(\mathbf{s}(t), s')$ *is* $\succsim_t$ *maximal on* $L_{s'} \cap X_t$, *where* $s' = \int \mathbf{s}$.

Proof. The proof is trivial. Since $\mathbf{s}$ is a CN equilibrium, it is true that for a.e. t $F(\mathbf{s}(t), s')$ is $\succsim_t$-maximal on $B_{s'} \cap X_t$. As the CN equilibrium is proper and $B_{s'}$ is a convex set containing 0, we have that for a.e. $t \in I$, $F(\mathbf{s}(t), s')$ belongs to the relative interior of $B_{s'}$. Now suppose that for some of those t we had $x \in L_{s'} \cap X_t$, $x \succ_t F(\mathbf{s}(t), s')$. As preferences are convex, we could take x in $B_{s'}$, but this is impossible. Hence the proposition is proved. ∎

Proposition 4 could be interpreted as saying that proper CN equilibria yield Walrasian allocations relative to the set of markets which, given the equilibrium strategies, are open. This is particularly clear if $L_{s'}$ belongs to a coordinate subspace. It can well happen (see Section 11) that the set of open markets is different at different CN equilibria.

One could ask about the efficiency properties of proper CN equilibria. Taking a cue from the previous paragraph, one could assert the efficiency of the equilibrium allocations relative to a properly defined subset of net trades allocations. It is not clear, however, if this would amount to much, as the really interesting property would be the efficiency of the CN net trade allocation relative to the net trade allocations which are feasible *within the given strategic market game*. Unfortunately, it is easy to convince oneself that proper CN equilibria are not necessarily efficient in this sense.

So, we are left with the following corollary of Proposition 5, which, being the main result of this paper, we call a theorem:

THEOREM 5. *Under the Anonymity, Continuity, Convexity, and Aggregation Axioms, every full* CN *equilibrium allocation is Walrasian, hence efficient.*

Remark 9. Let's restrict ourselves to strategic market games on $I = [0, 1]$ satisfying the Anonymity and Continuity Axioms. It will be convenient and admissible to identify a game with the continuous function $G : S \times \mathcal{M} \to R^l$. As we are in the continuum case, the public budget set is common to all traders. So, if the game G is understood, we let $B_\nu = G(S, \nu)$ for any $\nu \in \mathcal{M}$.

We show now how given any interval $T \subset R$, continuous function $g : T \to R$, and nondegenerate probability measure $\bar{\nu} \in \mathcal{M}$ satisfying $\int i d\bar{\nu} = 0$ and $\int g d\bar{\nu} = 0$, we can construct an anonymous strategic market game G for a two-commodity world with $I = [0, 1]$ and $B_{\bar{\nu}} = \text{Graph } g$.

Indeed, it suffices to define

$$G(s, \nu) = \left(\int |s|\, d\nu \Big/ \int |s|\, d\bar{\nu}\right)$$
$$\times \left(s - \frac{|s|}{\int |s|\, d\nu} \int s\, d\nu,\ g(s) - \frac{|s|}{\int |s|\, d\nu} \int g\, d\nu\right).$$

If $\int |s|\, d\nu = 0$ put $G(s, \nu) = 0$. The Continuity and Convexity Axioms are satisfied, but the Aggregation one is not. Also, if $g(0) = 0$, then $0 \in B_\nu$ for all $\nu \in \mathcal{M}$. Obviously, this G does not have any particular economic interpretation, which is not surprising, given the failure of aggregation.

It is therefore possible to have strategic market games satisfying the Anonymity, Continuity and Convexity Axioms and generating CN equilibria as the ones depicted in Figs. 2a, b, c, d. In every case, the economy is formed by two equal weight types with $X = R^2$ and the indicated preferences. The play distribution is in all cases $\nu(\{s_1\}) = \nu(\{s_2\}) = \frac{1}{2}$.

Figure 2a exhibits an improper equilibria. Figure 2b gives a proper, but not full equilibria. Both equilibria are inefficient. It is obvious that both pictures could be sustained by strategic market games satisfying the Aggregation Axiom. Figures 2c and 2d provide examples of full equilibria failing to be efficient on account of the lack of aggregation.

Remark 10. It is important to notice the interplay between the Aggregation axiom and the convexity hypothesis on the strategy space S. It is the latter which gives force to the first. Indeed, it is always formally possible, by expanding S, to satisfy approximately the aggregation axiom. But the expanded S may fail to be convex. For example, if $S = [0, 1]$ and the outcome net trades depend on plays through the mean and the variance of the message distribution, we could extend the message space to $S' = \{(s, s^2) : s \in [0, 1]\} \subset [0, 1]^2$. Then, the Aggregation Axiom is satisfied, but the Convexity Axiom is not.

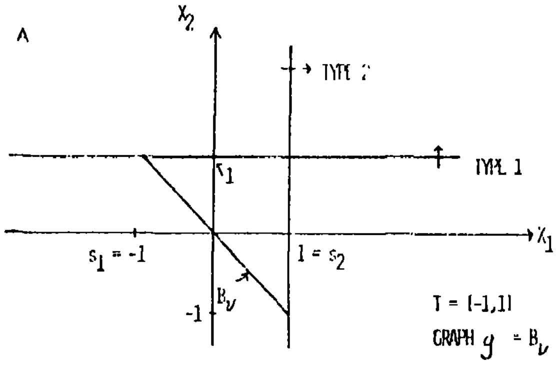

FIGURE 2a

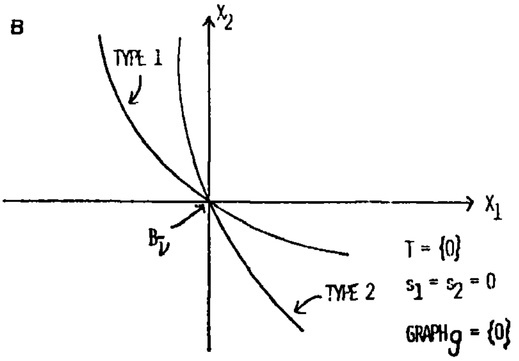

FIGURE 2b

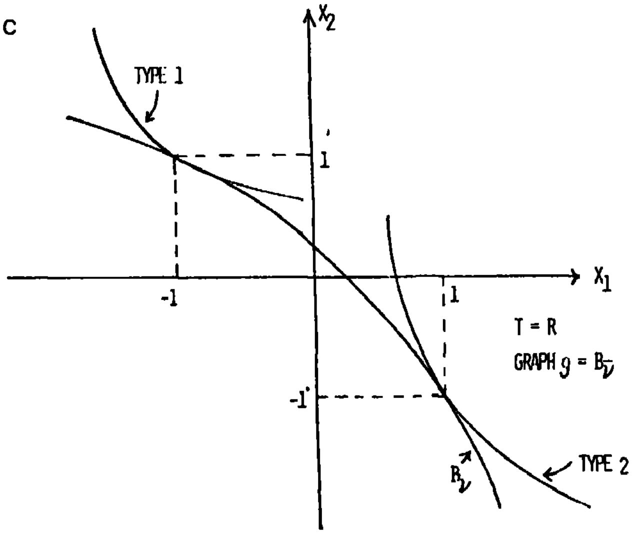

FIGURE 2c

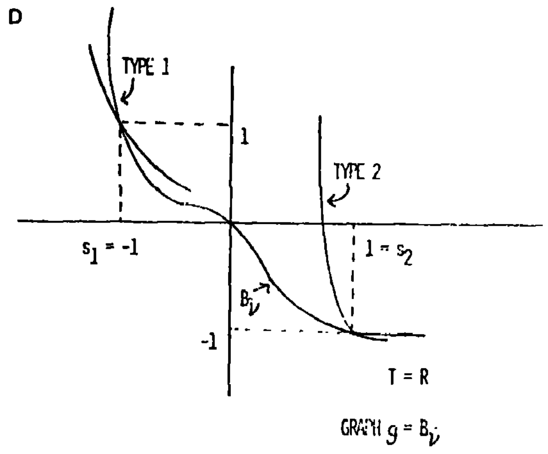

FIGURE 2d

Remark 11. Under the hypothesis made, the Theorem concludes that CN allocations are not just efficient, but in fact Walrasian. Essentially, this is a consequence of the Anonymity Axiom. Under quite general circumstances, it is the case that if the allocation derived from an anonymous continuum strategic market game is efficient then it is Walrasian. See Hammond (1979) for this point. Ch. Kahn (1979) has succeeded in formulating and proving an analog of the Theorem without the Anonymity Axiom. He gets that CN allocations are still efficient but not necessarily Walrasian. A common price system and linear budget sets still obtain, but lump-sum transfers at equilibrium are possible.

Remark 12. The Continuity Axiom is troublesome. It is basic to our approach since, as we saw in Section 7, we need something like it if the continuum model is to be of any use as a representation of a finite but large situation. But there is no denying that there are sensible economic models for which this axiom poses problems. As an instance, the outcome rule may be naturally multivalued (as in Example 2; more generally, this will tend to be the case if the outcome is itself the result of some equilibrium process). Or think of price competition along strict Bertrand lines, or on retaliation strategies, etc. See Green (1979) for a discussion and analysis of the continuity problem in a different (dynamic) context.

At any rate, if continuity fails the properties of equilibria for both the finite and the finite but large models may be very different from the conclusions of this paper. We may have models where equilibria are always efficient, even in the finite case (Bertrand-like models, for example) and models where there is no tendency for the equilibria to become efficient even if the economy is very large [see Green (1979)].

Once we place ourselves in the continuum, the Continuity Axiom plays only a minor role in the proofs of efficiency of a CN play $\mathbf{s}$. It is obvious from the proofs that it could be replaced by either of the hypotheses "$\int \mathbf{s} \in \operatorname{Int} S$" or "$F$ is continuous on $S \times \{\int \mathbf{s}\}$".

Remark 13. It is easy to convince oneself that we cannot dispense with the convexity of preferences. It is this hypothesis which, for each trader, makes a local optimum (and this is all we can prove for a CN equilibrium with our axioms) into a global maximizer in the budget set.

11. SOME EXAMPLES

We shall now examine some specific examples in the direction of the ones introduced by Shubik (1973). See also Shapley (1976) and Shapley and Shubik (1977).

The space of traders is I, finite or infinite.

As a first step, let there be only two commodities, $h = 1, 2$.

The simplest model calls for strategies which are nothing more than the single act of sending messages promising delivery of a certain amount of goods to either side of a single market, i.e., $S = R_+^2$ where by $(s_1, s_2) \in S$ we understand a commitment to supply s_h, $h = 1, 2$, of good h.

Let $\mathbf{s} : I \to S$ be a play. Which rules should we specify for the disbursal of $\bar{s}_1 = \int \mathbf{s}_1$, $\bar{s}_2 = \int \mathbf{s}_2$? Or, in other words, which conditions shall we impose on the outcome $\mathbf{x} : I \to R^2$ of the play? Consider the following three (postulated to be valid for every possible play *and* players space):

(i) Suppose we create a fictitious player space $[0, 2]$—endowed with Lebesgue measure—and a play $\hat{\mathbf{s}} : [0, 2] \to S$ defined by letting $\hat{\mathbf{s}}(r) = (\mathbf{s}_1(r), 0)$ if $r \leqslant 1$ and $\hat{\mathbf{s}}(r) = (0, \mathbf{s}_2(r - 1))$ if $1 \leqslant r \leqslant 2$. Let $\hat{\mathbf{x}}$ be the outcome of this play. Then $\mathbf{x}(t) = \hat{\mathbf{x}}(t) + \hat{\mathbf{x}}(t + 1)$. In words, there is complete separation of buying and selling and we can, without loss of generality, assume that $I = I_1 \cup I_2$ where

$$I_1 = \{t \in I : \mathbf{s}_2(t) = 0\}, I_2 = \{t \in I : \mathbf{s}_1(t) = 0\}.$$

(ii) If $t \in I_h$, $h = 1, 2$, then $\mathbf{x}_h(t) = -\mathbf{s}_h(t)$, i.e., the supply of a commodity offered to the market is always taken.

(iii) With the convention that 0/0 qualifies as any number, the ratio $\mathbf{x}_1(t)/\mathbf{x}_2(t)$ (resp. $\mathbf{x}_2(t)/\mathbf{x}_1(t)$) is independent of $t \in I_2$ (resp. $t \in I_1$). This can be interpreted as a kind of anonymity cum arbitrage-freeness.

Then, of course, the outcome is

$$\mathbf{x}(t) = \left(\frac{\bar{s}_1}{\bar{s}_2} \mathbf{s}_2(t) - \mathbf{s}_1(t), \frac{\bar{s}_2}{\bar{s}_1} \mathbf{s}_1(t) - \mathbf{s}_2(t)\right).$$

If $\bar{s}_2$ 0 (resp. $\bar{s}_1 = 0$) we put the first (resp. the second) entry equal to $-\mathbf{s}_1(t)$ (resp. $-\mathbf{s}_2(t)$). This is nothing but Example 1 in Section 4 (with an inessential difference: to enforce that $\mathbf{x}$ be always an allocation, we required there that $\mathbf{x} = 0$ whenever $\bar{s}_1 = 0$ or $\bar{s}_2 = 0$ while here if, say $\bar{s}_1 \neq 0$ and $\bar{s}_2$ 0, $\mathbf{x}$ will not be an allocation, since $\bar{s}_1$ is not redistributed to anyone). Note that the Anonymity, Convexity, and Aggregation Axioms are satisfied while Continuity may fail only at the point $\bar{s}_1 = \bar{s}_2 = 0$. It is Condition (iii) which yields anonymity and aggregation. Alternatively, we could have assumed the latter and derive the former. For example if there was a function $F : S \times S \to R$ such that $\mathbf{x}(t) = F(\mathbf{s}(t), \bar{s})$ for every possible play and players space, then, by Theorem 5, F will have to be linear on the first argument and we recover Condition (iii).

A multicommodity extension of the previous example is readily available. Let there be l commodities. We may define a *simple market* as one in which quantities of a single commodity i are exchanged for another single commodity j. A *market structure* composed of simple markets can be represented by a graph where the points represent commodities and the edges simple markets. See Fig. 3. We could call the edges of the graph *open markets* (and the nonexisting edges *closed markets*). A market structure is *complete* if its graph is connected. A commodity is a *money* if in the market structure there is an edge to every other commodity. So, every market structure with a money is complete. In Fig. 3, Structure 3a is not complete, while the rest are. Structures 3c, 3d, 3e have, respectively, 1, 2, and 4 money commodities.

Given a market structure M we could assume, to focus on the simplest case, that every market functions as the simple market previously discussed. Then the strategy space is $S = R_+^{2M}$ where for every $k \in M$ s_{1k}, s_{2k} represent the quantities offered to the market k of the two commodities transacted in that market. For a given play $\mathbf{s}$ we could let $M_{\mathbf{s}} \subset M$ be the active markets

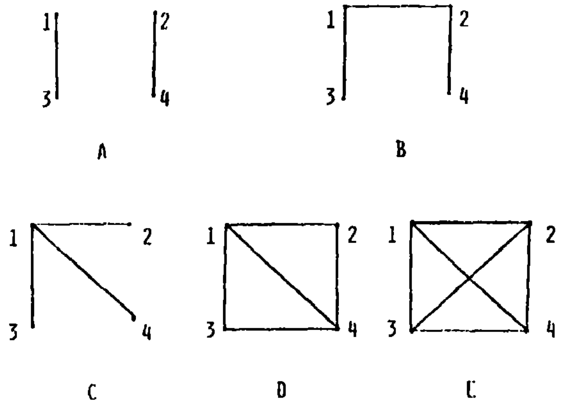

FIGURE 3

at $\mathbf{s}$, i.e., $M_{\mathbf{s}} = \{k \in M\colon \int \mathbf{s}_{1k}, \int \mathbf{s}_{2k} > 0\}$ and $S_{\mathbf{s}} = R_{+}^{2M_{\mathbf{s}}}$ the "active" strategy set.

Suppose $\mathbf{s}$ is a CN play. If we forget about nonactive strategies, i.e., imagine that our strategy set is $S_{\mathbf{s}}$ then each one of our axioms is satisfied (see end of Remark 12). So, the efficiency Theorem 5 applies if the equilibrium is full (note that the equilibrium is always proper). Now, the equilibrium will be full if and only if the market structure formed by the *active* markets $M_{\mathbf{s}}$ is complete. For this type of model an inactive market is no better than a closed one.

The reader familiar with Shubik's original model (1973) and with the multicommodity extensions due to Shapley (1976), Jaynes, Okuno and Schmeidler (1976), Postlewaite and Schmeidler (1978), Dubey and Shapley (1977), may wonder why we get efficiency of equilibria in the continuum case merely out of the connectedness of active markets when this was not at all the case in the previous references. The reason lies in the feasibility constraint we are presently using. We completely abstract from transactions problems and for equilibrium we only require the *net* trade to be individually feasible. So, it is possible to use the receipts in one market to buy in another market. If we visualize our simple markets as taking place simultaneously, this amounts to the absence of a liquidity constraint, i.e., to the presence of a perfect credit system.

Another observation is that the previous model will typically have multiplicity of equilibria. In fact if an existence theorem is available (it is, see references of the previous paragraph), we will have an equilibrium for every a priori specification of inactive markets.

The market strategies so far considered are rather extreme. They are commitments to supply a given amount of commodity i to the market in exchange for any quantity of commodity j. More generally, we could envision a supply message (interpretable, perhaps, as a decision rule) to a simple market as being a function $s_i(x_j)$ to be read as a promise to deliver $s_i(x_j)$ units of commodity i in exchange for x_j units of commodity j. In line with Conditions (i), (ii), and (iii) for the simpler case, we could require of market clearing and disbursal rules that at the transactions outcome of a play we had that for all i and j, $s_i(x_j)/x_j$ be independent of the particular supplier. Of course, this is nothing but market clearing at a price system, and if we further constrain every s^i to be nondecreasing and concave, we put ourselves in the anonymous and aggregate situation of Example 3 in Section 4. Indeed, it suffices to redefine a selling strategy as a 0-homogeneous function $\hat{s}_i(p_i, p_j)$ promising to deliver $\hat{s}_i(p_i, p_j)$ units of commodity i in exchange for $1/p_j\ p_i\hat{s}_i(p_i, p_j)$ units of commodity j. Clearly, every s_i strategy induces a $\hat{s}_i$ strategy and equilibrium prices are determined by the condition

$$p_j \int \hat{s}_j(p_i, p_j) = p_i \int \hat{s}_i(p_i, p_j). \qquad \text{See Fig. 4.}$$

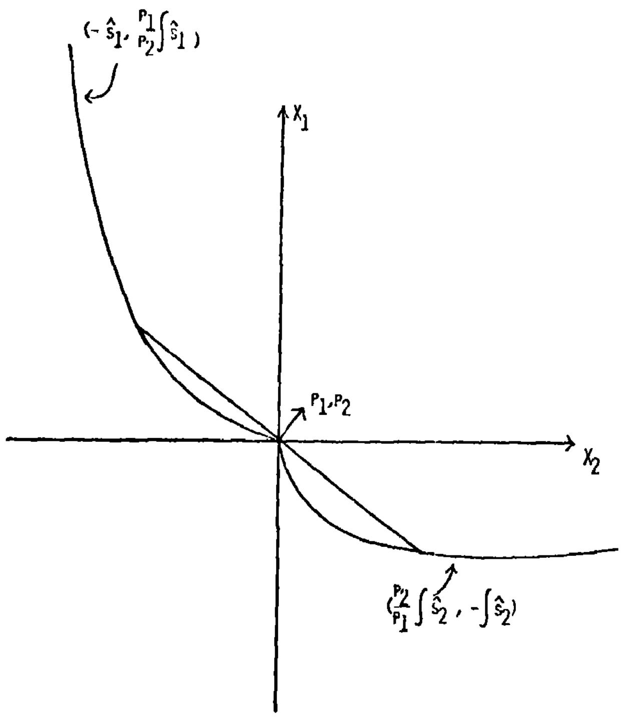

FIGURE 4

The example at the beginning of this section is a limit (the Cournot case) of this latter model. It corresponds to the case $s_i(x^j) = c$. The other limit, $s^i(x_j)$ linear homogeneous, is the Bertrand case. Since outcomes are in this case highly discontinuous on strategies, it is not easily amenable to our analysis.

12. ON THE CONCEPT OF STRICT COURNOT–NASH EQUILIBRIUM IN CONTINUUM ECONOMIES

Suppose that for $I = [0, 1]$ we have given an economy $\mathscr{E} : I \to \mathscr{A}$ and an anonymous trading game $G : S \times \mathscr{M} \to R^l$. From a noncooperative game theoretic standpoint, a CN play $\mathbf{s} : I \to S$ will be rather fragile *if* there were traders (strictly speaking, a nonnull set of traders) that could obtain a better net trade by pretending to be n different players (n finite but otherwise arbitrary). Indeed, in an anonymous mass market it may not be possible to prevent an individual trader from entering it several times or simply, using proxies. This observation prompts the following

DEFINITION. A play $\mathbf{s} : I \to S$ is a *strict Cournot–Nash equilibrium* if $x_{\mathbf{s}}$ is feasible and for a.e. $t \in [0, 1]$ $G(\mathbf{s}(t), \nu_{\mathbf{s}})$ is $\gtrsim_t$ maximal on $X_t \cap \bigcup_{n=1}^{\infty} nB_{\nu_{\mathbf{s}}}$.

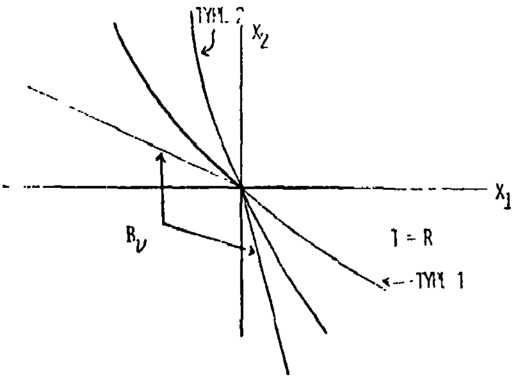

FIGURE 5

From Proposition 4 it is clear that

PROPOSITION 6. *If the Anonymity, Continuity, Convexity, and Aggregation Axioms are satisfied, then every proper CN equilibrium is strict.*

Without the Aggregation Axiom a proper CN equilibrium needs not to be strict. For example if at a CN distribution we have that $B_\nu \cap R^l_{++} \neq \varnothing$ (as in the example of Fig. 2c) then by the monotonicity of preferences, the distribution cannot be a strict CN.

Without the Aggregation Axiom, is every strict Cournot–Nash equilibrium Walrasian? Not necessarily. In Fig. 5 we can see an example where the CN equilibria are proper and full but not Walrasian (the two types indicated have the same weight). By using the technique of Remark 9 we can without difficulty construct economies and strategic market games supporting the figure. In the figure, B_ν fails to be smooth. An obvious result is:

PROPOSITION 7. *Suppose that $X_t = R^l$ for a.e. t and that the Anonymity Axiom holds. Let* $\mathbf{s}$ *be a strict CN equilibrium. If* $0 \in B_{\nu_\mathbf{s}}$ *and* $B_{\nu_\mathbf{s}}$ *is a* $(l-1)$ C^1 *manifold, then the equilibrium is Walrasian.*

Proof. Let T be the $(l-1)$ tangent plane to $B_{\nu_\mathbf{s}}$ at 0. It is easy to verify that for any $z \in T$, $\bigcup_{n=1}^{\infty} nB_{\nu_\mathbf{s}}$ contains net trades arbitrarily close to z. Henceforth by the continuity of preferences, for a.e. $t \in [0, 1]$, $\mathbf{x}_\mathbf{s}(t) \succsim_t z$ for any $z \in T$. Because of monotonicity of preferences, $T \cap R^l_{++} = \varnothing$. Hence we can pick $p \geqslant 0$ such that $T = \{z \in R : p \cdot z = 0\}$. With p as the candidate Walrasian price, the proof proceeds from this point on as in Proposition 4. ▌

As the example of Fig. 2d shows, Proposition 6 holds for strict CN equilibria but not for ordinary ones.

Remark 14. Given an anonymous strategic market game $G : S \times \mathcal{M} \to R^l$ it is possible to build a new one $G^* : S^* \times \mathcal{M}^* \to R^l$ in such a manner that the concept of ordinary and strict CN equilibria coincide in G^* and the ordinary equilibria of G^* correspond to the strict equilibria of G.

One would proceed as follows. Let S be a complete, separable subset of a Banach space. Denote by S^* the set of finite, nonnegative integer valued measures on S. Endow S^* with the weak-star topology and corresponding Borel σ-field. Let $\mathscr{M}^*$ be the (Borel) probability measures on S^* with bounded mean. A play is now an integrable $\mathbf{s}^* : I \dashrightarrow S^*$. The integral is defined by $(\int \mathbf{s}^*)(V) = \int \mathbf{s}^*(t)(V)$ for every Borel set $V \subset S$. It is a nonnegative finite measure on S [see (Mas-Colell, 1975, Sect. 3.1) for the mathematical technicalities of this remark]. Given $\mathbf{s}^*$ we let $\nu^*_{\mathbf{s}^*} \in \mathscr{M}^*$ be the Borel probability measure induced by $\mathbf{s}^*$ on S^*. An element $\nu^* \in \mathscr{M}^*$ induces a distribution ν on A by letting

$$\nu(V) = \frac{1}{\int s^*(A)\, d\nu^*} \int s^*(V)\, d\nu^*.$$

Given a play $\mathbf{s}^*$, one let $\nu_{\mathbf{s}^*}$ be the measure in S corresponding to $\nu^*_{\mathbf{s}^*}$. Clearly,

$$\nu_{\mathbf{s}^*} = \frac{1}{(\int \mathbf{s}^*)(A)} \left(\int \mathbf{s}\right).$$

Finally, one defines

$$G^*(s^*, \nu^*) = \int G(s, \nu)\, ds^*.$$

Note that in this new game the Aggregation Axiom is satisfied but S^* is not convex. Also, a proper strict CN equilibrium of G is an ordinary, but not necessarily proper, CN equilibrium of G^*. (See Fig. 5) where B_{ν^*} is the convex hull of B_ν).

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JOURNAL OF ECONOMIC THEORY 22, 363–376 (1980)

Nash Equilibria of Market Games: Finiteness and Inefficiency*

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Received August 1, 1979

1. INTRODUCTION

In this paper we make good the promise [6, Sect. 9] that, under appropriate conditions, the Nash Equilibria of finite-player strategic market games are generically inefficient. Our conditions are: (a) the dimension of each trader's strategy set is at most $l - 1$, where l is the number of commodities, (b) the mapping from strategies to net trades is sufficiently smooth, (c) so are the traders' preferences. For concreteness we work out a specific model—the so-called "sell-all" model, explored in [9]. But the same proof can easily be adapted to the general case.

Condition (c) is quite standard and not particularly restrictive (e.g., see [7]); Condition (b) is necessary and without it our conclusion may well fail. Indeed see [8, 5] for models with price-cutting strategies where the Nash and Walras Equilibria coincide. For a discussion of (c), see again Section 9 of [6].

Needless to say we take our cue from the analogous finiteness results obtained for the Walras Equilibria of markets by Debreu [2], Smale [11], and others. For a similar analysis of strategic games in general, see [4].

For any integer r, let $I_r = \{1, 2, ..., r\}$, and let Ω^r be the nonnegative orthant of the Euclidean space of dimension r. Let I_n be the set of traders and I_{k+1} the set of commodities in which they trade. Each trader $i \in I_n$ is characterized by an initial endowment $a^i \in \Omega^{k+1}$ and a utility function $u^i : \Omega^{k+1} \to R$. (Here a_j^i is the quantity of commodity j held by trader i.) We assume that $\sum_{i=1}^n a^i > 0$ and that $a_{k+1}^i > 0$ for all[1] $i \in I_n$.

To recast the market as a game in strategic form, we single out the $(k + 1)$st commodity as a money. There are k trading posts, one for each of the other commodities. Traders are required to put up all of their first k commodities

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* The research reported in this paper was partially supported by grant No. SOC77-27435 from the National Science Foundation, and by Contract N00014-77-C0518 issued by the Office of Naval Research under Contract Authority NR-047-006.

[1] If $a_{k+1}^i = 0$, then trader i is a "strategic dummy" in our game and can be ignored.

0022-0531/80/020363-14$02.00/0

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for sale in these trading posts, and use their endowment of commodity money for bidding on them. The strategy set S^i of trader i consists of bids on the k trading posts, but he is constrained to bid within a^i_{k+1}:

$$S^i = \left\{ b^i \in \Omega^k : \sum_{j \in I_k} b_j{}^i \leqslant a^i_{k+1} \right\}.$$

Given a choice of strategies $b = (b^1, ..., b^n)$, $b^i \in S^i$, prices $p(b) \in \Omega^k$ are formed in the trading posts and the markets cleared, with the final bundle x^i(b) accruing to i, according to the rules:

$$p_j(b) = \frac{\bar{b}_j}{\bar{a}_j} \qquad \left(\text{where } \bar{b}_j = \sum_{i \in I_n} b_j{}^i, \text{ etc.}\right)$$

$$x_j{}^i(b) = \frac{b_j{}^i}{p_j(b)} \qquad \text{if} \quad p_j(b) > 0$$

$$= 0 \qquad \text{if} \quad p_j(b) = 0$$

for $j = 1, ..., k$; and

$$x^i_{k+1}(b) = a^i_{k+1} - \sum_{j=1}^{k} b_j{}^i + \sum_{j=1}^{k} p_j(b)\, a_j{}^i.$$

(We may interpret $x_j{}^i(b) = 0$ to be a confiscation of goods in the absence of any bid. Consider a $\hat{b} = (\hat{b}^1, ..., \hat{b}^n) \in S = S' \times \cdots \times S^n$. $\hat{b}$ is called a *Nash Equilibrium* (N.E.) of this game if, for all $i \in I_n$,

$$u^i(x^i(\hat{b})) \geqslant u^i(x^i(\hat{b} \mid b^i)), \qquad b^i \in S^i,$$

where $(\hat{b} \mid b^i)$ is the same as $\hat{b}$ but with $\hat{b}^i$ replaced by b^i; it is called an *efficient* point of the game if there is no $b \in S$ such that

$$u^i(x^i(b)) \geqslant u^i(x^i(\hat{b})) \qquad \text{for all} \quad i \in I_n,$$

$$u^l(x^l(b)) > u^l(x^l(\hat{b})) \qquad \text{for some} \quad l \in I_n.$$

2. FINITENESS AND INEFFICIENCY

In our analysis we will, for convenience (see Remark 3, however), hold the traders' endowments fixed and vary their utilities only. Put $Q = \{x \in \Omega^{k+1} : x_j \leqslant \bar{a}_j\}$. Let U denote the linear space of all C^2 functions[2] from Q to R, endowed with the C^2-norm.[3] Thus we may think of a market game as given by a point $u = (u^1,\dots, u^n) \in (U)^n$. Denote the set of its N.E. by $\eta(u)$, and the set of its efficient points by $\epsilon(u)$.

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[2] That is, those which can be extended to a C^2 function on R^{k+1}.

[3] That is, $\| u \| = \sup\{\| u(x)\|, \| Du(x)\|, \| D^2u(x)\| : x \in Q\}$, where $\| \ \|$ denotes the maximum norm.

We will focus our attention on certain open sets of $(U)^n$. Let e be the unit vector in Ω^{k+1}, i.e., $e_j = 1$ for each j. Take two positive numbers σ and σ', $\sigma < \sigma'$. Define $U(\sigma, \sigma') = \{u \in U : \sigma e < Du < \sigma' e\}$. The manifold of games which we will consider will be open sets of the type $(U(\sigma, \sigma'))^n$. Finally let $S' = \{(b^1,\dots, b^n) \in S : b^i$ is a vertex of S^i for at least one $i \in I_n\}$. We are now ready to state our main result:

THEOREM. *Fix σ and σ', $0 < \sigma < \sigma'$. There is an open dense set E of $(U(\sigma, \sigma'))^n$ such that, for any $u \in E$,*

(a) *$\eta(u)$ is a finite set,*

(b) *$\eta(u) \cap \epsilon(u) \subset S'$.*

Remark 1. To obtain generic inefficiency, we could, for instance, confine our attention to $u = (u^1,\dots, u^n) \in (U(\sigma, \sigma'))^n$ such that:[4]

> for any i in I_n, there exist at least two distinct $j_1(i)$ and $j_2(i)$ in I_k with the property $u^{i^{-1}}(r) \cap \{x \in \Omega^{k+1}: x_{j_1(i)} = 0\} = \varnothing$ and $u^{i^{-1}}(r) \cap \{x \in \Omega^{k+1}: x_{j_2(i)} = 0\} = \varnothing$, whenever $r > u^i(0)$.

Such u form an open set $\tilde{E}$ in $(U(\sigma, \sigma'))^n$, and clearly imply $\eta(u) \cap S' = \varnothing$. Thus the theorem could be restated with $\tilde{E}$ in place of $(U(\sigma, \sigma'))^n$ and "$\eta(u) \cap \epsilon(u) = \varnothing$" in place of (b).

3. PROOF OF THEOREM

LEMMA 1. *Fix σ and σ', $0 < \sigma < \sigma'$. Then there exists a $\mu > 0$ such that for any $u \in (U(\sigma, \sigma'))^n$,*

$$\eta(u) \subset S_\mu = \{b \in S : \bar{b}_j > \mu \quad \textit{and} \quad b_j^i < \bar{b}_j \quad \textit{for all} \quad i \in I_n, j \in I_k\}.$$

Proof. Let $u \in (U(\sigma, \sigma'))^n$ and $b = (b^1,\dots, b^n) \in \eta(u)$. Then clearly $\bar{b}_j > 0$ for $j \in I_k$. Otherwise, if $\bar{b}_j = 0$ for some j, then any trader could bid an arbitrarily small ϵ on the jth trading post (if necessary by reducing some other bid). By this change of strategy, he acquires $\bar{a}_j$, while his other holdings change by amounts that go to zero with ϵ. Hence for small enough ϵ this improves his payoff, a contradiction.

Next, it is also clear that $b_j^i < \bar{b}_j$ for all i and j. For if $b_j^i = \bar{b}_j$, then i could reduce b_j^i and improve his payoff, a contradiction.

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[4] Intuitively this says that each trader "sufficiently desires" at least two commodities.

To establish the lower bound on $\bar{b}_j$ we consider two cases.

Case I. $\sum_{r=1}^{k} b_r^h < a_{k+1}^h$ for all $h \in I_n$. Let i be such that $b_j^i/\bar{b}_j < 1/2$. Now if i bids ϵ more on j, his increase in payoff for small ϵ is approximately:

$$\epsilon \cdot \left[\frac{\partial u^i}{\partial x_j}(x^i(b)) \cdot \frac{\bar{a}_j}{\bar{b}_j}\left(\frac{\bar{b}_j - b_j^i}{\bar{b}_j}\right) - \frac{\partial u^i}{\partial x_{k+1}}(x^i(b))\left(1 - \frac{a_j^i}{\bar{a}_j}\right)\right]$$

$$\geqslant \epsilon \cdot \left\{\frac{\sigma}{2} \cdot \frac{\bar{a}_j}{\bar{b}_j} - \sigma'\left[1 - \frac{a_j^i}{\bar{a}_j}\right]\right\}.$$

For b to be a N.E. we must have $a_j^i < \bar{a}_j$, and

$$\frac{\sigma \bar{a}_j}{2\bar{b}_j} \leqslant \sigma'\left[1 - \frac{a_j^i}{\bar{a}_j}\right],$$

i.e.,

$$\boxed{\bar{b}_j \geqslant \frac{\sigma \bar{a}_j}{2\sigma'\left[1 - \frac{a_j^i}{\bar{a}_j}\right]}}\,.$$

Case II. $\sum_{r=1}^{k} b_r^h = a_{k+1}^h$ for some $h \in I_n$. Then there is a $r^* \in I_k$ such that $b_{r^*}^h \geqslant a_{k+1}^h/k$. If $r^* = j$, then

$$\boxed{\bar{b} \geqslant \frac{a_{k+1}^h}{k}}\,.$$

If $r^* \neq j$, choose $i \in I_n$ to ensure that $b_j^i/\bar{b}_j < 1/2$. Now if i reduces his bid on r^* by ϵ and increases his bid on j by ϵ, then his increase in payoff for small ϵ is approximately:

$$\epsilon \cdot \left\{\frac{\partial u^i}{\partial x_j}(x^i(b)) \cdot \frac{\bar{a}_j}{\bar{b}_j}\left(\frac{\bar{b}_j - b_j^i}{\bar{b}_j}\right) - \frac{\partial u^i}{\partial x_{r^*}}(x^i(b)) \cdot \frac{\bar{a}_{r^*}}{\bar{b}_{r^*}}\left(\frac{\bar{b}_{r^*} - b_{r^*}^i}{\bar{b}_{r^*}}\right)\right.$$

$$\left. - \frac{\partial u^i}{\partial x_{k+1}}(x^i(b))\left(\frac{a_{r^*}^i}{\bar{a}_{r^*}} - \frac{a_j^i}{\bar{a}_j}\right)\right\}.$$

Now note

$$\frac{\partial u^i}{\partial x_{r^*}}(x^i(b)) \cdot \frac{\bar{a}_{r^*}}{\bar{b}_{r^*}}\left(\frac{\bar{b}_{r^*} - b_{r^*}^i}{\bar{b}_{r^*}}\right) \leqslant \sigma' \frac{\bar{a}_{r^*}k}{a_{k+1}^h},$$

$$\frac{\partial u^i}{\partial x_{k+1}}(x^i(b))\left[\frac{a_{r^*}^i}{\bar{a}_{r^*}} - \frac{a_j^i}{\bar{a}_j}\right] \leqslant \sigma' \frac{a_{r^*}^i}{\bar{a}_{r^*}},$$

$$\frac{\partial u^i}{\partial x_j}(x^i(b))\frac{\bar{a}_j}{\bar{b}_j}\left(\frac{\bar{b}_j - b_j^i}{\bar{b}_j}\right) \geqslant \frac{\sigma \bar{a}_j}{2\bar{b}_j}.$$

Hence the increase in i's payoff is at least:

$$\epsilon \cdot \left\{ \frac{\sigma \bar{a}_j}{2\bar{b}_j} - \sigma' \left[\frac{a^i_{r*}}{\bar{a}_{r*}} + \frac{\bar{a}_{r*}k}{a^h_{k+1}} \right] \right\}.$$

This must be nonpositive, from which we get

$$\boxed{\bar{b}_j \geqslant \frac{\sigma \bar{a}_j}{2\sigma' \left[\frac{a^i_{r*}}{\bar{a}_{r*}} + \frac{\bar{a}_{r*}k}{a^h_{k+1}} \right]}}\ .$$

Let

$$M^1 = \min\{\bar{a}_j : j \in I_k\}$$

$$M^2 = \min\{a^i_{k+1} : i \in I_n\}$$

$$M^3 = \max \left\{ 1 - \frac{a_j^i}{\bar{a}_j} : i \in I_n,\ j \in I_k,\ a_j^i < \bar{a}_j \right\}$$

$$M^4 = \max \left\{ \frac{a_j^i}{\bar{a}_j} + \frac{k\bar{a}_j}{a^h_{k+1}} : i \in I_n,\ h \in I_n,\ j \in I_k \right\}.$$

Set

$$\mu = \min \left[\frac{M^1}{2M^3\sigma'}, \frac{M^1}{2M^4\sigma'}, \frac{M^2}{k} \right]. \qquad \text{Q.E.D.}$$

Lemma 1 enables us to steer clear of the discontinuity in the payoff functions at $\bar{b}_j = 0$ for $j \in I_k$. This is important—see Remark 5—because in order to show the openness of E in the theorem we need to be able to extend the payoff functions smoothly in a neighborhood of S_μ.

Given Lemma 1, it suffices to prove the

AUXILIARY THEOREM. *There is an open dense set E of $(U(\sigma, \sigma'))^n$ such that for any $u \in E$*

(a) *$\eta(u) \cap S_\mu$ is a finite set*

(b) *$\eta(u) \cap \epsilon(u) \cap S_\mu \subset S'$.*

We proceed to prove the Auxiliary Theorem through a sequence of lemmas. Unfortunately, first we need to introduce some fairly cumbersome notation

V^i = the set of all the $k + 1$ vertices of S^i,

$\tilde{V}^i$ = the set of all nonempty subsets of V^i,

$\tilde{V} = \tilde{V}^1 \times \cdots \times \tilde{V}^n$.

For any $T^i \in \hat{V}^i$, $T_0^i = T^i/\{0^i\}$, where 0^i is the zero vertex of S^i.
For any $T = \{T^1, \ldots, T^n\} \in \hat{V}$ and $i \in N$:

$$\begin{aligned}
S^i(T) &= \text{convex hull of } T^i,\\
\mathring{S}^i(T) &= \text{relative interior of } S^i(T),\\
\mathring{S}(T) &= \mathring{S}^1(T) \times \cdots \times \mathring{S}^n(T),\\
\tilde{S}(T) &= \mathring{S}(T) \cap S_u\,,\\
S_j^i &= \left\{b \in S\colon a_j^i(\bar{b}_j - b_j^i) + \bar{a}_j b_j^i \left(\frac{a_j^i}{\bar{a}_j} - 1\right) = 0\right\},\\
\tilde{S}_j^i(T) &= \tilde{S}(T) \cap S_j^i,\\
N(T) &= \{i \in N\colon |\, T^i \,| > 1\},\\
\hat{T} &= \bigcup \{T_0^i\colon i \in N(T)\},\\
t^i &= |\, T_0^i \,|,\\
\hat{t} &= \sum_{i \in N(T)} t^i,\\
t &= |\, N(T)|,\\
R^{N\hat{T}} &= \text{Euclidean space of dimension } t\hat{t} \text{ whose axes}\\
& \text{are indexed by pairs } (i, j) \in N(T) \times \hat{T}.
\end{aligned}$$

For any $v \in R^{N\hat{T}}$, v_j^i will be its (i, j)th component. Also for any $L \subset \hat{T}$, v_L^i will be the vector in R^L (whose axes are indexed by elements of L) with components $\{v_j^i : j \in L\}$.

Note that there is a natural correspondence between elements of V_0^i and the variables $\{b_j^i : 1 \leqslant j \leqslant k\}$. Thus, without confusion, we will speak sometimes of the variable x_l, $l \in \hat{T}$.

We construct a mapping[5]

$$_T\psi : (U(\sigma, \sigma'))^n \times \tilde{S}(T) \to R^{N\hat{T}},$$

which will enable us to study $\eta(u)$ and $\epsilon(u)$. Letting $u = (u^1, \ldots, u^n) \in (U(\sigma, \sigma'))^n$ $b \in \tilde{S}(T)$, define (for $i \in N(T)$, $j \in \hat{T}$)

$$_T\psi^i(u, b) = \left(\frac{\partial u^i}{\partial x_l}\right)(b), \qquad i \in N(T), \quad l \in \hat{T}.$$

It is easy to compute that:

$$_T\psi_j^i(u, b) = \frac{\partial u^i}{\partial x_j}(x^i(b)) \cdot \left[\frac{\bar{a}_j(\bar{b}_j - b_j^i)}{(\bar{b}_j)^2}\right] + \frac{\partial u^i}{\partial x_{k+1}}(x^i(b)) \left[\frac{a_j^i}{\bar{a}_j} - 1\right]$$

---

[5] Assuming $\tilde{S}(T)$ is not empty.

if $j \in T_0^i$; and

$$_T\psi_j^i(u, b) = -\frac{\partial u^i}{\partial x_j}(x^i(b)) \cdot \left[\frac{\bar{a}_j b_j^i}{(\bar{b}_j)^2}\right] + \frac{\partial u^i}{\partial x_{k+1}}(x^i(b)) \left[\frac{a_j^i}{\bar{a}_j}\right]$$

if $j \notin T_0^i$.

Using $_T\psi$ we now define some other mappings. First, define the subspace $R^{\hat{T}}$ of $R^{N\hat{T}}$ by $R^{\hat{T}} = \{v \in R^{N\hat{T}} : v_j^i = 0 \text{ if } j \notin T_0^i\}$. Then define $_T\tilde{\psi} : (U(\sigma, \sigma'))^n \times S(T) \to R^{\hat{T}}$ by setting $_T\tilde{\psi}_j^i = 0$ if $j \notin T_0^i$. Finally define

$$_T\varnothing : (U(\sigma, \sigma'))^n \times \tilde{S}(T) \to R^{N\hat{T}} \times \tilde{S}(T)$$

and

$$_T\tilde{\varnothing} : (U(\sigma, \sigma'))^n \times \tilde{S}(T) \to R^{\hat{T}} \times \tilde{S}(T)$$

by

$$_T\varnothing(u, b) = ({}_T\psi(u, b), b),$$
$$_T\tilde{\varnothing}(u, b) = ({}_T\tilde{\psi}(u, b), b).$$

We next need to define two subsets of $R^{N\hat{T}}$. To this end, first let

$$T_a = \{i \in N(T) : 0^i \in T^i\},$$
$$T_b = \{i \in N(T); 0^i \notin T^i\}.$$

Then let

$$\Delta^1(T) = \{v \in R^{N\hat{T}} : \text{the projections of the } v^i,\ i \in N(T), \text{ on } S(T) \text{ are linearly dependent}\},$$

$$\Delta^2(T) = \{v \in R^{N\hat{T}} : v^i_{T_0^i} = 0 \quad \text{for} \quad i \in T_a;\quad v_j^i = v_l^i \quad \text{for} \quad i \in T_b, j \in T_0^i, l \in T_0^i\}.$$

Lemma 2. *Let M be any submanifold of $R^{\hat{T}} \times \tilde{S}(T)$. Then $_T\tilde{\varnothing}$ is transversal*[6] *to $M(({}_T\tilde{\varnothing} \pitchfork M))$.*

Proof. Consider any $(u, b) \in (U(\sigma, \sigma'))^n \times \tilde{S}(T)$ such that $_T\tilde{\varnothing}(u, b) = y \in M$. Take any $v \in R^{\hat{T}}$ and any $w \in$ Tangent space of $\tilde{S}(T)$. We will show that there is a differentiable path $\{({}_\tau u, {}_\tau b)\}_{\tau=0}^1$ in[7] $(U)^n \times \tilde{S}(T)$ such that

$$({}_0u, {}_0b) = (u, b),$$

$$\frac{d}{d\tau}[{}_T\tilde{\varnothing}({}_\tau u, {}_\tau b)]\Big|_{\tau=0} = (v, w).$$

---

[6] For the definition of "transversal," see Appendix.

[7] Clearly, for sufficiently small τ, this path will lie in $(U(\sigma, \sigma'))^n \times \tilde{S}(T)$.

To do this, let $\{{}_\tau b\}_{\tau=0}^{1}$ be a path in $\tilde{S}(T)$ such that[8]

$$
\begin{aligned}
{}_0 b &= 0,\\
\frac{d}{d\tau}\,{}_\tau b\,\Big|_{\tau=0} &= w
\end{aligned}
$$

and then let ${}_\tau u = \{{}_\tau u^i : i \in N(T)\}_{\tau=0}^{1}$ be given by:

$$
{}_\tau u^i(x) = u^i(x) + \sum_{j \in T_1{}^0} \tau \alpha_j{}^i x_j\,,
$$

where

$$
\begin{aligned}
\alpha_j{}^i &= \frac{1}{A_j{}^i}\,[v_j{}^i - A_j{}^i L_j{}^i - B_j{}^i Q_j{}^i - D_j{}^i],\\
A_j{}^i &= \frac{\bar{a}_j(\bar{b}_j - b_j{}^i)}{(\bar{b}_j)^2}\,,\\
L_j{}^i &= \lim_{\tau \to 0^+} \frac{d}{d\tau}\left[\frac{\partial u^i}{\partial x_j}(x({}_\tau b))\right],\\
B_j{}^i &= \lim_{\tau \to 0^+} \left[\frac{\partial u^i}{\partial x_j}(x({}_\tau b))\right],\\
Q_j{}^i &= \lim_{\tau \to 0^+} \frac{d}{d\tau}\left[\frac{\bar{a}_j({}_\tau\bar{b}_j - {}_\tau b_j{}^i)}{({}_\tau\bar{b}_j)^2}\right],\\
D_j{}^i &= \left(\frac{a_j{}^i}{\bar{a}_j} - 1\right) \lim_{\tau \to 0^+} \frac{d}{d\tau}\left[\frac{\partial u^i}{\partial x_{k+1}}(x({}_\tau b))\right].
\end{aligned}
$$

($A_j{}^i \neq 0$ since $b \in S''$, and these limits exist because u^i is C^2.) It is straightforward to verify that this path has all the requisite properties.

To complete the proof of the lemma, we need to establish that $(T_{(u,b)}\,{}_T\tilde{\varnothing})^{-1}(T_y M)$ splits. Since we have established that the derivative map is surjective, this follows from the finite dimensionality of the range $T_y M$.

Q.E.D.

LEMMA 3. *There exists an open dense set $E(T)$ in $(U(\sigma, \sigma'))^n$ such that, for any $u \in E(T)$,*

(a) $\eta(u) \cap \tilde{S}(T)$ is a finite set,

(b) $\eta(u) \cap \tilde{S}_j{}^i(T)$ *is empty (for $i \in N(T)$, $j \in T^i$).*

Proof. For any fixed u, denote by ${}^u_T\tilde{\varnothing}$ the mapping from $\tilde{S}(T)$ to $R^{\hat{T}} \times \tilde{S}(T)$ given by ${}^u_T\tilde{\varnothing}(b) = {}_T\tilde{\varnothing}(u, b)$. Consider the submanifold $(\Delta^2(T) \cap R^{\hat{T}}) \times \tilde{S}(T)$ of $R^{\hat{T}} \times \tilde{S}(T)$, and call it Z. By the Transversal Density and Openness

---

[8] Clearly such a path exists.

Theorems,[9] there is an open dense set $\hat{E}(T)$ of $(U(\sigma, \sigma'))^n$ such that for each $u \in \hat{E}(T)$, ${}^{u}_{T}\tilde{\varnothing} \pitchfork Z$. But (letting $t_b = | T_b |$) codim $Z = \hat{t} - t_b = \dim \tilde{S}(T)$. Hence, for such u, ${}^{u}_{T}\tilde{\varnothing}^{-1}(Z)$ has dimension zero. Being bounded, it must be a finite set.[10]

Next let $Z_j^i = (\Delta^2(T) \cap R^{\hat{T}}) \times \tilde{S}_j^i(T)$. Again there is an open dense set $E_j^i(T)$ of $(U(\sigma, \sigma'))^n$ such that for any $u \in E_j^i(T)$, ${}^{u}_{T}\tilde{\varnothing} \pitchfork Z_j^i$. But codim $Z_j^i > \hat{t} - t_b = \dim \tilde{S}(T)$. Hence, for such u, ${}^{u}_{T}\tilde{\varnothing}^{-1}(Z_j^i) = \varnothing$.

Let $E(T)$ be the intersection of $\hat{E}(T)$ and all the $E_j^i(T)$. Since $\eta(u) \cap \tilde{S}(T) \subset {}^{u}_{T}\tilde{\varnothing}^{-1}(Z)$ and $\eta(u) \cap \tilde{S}_j^i(T) \subset {}^{u}_{T}\tilde{\varnothing}^{-1}(Z_j^i)$, we have proved the lemma.

Q.E.D.

In Lemmas 4, 5, and 6 we will assume that $N(T) = N$. Consider $M(T) = \tilde{S}(T)/\cup\{\tilde{S}_j^i(T) : i \in N, j \in T^i\}$. Then $M(T)$ is a manifold of dimension $\hat{t} - t_b$.

LEMMA 4. *Consider any $M(T)$, and the mapping ${}_T\varnothing : (U(\sigma, \sigma'))^n \times M(T) \to R^{N\hat{T}} \times M(T)$. Then, for any $b \in M(T)$, and $i \in N$,*

(a) $L_T^i(b) = \{{}_T\varnothing^i(u, b) : u \in (U(\sigma, \sigma'))^n\}$ *is a manifold of dimension* $t^i + 1$.

(b) $L_T^i(b) \cap R^{T_0^i}$ is an open set in $R^{T_0^i}$ (where $R^{T_0^i} = \{v \in R^{N\hat{T}} : v_j^l \neq 0$ *if, and only if,* $l = i$ *and* $j \in T_0^i\}$).

Proof. (a) Take any $j \in T_0^i$ (w.l.o.g. $j = 1$) and consider the $(k + 1) \times (k + 1)$ matrix:

$$
\begin{array}{|c|c|c|c|c|c|c|}
\hline
\dfrac{\bar{a}_1(\bar{b}_1 - b_1^i)}{(\bar{b}_1)^2} & 0 & 0 & \cdots & 0 & 0 & \dfrac{a_1^i}{\bar{a}_1} - 1 \\
\hline
-\left(\dfrac{\bar{a}_1 b_1^i}{(\bar{b}_1)^2}\right) & 0 & 0 & \cdots & 0 & 0 & \dfrac{a_1^i}{\bar{a}_1} \\
\hline
0 & -\left(\dfrac{\bar{a}_2 b_2^i}{(\bar{b}_2)^2}\right) & 0 & \cdots & 0 & 0 & \dfrac{a_2^i}{\bar{a}_2} \\
\hline
\vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\
\hline
0 & 0 & 0 & \cdots & 0 & -\left(\dfrac{\bar{a}_k b_k^i}{(\bar{b}_k)^2}\right) & \dfrac{a_k^i}{\bar{a}_k} \\
\hline
\end{array}
$$

---

[9] See Appendix.

[10] This sentence is made rigorous in Remark 5.

Since $b \notin S_j^i(T)$ for any $j \in T_0^i$,

$$\det \begin{vmatrix} \dfrac{\bar{a}_1(\bar{b}_1 - b_1^i)}{(\bar{b}_1)^2} & \dfrac{a_1^i}{\bar{a}_1} - 1 \\ -\left(\dfrac{\bar{a}_1 b_1^i}{(\bar{b}_1)^2}\right) & \dfrac{a_1^i}{\bar{a}_1} \end{vmatrix} \neq 0$$

from which it is easily deduced that the rank of the matrix is $t^i + 1$. To prove (a) note (i) the mapping from $(U(\sigma, \sigma'))^n$ to $R^{N\hat{T}}$ given by $_T\varnothing^i(u, b)$ for fixed b is linear, (ii) the image of the mapping is obtained by linear combinations[11] of $(t^i + 1)$ nonzero vectors each with $\hat{t}$ components, (iii) the matrix displayed above (after removal of columns j, $j \notin T_0^i$) is a submatrix of the matrix (which, for future reference, will be denoted by $C_T^i(b)$) of these vectors.

(b) This is obvious.

Q.E.D.

By Lemma 4, $_T\varnothing((U(\sigma, \sigma'^n)) \times M(T))$ is a manifold $\tilde{M}(T)$ in $R^{N\hat{T}} \times M(T)$ (of dimension $\hat{t} + n + \hat{t} - t_b$). From now on, view the range of the mapping $_T\varnothing$ as $\tilde{M}(T)$.

Lemma 5. *Consider* $_T\varnothing : (U(\sigma, \sigma'))^n \times M(T) \to \tilde{M}(T)$. *Let* M' *be any submanifold of* $\tilde{M}(T)$. *Then* $_T\varnothing \pitchfork M'$.

Proof. This is along the same lines as the proof of Lemma 2. Let $_T\varnothing(u, b) = (v, b) \in \tilde{M}(T)$. Consider any differentiable path $({}_\tau v, {}_\tau b)$ in $\tilde{M}(T)$ for $\tau \in [0, 1]$, with $({}_0 v, {}_0 b) = (v, b)$. We will show that there is a differentiable path $({}_\tau u, {}_\tau b')$ in [12]$(U(\sigma, \sigma'))^n \times M(T)$ such that $_T\varnothing({}_\tau u, {}_\tau b') = ({}_\tau v, {}_\tau b)$ for $\tau \in [0, 1]$, and $({}_0 u, {}_0 b) = (u, b)$.

Write ${}_\tau v = ({}_\tau v^1, \dots, {}_\tau v^n)$, ${}_\tau u = ({}_\tau u^1, \dots, {}_\tau u^n)$. Since each $C_T^i({}_\tau b)$ has full rank there exists a unique $(t^i + 1)$-vector, ${}_\tau\alpha^i = \{{}_\tau\alpha_j^i : j \in T_0^i \text{ and } j = k + 1\}$ such that $C_T^i({}_\tau b){}_\tau\alpha^i = {}_\tau v^i$. Now construct the path $({}_\tau u, {}_\tau b')$ as follows:

$${}_\tau b' = {}_\tau b,$$

$${}_\tau u^i(x) = u^i(x) + \sum_{j \in T_i^0} {}_\tau\delta_j^i x_j + {}_\tau\delta_{k+1}^i x_{k+1},$$

where

$${}_\tau\delta_j^i = {}_\tau\alpha_j^i - \frac{\partial u^i}{\partial x_j}(x^i({}_\tau b))$$

for $j \in T_0^i$ and $j = k + 1$.

---

[11] Where the coefficients of the combination are picked from an open set in R^{t^i+1}.

[12] Again, for small enough τ, this path lies in $(U(\sigma, \sigma'))^n \times M(T)$.

It is easily checked that this path has the requisite properties. The rest of the proof is concluded as in Lemma 2. Q.E.D.

LEMMA 6. *For any T there is an open dense set $V(T)$ of $(U(\sigma, \sigma'))^n$ such that for $u \in V(T)$, $\eta(u) \cap \epsilon(u) \cap M(T)$ is empty.*

Proof. Define $M^*(T) = [R^{N\hat{T}} \cap \Delta^1(T) \cap \Delta^2(T)] \times M(T)$. Recall that $\Delta^1(T)$ is a finite union of submanifolds of $R^{N\hat{T}}$, each of which has codimension $\geqslant 1$. Piecing this with the previous lemma, we see that $M^*(T)$ is then a finite union of submanifolds of $\tilde{M}(T)$, say $M_1^*(T), \ldots, M_{r(T)}^*(T)$, with codimension $M_i^*(T) > \hat{t} - t_b$. But Lemma 5 again implies that there exist open dense sets $V_i(T)$ $(i = 1, \ldots, r(T))$ of $(U(\sigma, \sigma'))^n$ such that for any $u \in V_i(T)$, ${}_T^u\varnothing \pitchfork M_i^*(T)$. But then codim ${}_T^u\varnothing^{-1}(M_i^*(T)) =$ codim $M_i^*(T) >$ dim $M(T)$; hence ${}_T^u\varnothing^{-1}(M_i^*(T)) = \varnothing$. On the other hand, using the proposition from [10] $[\eta(u) \cap \epsilon(u) \cap M(T)] \subset {}_T^u\varnothing^{-1}(M_1^*(T) \cup \cdots \cup M_{r(T)}^*(T))$. Put $V(T) = \cap \{V_i(T) : i = 1, \ldots, r(T)\}$. Q.E.D.

Proof of Auxiliary Theorem. Put

$$E = [\cap\{E(T) : T \in \tilde{T}] \cap [\cap\{V(T) : T \in \tilde{T}, N(T) = N\}].$$

The proof follows from the observation that $\{\eta(u) \cap \tilde{S}(T) : T \in \tilde{T}\}$ constitutes a partition of $\eta(u) \cap S_u$. Q.E.D.

4. CONCLUDING REMARKS

Remark 2. Let $\hat{U} = \{[u^1, \ldots, u^n) \in (U(\sigma, \sigma'))^n :$ each u^i is concave$\}$. Then it can be shown—see [9]—that $\eta(u) \neq \varnothing$ for any $u \in \hat{U}$. Since $\hat{U}$ contains an open set of $(U(\sigma, \sigma'))^n$—for instance, consider those u^i which are strictly concave—our theorem is not vacuous.

Remark 3. We could have let the space of games be $(U)^n \times (\text{Int}\, \Omega^{k+1})$, allowing for the endowments to vary also. However, it would become necessary to restrict this variation to ensure that the N.E. are bounded away from the zero-price strategies, as in Lemma 1. Thus, for instance, we could state the theorem with $(U(\sigma, \sigma'))^n \times (\hat{Q})^n$, where $\hat{Q}$ is the interior of a cube which itself is in the interior of Ω^{k+1}.

Remark 4. The generic inefficiency of the N.E. attenuates in this model as the player-set is increased in the direction of a nonatomic continuum. For the continuum itself, the N.E.—under appropriate conditions (see [3])—are efficient, and coincide with the Walras Equilibria of the market.

Remark 5. We have been somewhat slipshod in parts of the proof in order to keep the main ideas clear. Now we redress this. Let $\hat{Q}$ be an open neighborhood of Q in R^{k+1} and $\hat{U}$ the manifold of all C^2 functions on $\hat{Q}$ whose restriction to Q is an element of $U(\sigma, \sigma')$. Note that if $\hat{E}$ is open (or dense) in $\hat{Q}$, then E, obtained by restricting the functions in $\hat{E}$ to the domain Q, is open (or dense) in E. This follows from the well-known fact that there is a $K > 0$ such that: each $u \in U(\sigma, \sigma')$ can be extended to a $\hat{u} \in \hat{U}$ with $\| u \| \leqslant K \| \hat{u} \|$. Next let $\hat{S}_\mu$ be an open neighborhood of S_μ in Ω^{nk} such that[13] $\hat{S}_\mu \cap \{(b^1, \ldots, b^n) \in \Omega^{nk} : \bar{b}_j = 0, \text{ some } j \in I_k\} = \varnothing$. $\hat{S}(T) = (\text{Aff}\, \tilde{S}(T)) \cap \hat{S}_\mu$, where Aff stands for "affine hull." Then define the mappings ${}_T\psi$, ${}_T\tilde{\psi}$, ${}_T\varnothing$ and ${}_T\tilde{\varnothing}$ on $(\hat{U})^n \times \hat{S}(T)$ exactly as before.[14] By adjusting $\hat{Q}$ to be large enough, and $\hat{S}_\mu$ to be small enough, there is no problem of definition. (There would have been a problem if $\hat{S}_\mu$ was not bounded away from points where $\bar{b}_j = 0$. Hence the importance of Lemma 1.)

Lemma 2, in fact, holds—and the proof really shows this—for this bigger mapping ${}_T\varnothing : (\hat{U})^n \times \hat{S}(T) \to R^{\hat{T}} \times \hat{S}(T)$. Now the proof of Lemma 3 may be reread as follows. First we apply the Transversal Density and Openness Theorems (see Appendix) to this ${}_T\tilde{\varnothing}$ with $A = (\hat{U})^n$, $X = \hat{S}(T)$, $K =$ Closure of $\tilde{S}(T)$, etc. This gives us the open dense set in $(\hat{U})^n$, which in turn induces the desired open dense set $\hat{E}(T)$ in $(U(\sigma, \sigma'))^n$, as explained above. Also $\eta(u) \cap \tilde{S}(T) \subset {}^{u}_{T}\varnothing^{-1}(Z) \cap K$, and the latter set is clearly finite since it is the intersection of a compact set and a 0-dimensional manifold.

Lemmas 4, 5, and 6 can similarly be reread to make the proof rigorous.

APPENDIX

We recall the results from [1] used in this paper.

"Let X and Y be C^1 manifolds, f: $X \to Y$ a C^1 map, and $W \subset Y$ a submanifold. We say that f is *transversal to W at a point* $x \in X$, in symbols: $f \pitchfork_x W$, iff, where $y = f(x)$, either $y \notin W$ or $y \in W$ and

(1) the inverse image $(T_x f)^{-1}(T_y W)$ splits, and

(2) the image $(T_x f)(T_x)$ contains a closed complement to $T_y W$ in $T_y Y$. We say f is *transversal to W*, in symbols: $f \pitchfork W$, iff $f \pitchfork_x W$ for every $x \in X$.

Let $\mathscr{A}$, X, and Y be C^r manifolds, $\mathscr{C}^r(X, Y)$ the set of C^r maps from X to Y, and $\rho : \mathscr{A} \to \mathscr{C}^r(X, Y)$ a map. For $a \in \mathscr{A}$ we write ρ_a instead of $\rho(a)$; i.e., $\rho_a : X \to Y$ is a C^r map. We say ρ is a *C^r representation* iff the evaluation map

$$\mathrm{ev}_\rho : \mathscr{A} \times X \to Y$$

---

[13] This is clearly possible.

[14] I.e., by the same formulas.

given by

$$\mathrm{ev}_\rho(a, x) = \rho_a(x)$$

for $a \in \mathscr{A}$ and $x \in X$ is a C^r map from $\mathscr{A} \times X$ to Y.

TRANSVERSAL DENSITY THEOREM. *Let $\mathscr{A}$, X, Y be C^r manifolds, $\rho : \mathscr{A} \to \mathscr{C}^r(X, Y)$ a C^r representation, $W \subset Y$ a submanifold (not necessarily closed), and $\mathrm{ev}_\rho : \mathscr{A} \times X \to Y$ the evaluation map. Define $\mathscr{A}_W \subset \mathscr{A}$ by*

$$\mathscr{A}_W = \{a \in \mathscr{A} \mid \rho_a \pitchfork W\}.$$

Assume that:

(1) *X has finite dimension n and W has finite codimension q in Y;*

(2) *$\mathscr{A}$ and X are second countable;*

(3) *$r > \max(0, n - q)$;*

(4) *$\mathrm{ev}_\rho \pitchfork W$.*

Then $\mathscr{A}_W$ is residual (and hence dense) in $\mathscr{A}$.

Openness of Transversal Intersection. Let $\mathscr{A}$, X, and Y be C^1 manifolds with X finite dimensional, $W \subset Y$ a closed C^1 submanifold, $K \subset X$ a compact subset of X, and $\rho : \mathscr{A} \to C^1(X, Y)$ a C^1 pseudorepresentation. Then the subset $\mathscr{A}_{KW} \subset \mathscr{A}$ defined by

$$\mathscr{A}_{KW} = \{a \in \mathscr{A} \mid \rho_a \pitchfork_x W \quad \text{for} \quad x \in K\}$$

is open. This holds even if X is not finite dimensional, provided that ρ is a C^1 representation."

For our purposes, it is enough to note that every C^1 representation is a C^1 pseudorepresentation. Also $T_y W$ is the tangent space to W at y; $T_x f : T_x X \to T_y Y$ is the derivative map of f at x. See [1] for detailed definitions.

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Printed in Belgium

JOURNAL OF ECONOMIC THEORY

Volume 22, Number 3, June 1980

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Published bimonthly at 37 Tempelhof, Bruges, Belgium
by Academic Press, Inc., 111 Fifth Avenue, New York, N.Y. 10003
1980: Volumes 22–23. Price per volume: $61.00 U.S.A.;
$73.00 outside U.S.A. All prices include postage and handling.
Information concerning personal subscription rates may be obtained
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All correspondence and subscription orders should be addressed to the office of the
Publishers at 111 Fifth Avenue, New York, N.Y. 10003.
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Second class postage paid at Jamaica, N.Y.
Air freight and mailing in the U.S.A. by Publications Expediting, Inc.,
200 Meacham Avenue, Elmont, New York 11003.

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Printed in Bruges, Belgium, by the St. Catherine Press, Ltd.

JOURNAL OF ECONOMIC THEORY 22, 377–394 (1980)

The Permanent Income Hypothesis and Long-Run Economic Stability*

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Received March 22, 1977; revised September 1979

1. INTRODUCTION

The work reported here has to do with the theoretical implications of a form of the permanent income hypothesis. This form of the hypothesis is formulated in a previous paper [1]. In a succeeding paper [2], it is proved that this hypothesis implies short-run price stability. The purpose of this paper is to emphasize that the hypothesis does not imply long-run economic stability.

The permanent income hypothesis of [1] is, roughly, that a consumer's notion of his average real income per period is constant. This hypothesis leads to a definition of consumer demand functions different from that usually used in equilibrium theory. The new definition states that demand is determined by constancy of the marginal utility of money. That is, a consumer spends money on a good up to the point at which the utility derived from the marginal purchase equals the utility of the money spent on it, and this marginal utility of money is independent of current income and prices.

This reformulation of consumer demand eliminates the income effects which make price instability possible in the *tâtonnement* process. In fact, an economy with the new form of demand functions has a unique equilibrium, which is stable in the sense that the following system of differential equations is globally stable. (See Part II of [2].)

$$\frac{dp_k(t)}{dt} = Z_k(p_1(t),\dots, p_L(t)), \qquad k = 1,\dots, L, \tag{1.1}$$

where, $p_k(t)$ is the price of good k at time t, and Z_k is the aggregate excess demand function for good k.

It makes sense in certain contexts to assume that a consumer's notion of his average real income is constant over the short run. For this reason, the implied stability of the above system should be thought of only as short-run

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* This paper was written at the University of Bonn, Bonn, Germany, in 1977.

0022-0531/80/030377-18$02.00/0

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stability. For the same reason, demand functions derived from the permanent income hypothesis are called short-run demand functions, and the equilibrium of supply and short-run aggregate demand is called short-run equilibrium.

Perverse income effects reappear as a source of instability, when consumers' notions of their average real income vary over the long run. This instability is manifested as instability in the notions of average real income themselves. It is imagined that these notions evolve in the following manner. The notions determine a short-run equilibrium, and consumers trade at this equilibrium. This trade is thought of as a continuous flow. In the short-run equilibrium, consumers may spend at a rate greater or less than the rate at which they earn. In order to bring spending into line with earnings, they (slowly) adjust their notions of average real income. It turns out that this process may be unstable.

This process is described more precisely as follows. A consumer's notion of his permanent income is reflected in his marginal utility of money. For $i = 0,\ldots, I$, let λ_i be the marginal utility of money of consumer i, let ω_i be his initial endowment, and let $\xi_i(p, \lambda_i)$ be his short-run demand as a function of λ_i and the vector of prices, p. Note that ω_i and ξ_i represent flows, that is, they are measured in units of commodity per period. $p(\lambda)$ denotes the short-term equilibrium price vector when the marginal utilities of money are $\lambda = (\lambda_0, \ldots, \lambda_I)$. $p(\lambda)$ is the unique price vector satisfying $\sum_{i=0}^{I} [\xi_i(p(\lambda), \lambda_i) - \omega_i] = 0$. The short-run net expenditure of consumer i is $\tau_i(\lambda) = p(\lambda) \cdot [\xi_i(p(\lambda), \lambda_i) - \omega_i]$. Presumably, a consumer would tend to raise or lower λ_i if $\tau_i(\lambda)$ were positive or negative, respectively. The precise process of adjustment of the λ_i is assumed to be

$$\frac{d\lambda_i(t)}{dt} = \tau_i(\lambda(t)), \qquad i = 0,\ldots, I, \tag{1.2}$$

where $\lambda_i(t)$ is the marginal utility of consumer i at time t and $\lambda(t) = (\lambda_0(t),\ldots, \lambda_I(t))$.

Observe that an equilibrium for the above system corresponds to the usual notion of equilibrium in general equilibrium theory. Such an equilibrium may be called a long-term equilibrium.

Mantel [8] has studied the above system (with a motivation different from that of this paper) and proved that for an economy described by Scarf in [12], the system is globally unstable. Scarf used his example to prove that the price adjustment process, (1.1), may be unstable. (Gale's example did so as well [5].) The instability of (1.1) was later made even more evident by the theorem of Sonnenschein, Mantel, and Debreu [3, 9–11, 14, 15], which states, roughly, that anything can happen in (1.1). More precisely, let $Z(p) = (Z_1(p),\ldots, Z_L(p))$ be an arbitrary continuous function, which is defined on a closed subset of the interior of the price simplex and satisfies Walras' law, $\sum_{k=1}^{L} p_k Z_k(p) = 0$. Then, there exists an economy with L consumers, the excess demand function of which equals Z. The consumers of the economy have continuous, monotone, and strictly convex preference orderings. (This is the theorem proved by Debreu [3].)

In this paper, the instability of (1.2) is made clear in a similar manner, by proving an analog of the Sonnenschein–Mantel–Debreu theorem for system (1.2). Let τ_i, $i = 0,..., I$, be twice continuously differentiable functions which are defined on a closed subset of the interior of the I-dimensional standard simplex and which satisfy $\sum_{i=0}^{I} \tau_i = 0$. It is proved that there exists an economy with $2I$ commodities in which the short-run net expenditure function of consumer i equals τ_i for all i. It is not known whether the restriction that there be $2I$ commodities can be relaxed.

This restriction on the number of commodities may appear to be disturbing, for one imagines that in reality there are many more consumers than commodities. However, the theorem still gives one a great deal of freedom in constructing counterexamples to stability. For example, suppose that an economy with $I + 1$ consumers and $2I$ goods were such that (1.2) was globally unstable. Then, the k-fold replica of the economy (in the sense of Debreu and Scarf [4]) would have $k(I + 1)$ consumers and $2I$ goods and would also be such that (1.2) was globally unstable.

The point of this paper is that an economy may not be able to find a long-run equilibrium spontaneously. To the extent that one takes this theoretical possibility seriously, it is inappropriate to apply general equilibrium theory in the context of long-run problems. Of course, as is shown in [2], the permanent income hypothesis lends plausibility to the application of equilibrium theory in certain short-run contexts. But the equilibrium appropriate in a short-run context is one with short-run demand functions, not the demand functions which usually appear in equilibrium theory.

2. REMARKS ON THE MARGINAL UTILITY OF MONEY ADJUSTMENT EQUATIONS

The system of differential equations (1.2) is studied because it is simple and seems to reflect sufficiently accurately the actual economic process under consideration. However, (1.2) cannot be claimed to be realistic or consistent with optimal consumer behavior. In these qualities, (1.2) is similar to the price adjustment system (1.1).

When interpreted literally, Eqs. (1.2) make little sense at all. They require that a consumer allow his rate of expenditure to differ from his income for an extended period of time, and there is nothing in the model that would induce him to do so.

In order to interpret (1.2), one must imagine that the demand and supply *flows represent long-term moving averages of randomly fluctuating demands and supplies, for the permanent income hypothesis makes sense only in the context of such fluctuations. In fact, suppose that prices, a consumer's needs, and his income fluctuate randomly and rapidly. It is proved in [1] that if these fluctuations were generated by a stationary process and if certain other* assumptions were satisfied, then it would be optimal for the consumer to keep his marginal utility of money nearly constant. The marginal utility of money should stay near a level which would make the long-term average rate of net expenditure nearly equal to zero. If a consumer did behave in this way, his current rate of net expenditure would fluctuate. For this reason, the consumer would only slowly become aware of any error in his expectations about long-term average net expenditures. To this extent, it makes sense to assume that in a random environment a consumer would adjust his marginal utility of money slowly in response to long-term net flows of money.

It seems pointless to make (1.2) more realistic and consistent by introducing randomness explicitly and having consumers behave optimally. Doing so would complicate matters enormously and would surely not increase the stability of (1.2). By demonstrating the instability of (1.2), one effectively demonstrates the instability of the vaguely defined and complicated system it reflects.

3. DEFINITIONS AND NOTATION

An *economy*, $\mathscr{E}$, with $I + 1$ agents and L commodities will be defined by $\mathscr{E} = \{(u_i, \omega_i): i = 0, \ldots, I\}$, where $u_i: R_+^L \to R$ and $\omega_i \in R_+^L$. u_i is the utility function of agent i, and ω_i is his initial endowment. R^L denotes L-dimensional Euclidean space. $R_+^L = \{x \in R^L \mid x_i \geqslant 0, \text{ for all } i\}$.

A function $f: U \to R^k$, where U is a subset of a Euclidean space, is said to be $\mathscr{C}^m$ if its first m derivatives exist and are continuous. If $u: R_+^L \to R$, $Du(x)$ and $D^2u(x)$ will denote the first and second derivatives, respectively of u at x. u will be said to be *differentiably strictly monotone* if it is differentiable, and $\partial u(x)/\partial x_k > 0$, for all k and x. u will be said to be *differentiably strictly concave* if it is twice differentiable, and $D^2u(x)$ is negative definite for all x. (These last two definitions correspond, approximately, to definitions given in [13].)

Let $u: R_+^L \to R$ be the utility function of a consumer and assume that u is $\mathscr{C}^2$ and differentiably strictly monotone and concave. The *short-run demand* of the consumer is denoted by $\xi(p, \lambda)$, where λ is his marginal utility of money and $p \in R_+^L$ is the vector of prices. $\xi(p, \lambda)$ satisfies the following inequalities.

$$\frac{\partial u(x)}{\partial x_k} \leqslant \lambda p_k, \qquad \text{with equality if } x_k > 0, \quad k = 1, \ldots, L. \tag{3.1}$$

If λ is positive and the components of p are sufficiently large, then (3.1) has a unique solution, so that $\xi(p, \lambda)$ is well defined.

Let $\mathscr{E} = \{(u_i, \omega_i): i = 0,\ldots, I\}$ be such that the u_i are $\mathscr{C}^2$ and differentiably strictly concave and monotone. Let $\xi_i(p, \lambda_i)$ be the short-run demand function of agent i, and let $\lambda = (\lambda_0, \ldots, \lambda_I)$ denote a vector of marginal utilities of money. $\xi(p, \lambda) \equiv (\xi_0(p, \lambda_0),\ldots, \xi_I(p, \lambda_I))$ and $p \in R_+^L$ will be said to form a *short-run equilibrium* if $\sum_{i=0}^{I} \xi_i(p, \lambda_i) = \sum_{i=0}^{I} \omega_i$. In Section II of [2], it is proved that to every vector, λ, with positive components, there corresponds a unique short-run equilibrium, $(p(\lambda), \xi(p(\lambda), \lambda))$. Hence, $p(\lambda)$ and $\xi(p(\lambda), \lambda)$ are well defined functions on int $\Delta^I = \{\lambda \in R^{I+1} \mid \sum_{i=0}^{I} \lambda_i = 1$ and $\lambda_i > 0$, for all $i\}$.

The (short-run) *net expenditure* of agent i at the short-run equilibrium $(p(\lambda), \xi(p(\lambda), \lambda))$ is $\tau_i(\lambda) = p(\lambda) \cdot [\xi_i(p(\lambda), \lambda_i) - \omega_i]$. The function $\tau: \Delta^I \to R^{I+1}$ defined by $\tau(\lambda) = (\tau_0(\lambda),\ldots, \tau_I(\lambda))$ will be called the *net expenditure function of* $\mathscr{E}$.

Note that $\sum_{i=0}^{I} \tau_i(\lambda) = 0$, for all λ, so that $\tau(\lambda)$ is tangent to int Δ^I. It follows that the differential equations (1.2) form a differential system on int Δ^I.

4. THE THEOREM

Define Δ_ϵ^I, for $\epsilon > 0$, by $\Delta_\epsilon^I = \{\lambda \in R^{I+1} \mid \sum_{i=0}^{I} \lambda_i = 1$ and $\lambda_i \geqslant \epsilon$, for all $i\}$. Let $T: \Delta_\epsilon^I \to R^{I+1}$ be such that $\sum_{i=0}^{I} T_i(\lambda) = 0$, for all λ. An economy, $\mathscr{E}$, will be said to realize T if $T(\lambda) = \tau(\lambda)$, for all $\lambda \in \Delta_\epsilon^I$, where τ is the net expenditure function of $\mathscr{E}$.

THEOREM. *For $I \geqslant 1$, let $T: \Delta_\epsilon^I \to R^{I+1}$ be any $\mathscr{C}^2$ function such that $\sum_{i=0}^{I} T_i(\lambda) = 0$, for all λ. Then, T is realized by an economy, $\mathscr{E}$, with $2I$ commodities (and $I + 1$ consumers) and with utility functions which are $\mathscr{C}^2$ and differentiably strictly concave and strictly monotone.*

5. MORE DEFINITIONS AND NOTATION

Let $\mathscr{E} = \{(u_i, \omega_i): i = 0, 1,\ldots, I\}$ be an economy with L commodities and $I + 1$ consumers. An *allocation* for $\mathscr{E}$ is a vector $x = (x_0, x_1,\ldots, x_I)$, where $x_i \in R_+^L$, for all i. x_i is the allocation to agent i. x is said to be *feasible* if $\sum_{i=0}^{I} x_i = \sum_{i=0}^{I} \omega_i$. x is said to be *Pareto optimal* if it is feasible and there exists no feasible allocation x' such that $u_i(x_i') \geqslant u_i(x_i)$, for all i, and $u_i(x_i') > u_i(x_i)$, for some i.

Now assume that the utility functions, u_i, of $\mathscr{E}$ are $\mathscr{C}^k$, $k \geqslant 2$, and are differentiably strictly concave and monotone. The short-run equilibrium allocation, $\xi(p(\lambda), \lambda)$, corresponding to $\lambda \in \operatorname{int} \Delta^I$ is Pareto optimal and $p(\lambda)$ is precisely the price vector that makes this allocation an equilibrium with transfer payments. The transfer payments are the net expenditures, $\tau_i(\lambda)$. Let X: $\operatorname{int} \Delta^I \to R_+^{(I+1)L}$ be defined by $X(\lambda) = \xi(p(\lambda), \lambda)$. X will be called the *Pareto allocation function.*

Now drop the assumptions that the utility functions of $\mathscr{E}$ are strictly concave and monotone, but retain the assumption that they are $\mathscr{C}^k$. An allocation, x, is said to be a *first-order optimum* if there exist positive numbers, $\lambda_0, \ldots, \lambda_I$, such that $\lambda_i^{-1} Du_i(x_i) = \lambda_0^{-1} Du_0(x_0) \neq 0$, for $i = 1, \ldots, I$. (This definition corresponds, roughly, to one given by Smale in [13, p. 11].) The vector $\lambda_i^{-1} Du_i(x_i)$ may be thought of as the short-run equilibrium price vector corresponding to x. These definitions are introduced only because use will be made of utility functions which are not concave. If the utility functions were concave, then, of course, every first-order optimum would be a Pareto optimum.

Let $\mathscr{P}(\mathscr{E}) = \{x \mid x \text{ is a first-order optimum and } x_{ik} > 0, \text{ for } i = 0, \ldots, I \text{ and } k = 1, \ldots, L\}$. For each $x \in \mathscr{P}(\mathscr{E})$, there exists a unique vector $\Lambda(x) \in \Delta^I$ such that for all i, $(\Lambda_i(x))^{-1} Du_i(x_i) = (\Lambda_0(x))^{-1} Du_0(x_0)$. The function Λ: $\mathscr{P}(\mathscr{E}) \to \Delta^I$ will be called the *relative marginal utilities function* of $\mathscr{E}$.

Now assume again that the utility functions are differentiably strictly concave and monotone, so that the Pareto allocation function, X, is well defined. Then, $X(\Lambda(x)) = x$, for all $x \in \mathscr{P}(\mathscr{E})$, so that Λ is well defined and its inverse is the restriction of X to the range of Λ.

Use will be made of the concept of *diffeomorphism*. Let f: $U \to R^p$, where $U \subset R^q$. f will be said to map U $\mathscr{C}^k$ diffeomorphically onto its image if f is $\mathscr{C}^k$ and f has an inverse, f^{-1}: $f(U) \to U$, which is also $\mathscr{C}^k$.

6. THE IDEA OF THE PROOF IN THE CASE OF TWO AGENTS

The idea of the construction in the case of two agents can be visualized with the aid of the Edgeworth box. First one defines an economy $\mathscr{E}_C = \{(u_{Ci}, \omega_{Ci}): i = 0, 1\}$, where $\omega_{C0} = (1, 0)$, $\omega_{C1} = (0, 1)$, and the u_{Ci} are defined on *all* of R^2 and are $\mathscr{C}^\infty$ and differentiably strictly concave and monotone. The u_{Ci} are so constructed that the following three conditions are satisfied. (1) The set of Pareto optima corresponds to the diagonal of the Edgeworth box. (2) The relative marginal utilities map, Λ_C, of $\mathscr{E}_C$ maps the set of Pareto optima onto Δ_ϵ^1. (3) The indifference curves intersect the diagonal of the box orthogonally. N will denote the set of Pareto optima of $\mathscr{E}_C$. The box diagram for $\mathscr{E}_C$ appears in Fig. 1.

In the next step, the utility functions of $\mathscr{E}_C$ are perturbed in such a way that the new economy realizes T. However, this perturbation may destroy the concavity and monotonicity of the utility functions. The final step

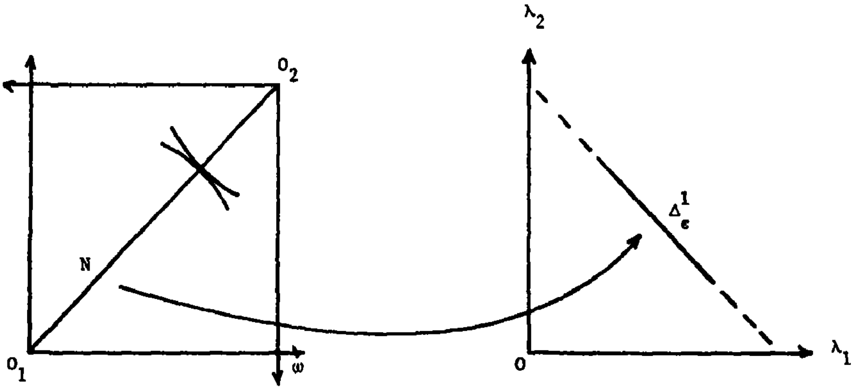

FIGURE 1

overcomes this problem by making the perturbations so small that concavity and monotonicity are preserved.

The problem may be visualized with the help of Fig. 2. The perturbations of the u_{Ci} change the tangent line at a point $x \in N$ from L to L'. The magnitude of the perturbations has to do with the size of the angle between L and L'. The size of the change in net expenditures has to do with the distance d. The problem, therefore, is to decrease the angle corresponding to a given value of d, that is, to increase the distance between N and ω.

Suppose now that the endowments of both agents were greatly increased and the utility functions, u_{Ci}, suitably translated, so that N would correspond to a segment of Pareto optima in the middle of a much enlarged Edgeworth box, as in Fig. 3. (The small box inside corresponds to the box of Fig. 2.) Then, the angle corresponding to a given distance d would be greatly reduced. In the final step, the Edgeworth box is expanded in this way to such an extent that the perturbations preserve concavity and monotonicity.

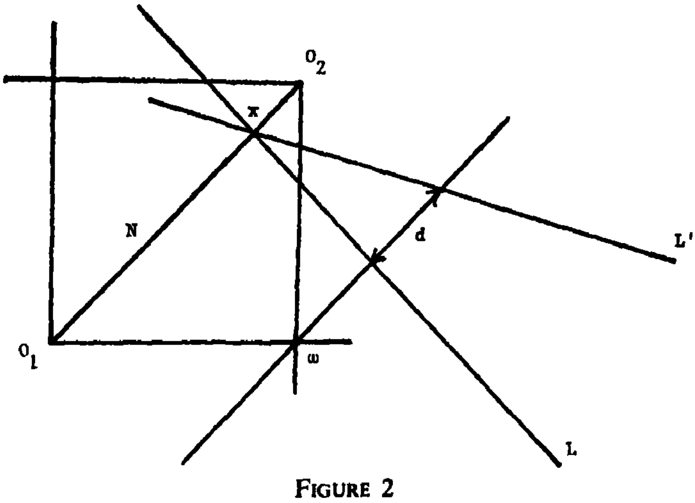

FIGURE 2

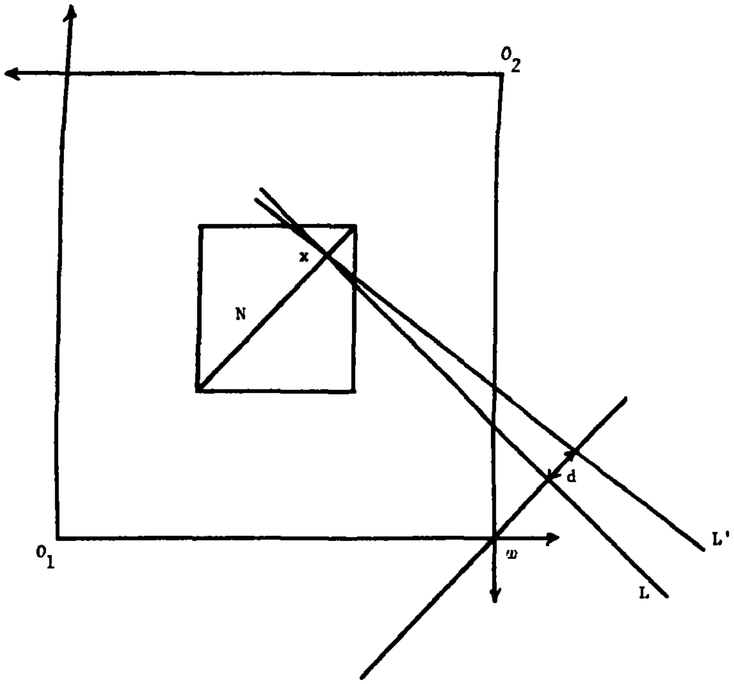

FIGURE 3

7. INTRODUCTION TO THE PROOF IN THE CASE OF ARBITRARY I

An essential step in the proof is completely obscured in the case of two agents because it is so easy in this case. This step is to define an economy, $\mathscr{E}_C$, with a Pareto allocation function, $X_C = (X_{C0}, \ldots, X_{Cl})$, such that each x_{Ci} maps Δ_ϵ^l diffeomorphically onto its image. (Here, of course, X_{Ci} gives the allocation of agent i.)

The first step is accomplished in Sections 8, 9, and 10. In Section 8, an economy $\mathscr{E}$ with l commodities is defined, which has a Pareto allocation function, $X_A = (X_{A0}, \ldots, X_{Al})$, such that each X_{Ai} maps a neighborhood of $((l+1)^{-1}, \ldots, (l+1)^{-1})$ in Δ^l diffeomorphically onto its image. In Section 9, $\mathscr{E}_A$ is embedded in an economy, $\mathscr{E}_B$, with $2l$ commodities. In Section 10, the utility functions of $\mathscr{E}_B$ are distorted so that each X_{Ci} maps all of Δ_ϵ^l diffeomorphically onto its image, where $X_C = (X_{C0}, \ldots, X_{Cl})$ is the Pareto allocation map of $\mathscr{E}_C$.

Clearly, the first step requires only l commodities. An additional l commodities are introduced in order to make it possible to distort the utility functions of $\mathscr{E}_C$ so as to realize T. One needs the additional l-degrees of freedom in order to be able to associate an arbitrary vector of net expenditures with each Pareto optimal allocation.

In Section 11, a sequence of economies is defined, $\mathscr{E}_M$, $M = 1, 2, \ldots$, each of which realizes T. The utility functions of $\mathscr{E}_M$ are perturbations of those of $\mathscr{E}_C$, and the perturbations become arbitrarily small as M goes to infinity. These economies correspond to the expansion of the Edgeworth box described in Section 6.

In Section 12, it is pointed out that for M large enough, the utility functions of $\mathscr{E}_M$ are differentiably strictly concave and monotone. This completes the proof.

8. $\mathscr{E}_A$ (AN ECONOMY WITH I COMMODITIES)

Let $\mathscr{E}_A = \{(u_{Ai}, \omega_{Ai}): i = 0, 1, \ldots, I\}$, where $\omega_{Ai} = (1, \ldots, 1) \in R^I$ for all i, and the u_{Ai}: $R^I \to R$ are defined by

$$u_{A0}(y) = -\frac{1}{2} \sum_{k=1}^{I} (I + 1 - y_k)^2 + \frac{I^3}{2},$$

and

$$u_{Ai}(y) = -\frac{2I-1}{4I} \sum_{k \neq i} (I + 1 - y_k)^2 - \frac{2I-1}{2I} (I + 1 - y_i)^2 + \frac{I^3}{2},$$

$$\text{for} \quad i \geq 1.$$

By symmetry, the price vector $(1, \ldots, 1)$ is an equilibrium price vector for $\mathscr{E}_A$ in the usual Walrasian sense of equilibrium. The corresponding equilibrium allocation is $\bar{x}$, defined as follows. $\bar{x}_{0k} = 1$ for all k. For $i = 1, \ldots, I$, $\bar{x}_{ii} = (I^2 + I - 1)/(2I - 1)$ and $\bar{x}_{ik} = (I - 1)/(2I - 1)$, if $k \neq i$. Since all components of $\bar{x}$ are positive, $\bar{x} \in \mathscr{P}(\mathscr{E}_A)$, so that $\mathscr{P}(\mathscr{E}_A)$ is not empty.

Let Λ_A and $X_A = (X_{A0}, \ldots, X_{AI})$ be, respectively, the relative marginal utilities and Pareto allocation functions of $\mathscr{E}_A$. $\Lambda_A(\mathscr{P}(\mathscr{E}_A))$ will denote the range of Λ_A. Let

$$\mathscr{P}_i(\mathscr{E}_A) = \{x_i \mid (x_0, \ldots, x_I) \in \mathscr{P}(\mathscr{E}_A)\}, \qquad i = 0, \ldots, I.$$

I now prove that

$$X_{Ai}: \Lambda_A(\mathscr{P}(\mathscr{E}_A)) \to \mathscr{P}_i(\mathscr{E}_A) \text{ is a } \mathscr{C}^\infty \text{ diffeomorphism,} \qquad \text{for all } i. \tag{8.1}$$

It is easy to see that X_{Ai} is a $\mathscr{C}^\infty$ function on $\Lambda_A(\mathscr{P}(\mathscr{E}_A))$. Hence in order to prove (8.1), it is sufficient to prove that if $x \in \mathscr{P}(\mathscr{E}_A)$ and $(\lambda_0, \ldots, \lambda_I) \in \Lambda_A(x)$, then $\lambda_1\lambda_0^{-1}, \ldots, \lambda_I\lambda_0^{-1}$ may be expressed algebraically as a function of $x_{i1}, \ldots, x_{iI}$. In fact, for $i = 0$ one has

$$\lambda_m\lambda_0^{-1} = 2 \sum_{k=1}^{I} \frac{I^2 - 1 + x_{0k}}{I(I + 1 - x_{0k})} - \frac{(2I - 1)(I^2 - 1 + x_{0m})}{I(I + 1 - x_{0m})}, \qquad m = 1, \ldots, I.$$

For $i = 1,\ldots, I$, one has

$$\lambda_i\lambda_0^{-1} = I^{-1}\left\{-1 + \frac{I(I+1)}{2I-1}\left[\sum_{k\neq i}\frac{4}{I+1-x_{ik}} - \frac{2I-3}{I+1-x_{ii}}\right]\right\}^{-1},$$

and

$$\lambda_m\lambda_0^{-1} = \left[1 - \frac{I(I+1)}{I+1-x_{ii}} - \frac{2I(I+1)}{I+1-x_{im}}\right]\lambda_i\lambda_0^{-1}, \qquad \text{for} \quad m \neq i.$$

These formulas may be checked by substitution into the following formulas for the x_{ik} in terms of the $\lambda_i\lambda_0^{-1}$.

$$x_{0k} = (I+1)\left[1 - \frac{I(2I-1)}{2I\sum_{m\neq k}\lambda_m\lambda_0^{-1} + I\lambda_k\lambda_0^{-1} + 2I-1}\right],$$
$$\text{for} \quad k = 1,\ldots, I.$$

For $i = 1,\ldots, I$,

$$x_{ik} = (I+1)\left[1 - \frac{2I^2\lambda_i\lambda_0^{-1}}{2I\sum_{m\neq k}\lambda_m\lambda_0^{-1} + I\lambda_k\lambda_0^{-1} + 2I-1}\right],$$
$$\text{for all} \quad k \neq i,$$

$$x_{ii} = (I+1)\left[1 - \frac{I^2\lambda_i\lambda_0^{-1}}{2I\sum_{m\neq i}\lambda_m\lambda_0^{-1} + I\lambda_i\lambda_0^{-1} + 2I-1}\right].$$

These formulas may in turn be checked by direct substitution into the utility functions u_{Ai} and verification that $(\lambda_0,\ldots,\lambda_I)\in\Lambda_A(x)$. That is, one must check that

$$\frac{\partial u_{Ai}(x_i)}{\partial y_k} = \lambda_i\lambda_0^{-1}\frac{\partial u_{A0}(x_0)}{\partial y_k}, \qquad \text{for} \quad i, k = 1,\ldots, I,$$

and

$$\sum_{i=0}^{I} x_{ik} = I+1, \qquad \text{for} \quad k = 1,\ldots, I.$$

This completes the proof of (8.1).

Note that by (8.1), the projection from $\mathscr{P}(\mathscr{E}_A)$ onto $\mathscr{P}_i(\mathscr{E}_A)$ is a $\mathscr{C}^\infty$ diffeomorphism. Hence there exists $\delta > 0$ so small that if $x\in\mathscr{P}(\mathscr{E}_A)$ and $|x_{0k} - \bar{x}_{0k}| \leqslant \delta$, for all k, then $|x_{ik} - \bar{x}_{ik}| \leqslant (10)^{-1}$, for all i and k. Let $\delta > 0$ be so small and let $N_A = \{x\in\mathscr{P}(\mathscr{E}_A)\colon |x_{0k} - \bar{x}_{0k}| \leqslant \delta,$ for all $k\}$. Let $N_{Ai} = \{x\in R_+^L \mid x = (x_0,\ldots, x_I)\in N_A\}$, $i = 0, 1,\ldots, I$, and let $\pi_{Ai}\colon N_A \to N_{Ai}$ be the projection. By (8.1),

$$\pi_{Ai} \text{ is a } \mathscr{C}^\infty \text{ diffeomorphism for all } i. \tag{8.2}$$

Use will be made of the following fact, which may be verified by a short calculation.

$$\text{If } x \in N_A, \text{ then } \frac{I}{2} < \frac{\partial u_{Ai}(x_i)}{\partial y_k} < 2I, \qquad \text{for all } i \text{ and } k. \tag{8.3}$$

9. $\mathscr{E}_B$ (AN ECONOMY WITH $2I$ COMMODITIES)

The utility functions of $\mathscr{E}_A$ are not monotone everywhere. This problem is corrected in this section at the same time that an additional I commodities are adjoined to $\mathscr{E}_A$.

The definition of $\mathscr{E}_B$ requires a good deal of new notation. Let $c\colon R \to R$ be any $\mathscr{C}^\infty$ function such that

$$\text{(i)}\quad c(0) = 0 \quad \text{and} \quad \frac{dc(0)}{dt} = 0,$$

$$\text{(ii)}\quad \frac{d^2c(s)}{dt^2} < 0, \qquad \text{for all } s,$$

$$\text{(iii)}\quad \left| \frac{dc(s)}{dt} \right| < \frac{I}{2}, \qquad \text{for all } s.$$

For $i = 0, 1, \ldots, I$ and $k = 1, \ldots, I$, let $b_{ik}\colon R \to R$ be any $\mathscr{C}^\infty$ function such that

$$\text{(i)}\quad \frac{db_{ik}(s)}{dt} > \frac{I}{2} \quad \text{and} \quad \frac{d^2b_{ik}(s)}{dt^2} < 0, \qquad \text{for all } s,\ i, \text{ and } k,$$

$$\text{(ii)}\quad b_{0k}(s) = -\frac{1}{2}(I+1-s)^2, \qquad \text{if } |s - \bar{x}_{0k}| \leqslant (10)^{-1} \text{ and } k = 1, \ldots, I,$$

$$\text{(iii)}\quad b_{ii}(s) = -\frac{2I-1}{2I}(I+1-s)^2, \qquad \text{if } |s - \bar{x}_{ii}| \leqslant (10)^{-1} \text{ and } i = 1, \ldots, I,$$

$$\text{(iv)}\quad b_{ik}(s) = -\frac{2I-1}{4I}(I+1-s)^2, \qquad \text{if } |s - \bar{x}_{ik}| \leqslant (10)^{-1},\ k \neq i, \text{ and } i = 1, \ldots, I.$$

Statement (8.3) implies that condition (i) of the definition of b_{ik} does not contradict conditions (ii), (iii), or (iv). (There is no question that the b_{ik} exist. See, for example, [6, Chap. 6].)

Let $v_i = e_i + e_{I+i}$ and $w_i = -e_i + e_{I+1}$, $i = 1, \ldots, I$, where e_k is the kth standard basis vector of R^{2I}. $(v_1, \ldots, v_I, w_1, \ldots, w_I)$ forms an orthogonal basis of R^{2I}. $u_{Bi}\colon R^{2I} \to R$, $i = 0, 1, \ldots, I$, is defined by

$$u_{Bi}\left(\sum_{k=1}^{I} s_k v_k + \sum_{k=1}^{I} t_k w_k\right) = \sum_{k=1}^{I} b_{ik}(s_k) + \sum_{k=1}^{I} c(t_k) + \frac{I^3}{2},$$

Since $c(0) = 0$, it follows that

$$u_{Bi}\left(\sum_{k=1}^{I} s_k v_k\right) = u_{Ai}(s_1,\ldots,s_I), \qquad \text{for all } s_1,\ldots,s_I \text{ and for all } i. \qquad (9.1)$$

The u_{Bi} are clearly $\mathscr{C}^\infty$ and differentiably strictly concave. It is next shown that they are differentiably strictly monotone. Fix i, $i = 0, 1,\ldots, I$, and let $y = \sum_{k=1}^{I} s_k v_k + \sum_{k=1}^{I} t_k w_k$. If $1 \leq k \leq I$, then

$$e_k = \frac{v_k - w_k}{2},$$

so that

$$\frac{\partial u_{Bi}(y)}{\partial y_k} = \frac{1}{2}\left(\frac{db_{ik}(s_k)}{ds} - \frac{dc(t_k)}{ds}\right) > \frac{1}{2}\left(\frac{I}{2} - \frac{I}{2}\right) = 0.$$

The inequality follows from condition (iii) on c and condition (i) on b_{ik}. Similarly, $\partial u_{Bi}(y)/\partial y_k > 0$, if $k > I$. This completes the proof that the u_{Bi} are differentiably strictly monotone.

Let $\omega_{Bi} = \sum_{i=1}^{I} v_i$, $i = 0, 1,\ldots, I$, and let $\mathscr{E}_B = \{(u_{Bi}, \omega_{Bi}) : i = 0, 1,\ldots, I\}$. $\mathscr{E}_B$ has a set of Pareto optima corresponding to the set N_A of optima for $\mathscr{E}_A$. This set will be denoted by N and is defined as follows. Let $f: R^I \to R^I \times R^I$ be defined by $f(y) = (y, y)$. Then,

$$N = \{(f(x_0),\ldots,f(x_I)) \in R^{2I(I+1)} \mid (x_0,\ldots,x_I) \in N_A\}.$$

Let $\zeta: N \to N_A$ be the map defined by $\zeta(x_0,\ldots,x_I) = (f^{-1}(x_0),\ldots,f^{-1}(x_I))$.

It is now shown that

$$N \text{ consists of Pareto optima for } \mathscr{E}_B \text{ and } \Lambda_B = \Lambda_A \circ \zeta, \qquad (9.2)$$

where $\Lambda_B: N \to \Delta^I$ is the restriction to N of the relative marginal utilities map of $\mathscr{E}_B$. Clearly, the allocations in N are feasible for $\mathscr{E}_B$. Hence (9.2) follows from the following.

If $x = (x_0,\ldots,x_I) \in N$, then

$$\begin{aligned}
(\Lambda_{A0}\circ\zeta(x))^{-1} Du_{B0}(x_0) &= (\Lambda_{A0}\circ\zeta(x))^{-1} Du_{B0}\left(\sum_{k=1}^{I} x_{0k} v_k\right)\\
&= \frac{1}{2}(\Lambda_{A0}\circ\zeta(x))^{-1}\sum_{k=1}^{I}\frac{\partial u_{A0}(f^{-1}(x_0))}{\partial y_k} v_k\\
&= \frac{1}{2}(\Lambda_{Ai}\circ\zeta(x))^{-1}\sum_{k=1}^{I}\frac{\partial u_{Ai}(f^{-1}(x_i))}{\partial y_k} v_k\\
&= (\Lambda_{Ai}\circ\zeta(x))^{-1} Du_{Bi}(x_i), \qquad i = 1,\ldots, I.
\end{aligned}$$

The second and fourth equalities above follow from the fact that $dc(0)/ds = 0$. The third equality follows from the fact that $\zeta(x)$ is a Pareto optimal allocation for $\mathscr{E}_A$.

Let $N_i = \{x_i \in R_+^{2I} \mid (x_0, \ldots, x_I) \in N\}$ and let $\pi_i \colon N \to N_i$ be the projection, $i = 0, 1, \ldots, I$. By (8.2)

$$\pi_i \text{ is a } \mathscr{C}^\infty \text{ diffeomorphism, for all } i. \tag{9.3}$$

Note that by (8.3),

$$\text{if } x \in N, \text{ then } \frac{I}{4} < \frac{\partial u_{Bi}(x_i)}{\partial y_k} < I, \qquad \text{for all } i \text{ and } k \tag{9.4}$$

10. $\mathscr{E}_C$ (MATHEMATICAL CONTROL OF EXPENDITURES IS EXTENDED TO ALL OF Δ_ϵ^I

$\mathscr{E}_C$ will be of the form $\{(h \circ u_{Bi}, \omega_{Bi}) \colon i = 0, 1, \ldots, I\}$, where $h \colon R \to R$ is $\mathscr{C}^\infty$ and is differentiably strictly concave and monotonely increasing. It follows that $\mathscr{P}(\mathscr{E}_C) = \mathscr{P}(\mathscr{E}_B)$, so that $N \subset \mathscr{P}(\mathscr{E}_C)$. h will be such that $h'(t)$ decreases abruptly as t crosses zero. Since $u_{Bi}(f(\bar{x}_i))$ $(= u_{Ai}(\bar{x}_i)) = 0$, for all i, this property means that as x varies over N, the relative marginal utilities corresponding to x vary over a wide range in Δ^I. h will be chosen so that

$$\Delta_\epsilon^I \subset \Lambda_C(\text{int } N), \tag{10.1}$$

where Λ_C is the relative marginal utilities map of $\mathscr{E}_C$ and int N is the topological interior of N.

In defining h, use will be made of the map $u_B \colon N \to R^{I+1}$ defined by $u_B(x) = (u_{B0}(x_0), \ldots, u_{BI}(x_I))$. u_B is a $\mathscr{C}^\infty$ diffeomorphism onto its image [13, p. 5]. Let $\bar{y} = (f(\bar{x}_0), \ldots, f(\bar{x}_I))$. $u_B(\bar{y}) = 0$, so that for $\alpha > 0$, $Z = \{(t_0, \ldots, t_I) \in u_B(N) \mid t_i \leqslant \alpha, \text{ for all } i\}$ is a neighborhood of $u_B(\bar{y})$ in $u_B(\mathscr{P}_B)$. If α is sufficiently small, then

$$\begin{gathered} Z \text{ is diffeomorphic to } \Delta^I \text{ and the boundary of } Z \text{ is} \\ \{(t_0, \ldots, t_I) \in Z \mid t_i = \alpha, \text{ for some } i\}. \end{gathered} \tag{10.2}$$

Assume that α is so small.

Let $h \colon R \to R$ be any $\mathscr{C}^\infty$ function satisfying the following conditions:

(i) $h(t) = t, \quad \text{for} \quad t \leqslant 0,$

(ii) $\dfrac{dh(t)}{ds} > 0 \quad \text{and} \quad \dfrac{d^2h(t)}{ds^2} < 0, \quad \text{for all} \quad t > 0,$

(iii) $\dfrac{dh(\alpha)}{ds} = \dfrac{\epsilon}{4}.$

Let $u_{Ci} = h \circ u_{Bi}$, $i = 0, 1, \ldots, I$, and let $\mathscr{E}_C = \{(u_{Ci}, \omega_{Bi}): i = 0, 1, \ldots, I$

We now prove (10.1). Let $\Psi: \Lambda_C \circ u_B^{-1}: Z \to \Delta^I$. Statement (10.1) follow easily from (10.2) and the following.

$$\text{If} \quad (t_0, \ldots, t_I) \in Z \qquad \text{and} \qquad t_i = \alpha, \qquad \text{then} \quad \Psi_i(t_0, \ldots, t_I) < \epsilon. \tag{10.3}$$

Suppose that $(t_0, \ldots, t_I) \in Z$ and that $t_i = \alpha$. Let $x = u_B^{-1}(t_0, \ldots, t_I) \in N$. Then,

$$\frac{\partial u_{Ci}(x_i)}{\partial y_1} = \frac{dh(u_{Bi}(x_i))}{ds} \frac{\partial u_{Bi}(x_i)}{\partial y_1} = \frac{dh(\alpha)}{ds} \frac{\partial u_{Bi}(x_i)}{\partial y_1} < \frac{\epsilon I}{4}.$$

The inequality above follows from (9.4) and from condition (iii) on h.

Since $\bar{y}$ is a Pareto optimum for $\mathscr{E}_B$ and since $u_B(\bar{y}) = 0$, it follows that $u_{Bj}(x_j) < 0$, for some consumer j. For this j,

$$\frac{\partial u_{Cj}(x_j)}{\partial y_1} = \frac{dh(u_{Bj}(x_j))}{ds} \frac{\partial u_{Bj}(x_j)}{\partial y_1} = \frac{\partial u_{Bj}(x_j)}{\partial y_1} > \frac{I}{4}.$$

The inequality here follows from (9.4). The second equality follows from condition (i) on h.

From the above inequalities, it follows that

$$\Psi_i(t_0, \ldots, t_I) = \left[\sum_{k=0}^{I} \frac{\partial u_{Ck}(x_k)}{\partial y_1}\right]^{-1} \frac{\partial u_{Ci}(x_i)}{\partial y_1} < \frac{4}{I}\left(\frac{\epsilon I}{4}\right) = \epsilon.$$

This proves (10.3) and hence (10.1).

11. $\mathscr{E}_M$, $M = 1, 2, \ldots$ (T IS REALIZED)

Still more notation is needed in order to define $\mathscr{E}_M$. Let $N_\epsilon = \Lambda_C^{-1}(\Delta_\epsilon^I)$ and let $N_{\epsilon i} = \pi_i(N_\epsilon)$, $i = 0, 1, \ldots, I$, where π_i is as in (9.3). Let τ_C be the net expenditure function of $\mathscr{E}_C$, and let σ_{ik}, for $i = 0, 1, \ldots, I$ and $k = 1, \ldots, I$, be any real-valued $\mathscr{C}^2$ function defined on the linear span of $v_1, \ldots, v_I$ and such that

(i) $\sigma_{ik}(y) = (\Lambda_{Ci} \circ \pi_i^{-1}(y))[\tau_{Ck} \circ \Lambda_C \circ \pi_i^{-1}(y) - T_k \circ \Lambda_C \circ \pi_i^{-1}(y)]$, if $y \in N_{\epsilon i}$, and

(ii) $\sigma_{ik}(y) = 0$, if $y \notin N_i$.

Such a function exists, since (by (10.1)) $N_{\epsilon i}$ is a closed set contained in the interior of N_i. Observe that

$$\text{if} \quad x = (x_0, \ldots, x_I) \in N_\epsilon, \quad \text{then}$$

$$(\Lambda_{Ci}(x))^{-1} \sigma_{ik}(x_i) = (\Lambda_{C0}(x))^{-1} \sigma_{0k}(x_0), \qquad \text{for all } i \text{ and } k. \tag{11.1}$$

Finally, let $g\colon R \to R$ be any $\mathscr{C}^2$ function such that $g(0) = 0$, $dg(0)/(ds) = 1$, and $g(t) = 0$ if $|t| \geqslant 1$.

Let $\mathscr{E}_M = \{(u_{Mi}, \omega_{Mi})\colon i = 0, 1,\ldots, I\}$, where the ω_{Mi} and u_{Mi} are defined as follows.

$$\omega_{M0} = M\sum_{j=1}^{I} v_j - M\sum_{j=1}^{I} w_j + \omega_{C0},$$

$$\omega_{Mi} = M\sum_{j=1}^{I} v_j + Mw_i + \omega_{Ci}, \qquad \text{for} \quad i = 1,\ldots, I.$$

$$u_{Mi}\left(\sum_{j=1}^{I} (s_j + M)\, v_j + \sum_{j=1}^{I} t_j w_j\right)$$

$$= u_{Ci}\left(\sum_{j=1}^{I} s_j v_j + \sum_{j=1}^{I} t_j w_j\right) + M^{-1}\sum_{k=1}^{I} \sigma_{ik}\left(\sum_{j=1}^{I} s_j v_j\right) g(t_k),$$

$$\text{for} \quad i = 0, 1,\ldots, I.$$

Note that the functions u_{Ci} are $\mathscr{C}^2$.

Let

$$N_{\epsilon M} = \left\{\left(x_0 + M\sum_{j=1}^{I} v_j ,\ldots, x_I + M\sum_{j=1}^{I} v_j\right) \,\middle|\, (x_0,\ldots, x_I) \in N_\epsilon\right\}.$$

It is next shown that

$N_{\epsilon M}$ consists of first-order optima for $\mathscr{E}_M$, and if $x \in N_\epsilon$, then

$$\Lambda_M\left(x_0 + M\sum_{j=1}^{I} v_j ,\ldots, x_I + M\sum_{j=1}^{I} v_j\right) = \Lambda_C(x), \tag{11.2}$$

where $\Lambda_M\colon N_{\epsilon M} \to \Delta^I$ gives the relative marginal utilities in $\mathscr{E}_M$. Equation (11.2) follows immediately from the following.

If $x \in N_\epsilon$, then

$$(\Lambda_{C0}(x))^{-1}\, Du_{C0}\left(x_0 + M\sum_{j=1}^{I} v_j\right)$$

$$= (\Lambda_{C0}(x))^{-1}\, Du_{C0}(x_0) + \frac{(\Lambda_{C0}(x))^{-1}}{2M}\sum_{k=1}^{I} \sigma_{0k}(x_0)\, w_k$$

$$= (\Lambda_{Ci}(x))^{-1}\, Du_{Ci}(x_i) + \frac{(\Lambda_{Ci}(x))^{-1}}{2M}\sum_{k=1}^{I} \sigma_{ik}(x_i)\, w_k$$

$$= (\Lambda_{Ci}(x))^{-1}\, Du_{Mi}\left(x_i + M\sum_{j=1}^{I} v_j\right). \tag{11.3}$$

The first equality makes use of the fact that $g(0) = 0$. The second equality above follows from the fact that x is a first-order optimum for $\mathscr{E}_C$ and from (11.1).

Note that (11.2) implies that

$$\Delta_\epsilon^I = \Lambda_M(N_{\epsilon M}). \tag{11.4}$$

It is next shown that

net expenditures in $\mathscr{E}_M$ at allocations in $N_{\epsilon M}$
are given by $T \circ \Lambda_M \colon N_{\epsilon M} \to R^{I+1}$. (11.5)

Let $x = (x_0, \ldots, x_I) \in N_\epsilon$. If $i = 1, \ldots, I$, the net expenditure of agent i at $(x_0 + M \sum_{j=1}^I v_j, \ldots, x_I + M \sum_{j=1}^I v_j) \in N_{\epsilon M}$ is

$$\begin{aligned}
&(\Lambda_{Ci}(x))^{-1} Du_{Mi}\left(x_i + M \sum_{j=1}^I v_j\right) \cdot \left(x_i + M \sum_{j=1}^I v_j - \omega_{Mi}\right) \\
&\quad = (\Lambda_{Ci}(x))^{-1} \left[Du_{Ci}(x_i) + (2M)^{-1} \sum_{k=1}^I \sigma_{ik}(x_i)\, w_k\right] \cdot (x_i - Mw_i - \omega_{Ci}) \\
&\quad = (\Lambda_{Ci}(x))^{-1} Du_{Ci}(x_i) \cdot (x_i - \omega_{Ci}) - (\Lambda_{Ci}(x))^{-1} \sigma_{ii}(x_i) \\
&\quad = \tau_{Ci} \circ \Lambda_C(x) - (\tau_{Ci} \circ \Lambda_C(x) - T_i \circ \Lambda_C(x)) \\
&\quad = T_i \circ \Lambda_C(x).
\end{aligned}$$

In the second equality above, use is made of the facts that $Du_{Ci}(x_i) \cdot w_i = 0$ and $w_k \cdot (x_i - \omega_{Ci}) = 0$, for all k.

It has been proved that the expenditures of the last I agents are given by $T_i \circ \Lambda_M$. Since the net expenditures add to zero, it is also true that the net expenditure of agent 0 is given by $T_0 \circ \Lambda_M$. This proves (11.5).

If the utility functions of $\mathscr{E}_M$ were differentiably strictly concave and monotone, then $\mathscr{E}_M$ would have a well-defined net expenditure function, and (11.4) and (11.5) would imply that this function equalled T on Δ_ϵ^I.

12. CONCAVITY AND MONOTONICITY

It remains to be proved that

if M is sufficiently large, then u_{Mi} is
differentiably strictly concave and monotone for all i. (12.1)

Let $\hat{u}_{Mi}\colon R^{2I} \to R$ be defined by $\hat{u}_{Mi}(x) = u_{Mi}(x + M \sum_{j=1}^{I} v_j)$. Then,

$$\hat{u}_{Mi}\left(\sum_{j=1}^{I} s_j v_j + \sum_{j=1}^{I} t_j w_j\right)$$

$$= u_{Ci}\left(\sum_{j=1}^{I} s_j v_j + \sum_{j=1}^{I} t_j w_j\right) + M^{-1} \sum_{k=1}^{I} \sigma_{ik} \left(\sum_{j=1}^{I} s_j v_j\right) g(t_k).$$

Observe that the function

$$H\left(\sum_{j=1}^{I} s_j v_j + \sum_{j=1}^{I} t_j w_j\right) \equiv \sum_{k=1}^{I} \sigma_{ik} \left(\sum_{j=1}^{I} s_j v_j\right) g(t_k)$$

is zero outside the compact set

$$\left\{\sum_{j=1}^{I} s_j v_j + \sum_{j=1}^{I} t_j w_j \,\middle|\, \sum_{j=1}^{I} s_j v_j \in N_i \text{ and } | t_j | \leq 1, \text{ for all } j\right\}.$$

Since $\hat{u}_{Mi} = u_{Ci} + M^{-1}H$ and u_{Ci} is differentiably strictly concave and monotone, it follows by an obvious compactness and continuity argument that if M is sufficiently large then $\hat{u}_{Mi}$ is also differentiably strictly concave and monotone. Obviously, if $\hat{u}_{Mi}$ is concave and monotone, then so is u_{Mi}. This proves (12.1) and hence the theorem. Q.E.D.

ACKNOWLEDGMENT

The author is most grateful to Andreu Mas-Colell for encouragement and advice. It was he who suggested that I make my point by proving a Sonnenschein-like theorem, as is done here.

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JOURNAL OF ECONOMIC THEORY 22, 395–406 (1980)

Expectations Equilibrium with Expectations Conditioned on Past Data*

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Received January 16, 1978

I. INTRODUCTION

The theory of economic prediction has recently been complicated by the discovery that if current predictions of future events are based on current market variables, statistically correct prediction can be inconsistent with market equilibrium, even in otherwise well-behaved (classical) economies.[1] For example, suppose that the future variable to be predicted is systematically related to a current exogenous variable which is not observed directly. Suppose that the exogenous variable can take two possible values, a and b; and that predictions are to be based on a currently determined equilibrium price. If the price differs between the states a and b, agents will have different expectations in the two states. However, it may happen that the equilibrium prices for the two excess demand functions are the same, so that no distinction between the states can be observed. Then expectations must be the same in both states. But then the excess demand functions will not be the same as those mentioned above, and the equilibrium prices for the new excess demand functions may differ between the two states, making the distinction observable. Thus there is no equilibrium with prediction.

In response to this difficulty it is natural to suppose that predictions are based on past rather than current market data. Hellwig and Rothschild [2] have presented a securities market model in which, under the assumption of constant absolute risk-aversion, equilibrium exists when expectations are conditioned on previous prices. Since an agent's current demand must be sensitive to current prices, it may seem highly artificial to suppose that his current expectations are not. However, if agents rely on published predictions,

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* This research was supported by a Summer Research Appointment at the University of Minnesota Graduate School and NSF Grant SOC-77-07852. We would like to thank the referees for several useful suggestions.

[1] As far as we are aware, this problem was first mentioned by Radner [7, p. 5], and the first examples were constructed by Green [1] and Kreps [5].

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for example, some lag in the formation of expectations is inevitable. In the macroeconomic rational expectations literature, it is conventional to suppose that current expectations are not directly influenced by current endogenous variables (e.g., [9, pp. 18–19]).

Unfortunately, constraining current expectations to be insensitive to current market variables, besides seeming unnatural, can easily worsen the existence problem. This paper develops a very simple three-period model in which agents who are not exogenously informed of the state rely on previous market data. Thus, in period 1 they have no information, and in period 2 their expectations about period 3 are based on data generated as some function of first period market variables. The main result is that, under the assumption that these data functions are continuous, the only data functions which admit the general existence of equilibrium are constant functions. Stating the conclusion somewhat differently, the general existence of an equilibrium with prediction, which will be called an *expectations equilibrium*, cannot be assured for any nontrivial previous market data. This result may be contrasted with the results in [3, 4] which indicate that if current expectations are conditioned on current market data, there do exist some nonconstant data functions which admit the general existence of expectations equilibrium.

II. THE MODEL

This section describes a model of a three-period exchange economy. In period 1 each agent receives an endowment of a current consumption good and a durable good, which it will be convenient to call money, which is not consumed until period 3. In period 2, each agent receives an endowment of current consumption but no additional endowment of money. Each agent's utility is additively separable over time. There are two states of the world. The state may influence an agent's consumption endowment and final period utility function. (For simplicity, each agent's money endowment and first two-period utility functions will be assumed to be state independent.) In each of the first two periods there are markets for exchanging the consumption good and money. In the final period each agent consumes his holding of money. There are no markets for contingent claims. Agents are permitted to have different information about the state of the world. In particular, an agent may either know the state or the objective probability of the state. An agent who does not know the state in period 1 may come to know the state in period 2. Each agent is, however, assumed to know the market clearing price in each state and period.

The model can be viewed as a two-period slice of an ongoing monetary exchange economy where the final-period utility of money can be viewed as an indirect utility function. Nonmonetary interpretations are also possible.

2.1. DEFINITIONS. Periods are indexed by the subscript t, $t = 1, 2, 3$; and states are indexed by the subscript s, $s = a, b$. There are N agents indexed by the superscript i. The consumption good is denoted by c, money by m, with the appropriate subscripts and superscripts. Since consumption in period 3 is the agent's final money holding at the end of period 2, we will denote third-period consumption by m_{s2}^i.

Agents are partially characterized by their endowments and their utility functions. Denote by $e^i = (\bar{c}_{a1}^i, \bar{c}_{b1}^i, \bar{m}^i; \bar{c}_{a2}^i, \bar{c}_{b2}^i; u_1{}^i, u_2{}^i; u_{a3}^i, u_{b3}^i)$ a tuple of the ith agent's characteristics. The first five components represent endowments, and the utility function in state s is given by $u_1{}^i + u_2{}^i + u_{s3}^i$. It will be assumed that $\bar{c}_{st}^i > 0$ for all i, s, t, and $\bar{m}^i > 0$ for all i. For each i, each of the last four components is a real-valued function u, defined on the strictly positive real axis, satisfying

(i) $u \in C^\infty$ and $Du(c) > 0$, $D^2u(c) < 0$ for all $c > 0$, and

(ii) $\lim_{c \to 0} Du(c) = \infty$.

The set of such tuples for the ith agent is denoted E^i. An element of E^i partially characterizes the ith agent.

2.2. *Remarks.* The endowments $(\bar{c}_{s1}^i, \bar{m}^i)$ are realized before trade in period 1, so if $\bar{c}_{a1}^i \neq \bar{c}_{b1}^i$, the ith agent can infer the state directly from his endowment. If $\bar{c}_{a1}^i = \bar{c}_{b1}^i$, we will assume that the ith agent has no exogenous source of information, so his first-period trades must be chosen in ignorance of the state. If $\bar{c}_{a1}^i = \bar{c}_{b1}^i$ and $\bar{c}_{a2}^i \neq \bar{c}_{b2}^i$, the agent would become exogenously informed at the beginning of period 2. For simplicity, we will remove this case by assuming that for each i, if $\bar{c}_{a1}^i = \bar{c}_{b1}^i$ then $\bar{c}_{a2}^i = \bar{c}_{b2}^i$. Thus, with respect to exogenous information, each agent is either informed or uninformed in both of the first two periods. These assumptions are introduced formally in the definitions below.

2.3. *Initial information.* Let $I = \{0, 1\}$, with generic element η_1. If $\eta_1{}^i = 1$, the ith agent is said to be *initially informed*; and if $\eta_1{}^i = 0$, the ith agent is said to be *initially uninformed.* Let $E = \{[\pi; (\eta_1{}^1, e^1), \ldots, (\eta_1{}^N, e^N)] \in (0, 1) \times \prod_i (I \times E^i)$: for each i, $\eta_1{}^i = 0$ only if $\bar{c}_{a1}^i = \bar{c}_{b1}^i$ and $\bar{c}_{a2}^i = \bar{c}_{b2}^i$; and $\eta_1{}^i = 1$ only if $\bar{c}_{a1}^i \neq \bar{c}_{b1}^i\}$, with generic element e. The set E is the set of environments, and the first coordinate, π, of an environment represents the probability of state a.

III. EXPECTATIONS EQUILIBRIUM

However expectations are formed, an equilibrium will somehow associate prices, p_{s1}/p_m and p_{s2}/p_m, of current consumption in terms of money in periods 1 and 2, respectively, with each state s. As in [7], each agent will be *assumed to know the joint distribution of states and prices. Then if $\eta_1^i = 1$, the ith agent knows in period 1 the price he will face in period 2 and his final-period utility function. Such an agent will thus choose his demands in periods 1 and 2 in each state as though he were participating in a three-*commodity static exchange environment, with the endowment $(\bar{c}^i_{s1}, \bar{c}^i_{s2}, \bar{m}^i)$ and the utility function $u_1^i(c^i_{s1}) + u_i^2(c^i_{s2}) + u^i_{s3}(m^i_{s2})$. These remarks are made precise in Section 3.1 below.

An initially uninformed agent does not know in period 1 whether he will face the price p_{a2} or P_{b2} in period 2. Thus if $\eta_1^i = 0$, agent i is faced with a stochastic dynamic programming problem in period 1. In period 2, the ith agent's demand will depend on whether he has become informed by data generated in period 1. The behavior of an initially uninformed agent is derived in Section 3.2 below, where the variable η_2^i is introduced to represent second period information.

3.1. *The Behavior of Initially Informed Agents*

Let P be the set of positive real numbers. Given $(p_{s1}, p_{s2}, p_m) \in P^3$, which denotes the vector of prices of consumption goods in state s in each of the first two periods and the price of money, an informed agent in state s will choose (c_{s1}, c_{s2}, m_{s2}) (suppressing the superscript i) for each s so as to

$$\begin{aligned}&\text{maximize } u_1(c_{s1}) + u_2(c_{s2}) + u_{s3}(m_{s2})\\ &\text{subject to } p_{s1}c_{s1} + p_{s2}c_{s2} + p_m m_{s2} = p_{s1}\bar{c}_{s1} + p_{s2}\bar{c}_{s2} + p_m\bar{m}.\end{aligned}$$

(Writing the budget constraint as an equality and requiring consumption to be strictly positive is not a restriction for the class of utility functions considered. All maxima will be interior in any event.)

It is convenient for what follows to introduce the variable

$$m_{s1} = \frac{p_{s1}(\bar{c}_{s1} - c_{s1}) + p_m\bar{m}}{p_m},$$

which is just the net saving in the first period. The budget constraint can then be written

$$p_{s1}c_{s1} + p_m m_{s1} = p_{s1}\bar{c}_{s1} + p_m\bar{m} \quad \text{and} \quad p_{s2}c_{s2} + p_m m_{s2} = p_{s2}\bar{c}_{s2} + p_m m_{s1}.$$

A negative value of m_{s1} indicates borrowing in the first period.

3.2. *The Behavior of Initially Uninformed Agents*

For each i let $E^{0i} = \{(\eta_1^i, \eta_2^i, e^0) \in I^2 \times E^i : \eta_1^i = 0, \bar{c}^i_{a1} = \bar{c}^i_{b1}, \text{and } \bar{c}^i_{a2} = \bar{c}^i_{b2}\}$. Suppressing redundant coordinates, the typical element of E^{0i} is $(\eta_2^i, e^i) = (\eta_2^i; \bar{c}_1^i, \bar{m}^i, \bar{c}_2^i; u_1^i, u_2^i u^i_{a3}, u^i_{b3})$.

Given prices $(p_{a1}, p_{b1}; p_{a2}, p_{b2}; p_m)$, in state s, if $\eta_2^i = 1$ then the ith agent will choose (c_{s1}, m_{s1}) (again supressing superscripts) so as to

$$\begin{aligned}
&\underset{c_{s1}, m_{s1}}{\text{maximize}}\ u_1(c_{s1}) + \pi[\max_{c,m} \{u_2(c) + u_{a3}(m): \\
&\qquad\qquad\qquad\qquad p_{a2}c + p_m m = p_{a2}\bar{c}_2 + p_m m_{s1}\}] \\
&\qquad + (1 - \pi)[\max_{c,m} \{u_2(c) + u_{b3}(m): \\
&\qquad\qquad\qquad\qquad p_{b2}c + p_m m = p_{b2}\bar{c}_2 + p_m m_{s1}\}] \\
&\text{subject to } p_{s1}c_{s1} + p_m m_{s1} = p_{s1}\bar{c}_1 + p_m \bar{m}.
\end{aligned}$$

Then in the second period, since $\eta_2 = 1$ the agent becomes informed and chooses (c_{s2}, m_{s2}) to

$$\begin{aligned}
&\text{maximize } u_2(c_{s2}) + u_{s3}(m_{s2}) \\
&\text{subject to } p_{s2}c_{s2} + p_m m_{s2} = p_{s2}\bar{c}_2 + p_m m_{s1}\,.
\end{aligned}$$

If on the other hand $\eta_2 = 0$ and the agent never becomes informed, then in state s, (c_{s1}, m_{s1}) will be chosen to

$$\begin{aligned}
&\underset{c_{s1}, m_{s1}}{\text{maximize}}\ u_1(c_{s1}) + \pi[\max_{c,m} \{u_2(c) + \pi u_{a3}(m) + (1 - \pi)\, u_{b3}(m): \\
&\qquad\qquad\qquad\qquad p_{a2}c + p_m m = p_{a2}\bar{c}_2 + p_m m_{s1}\}] \\
&\qquad + (1 - \pi)[\max_{c,m} \{u_2(c) + \pi u_{a3}(m) + (1 - \pi)\, u_{b3}(m): \\
&\qquad\qquad\qquad\qquad p_{b2}c + p_m m = p_{b2}\bar{c}_2 + p_m m_{s1}\}] \\
&\text{subject to } p_{s1}c_{s1} + p_m m_{s1} = p_{s1}\bar{c}_1 + p_m \bar{m}.
\end{aligned}$$

In the second period (c_{s2}, m_{s2}) is chosen to

$$\begin{aligned}
&\text{maximize } u_2(c_{s2}) + \pi u_{a3}(m_{s2}) + (1 - \pi)\, u_{b3}(m_{s2}) \\
&\text{subject to } p_{s2}c_{s2} + p_m m_{s2} = p_{s2}\bar{c}_2 + p_m m_{s1}\,.
\end{aligned}$$

3.3. *Data Structures*

Let $Y = \{(y^1, \ldots, y^N) \in R^N: \sum_i y^i = 0\}$ be the set of possible net trades of the consumption good in a given period. For each i let D^i be a Hausdorff space. A *data structure* is a function $f: P \times Y \to \prod_i D^i$. A data structure f is said to be a continuous data structure if it is a continuous function, and is said to be trivial if it is a constant function. A data structure associates with each first-period price ratio p_{s1}/p_m and set of net trades a datum for each agent.

3.4. *Expectations Equilibrium*

An *expectations equilibrium* for an environment $e \in E$ and a data structure f is an element

$$(\eta_2 ; p_{a1}, y_{a1}, p_{a2}, y_{a2} ; p_{b1}, y_{b1}, p_{b2}, y_{b2}) \in I^N \times (P \times Y \times P \times Y)^2$$

such that

(i) given η_1^i and η_2^i each $y_{st}^i = c_{st}^i - \bar{c}_{st}^i$ is chosen according to Sections 3.1 and 3.2.

(ii) for each i, $\eta_2^i = 1$ if and only if $f^i(p_{a1}, y_{a1}) \neq f^i(p_{b1}, y_{b1})$ or $\eta_1^i = 1$.

(The prices have been normalized so that $p_m = 1$.) A data structure f is said to be *admissible* for the class of environments E if for each $e \in E$, there exists an expectations equilibrium for (e, f).

3.5. *Remarks*

This notion of equilibrium is seemingly artificial in that agents know the joint distribution of equilibrium prices but may not use that information in the first period to infer the state even if $p_{a1} \neq p_{a2}$, yet in period 2 they must be able to distinguish $f^i(p_{a1}, y_{a1})$ and $f^i(p_{b1}, y_{b1})$. Also, uninformed agents must guess whether they will become informed in period 2 since Section 3.2 requires them to know η_2^i in period 1. In equilibrium these guesses will be correct.

The following theorem, which is the main result of the paper, states that the only admissible continuous data structures are trivial. In [3, Theorem 3.5] admissibility is characterized without the continuity hypothesis in a model in which current expectations are based on current data. It seems likely that the continuity hypothesis can be dropped from the present theorem also.

The nonexistence of expectations equilibria for nontrivial data structures can be motivated in the following way. Given an environment e and a data structure f, choose (p_{a1}, y_{a1}) and (p_{b1}, y_{b1}) in $P \times Y$, and let η_2 be consequently determined by f. Given η_2, and the second period distribution of money determined by initial endowments and the above first-period prices and trades, second-period equilibrium prices p_{a2} and p_{b2} can be obtained for e. Given η_2, p_{a2}, and p_{b2}, first-period equilibria (p'_{a1}, y'_{a1}), and (p'_{b2}, y'_{b2}) can be derived. An expectations equilibrium is a fixed point of this process. However, the presence of the discontinuous variable η_2 suggests that if f is nontrivial, a fixed point may fail to exist.

3.6. THEOREM. *A continuous data structure is admissible if and only if it is trivial.*

IV. PROOF OF THEOREM 3.6

The sufficiency of triviality is immediate. The necessity follows from Proposition 4.3 below. The failure of existence of expectations equilibrium for nontrivial data structures stems from the lack of a consistent information structure. This motivates the following definition.

4.1. DEFINITION. Let $(e, \eta_2) \in E \times I^N$. An *equilibrium* for (e, η_2) is a tuple $(p_{a1}, y_{a1}, p_{a2}, y_{a2}; p_{b1}, y_{b1}, p_{b2}, y_{b2}) \in (P \times Y \times P \times Y)^2$, where $(y^i_{a1}, y^i_{a2}, y^i_{b1}, y^i_{b2})$ are chosen according to Sections 3.1 and 3.2 given the values of $\eta_1{}^i$ and $\eta_2{}^i$.

The following Lemma is an immediate consequence of the above definition and the definition of expectations equilibrium.

4.2. LEMMA. *Let $e \in E$ with $\eta_1{}^i = 0$ and $\eta_1{}^j = 1$ for all $j \neq i$, let $\eta_2 \in I^N$ with $\eta_2{}^i = 1$, and let $\eta'_2 \in I^N$ with $\eta'^i_2 = 0$. Suppose that (e, η_2) has unique equilibrium $(p_{a1}, y_{a1}, p_{a2}, y_{a2}; p_{b1}, y_{b1}, p_{b2}, y_{b2})$ with $(p_{a1}, y_{a1}) = (p_{b1}, y_{b1})$. If f is an admissible data structure then (e_2, η'_2) has equilibrium $(p'_{a1}, y'_{a1}, p'_{a2}, y'_{a2}; p'_{b1}, y'_{b1}, p'_{b2}, y'_{b2})$ with $f^i(p'_{a1}, y'_{a1}) = f^i(p'_{b1}, y'_{b1})$.*

4.3. PROPOSITION. *Let f be a data structure such that (e, f) has an expectations equilibrium for each $e \in E$. Then for each i, if f^i is continuous then f^i is constant on $P \times Y$.*

Proof. Since E is symmetric with respect to agents, it suffices to prove the assertion for $i = 1$, so suppose f^1 is continuous. Let $L = \{e \in E\colon \eta_1{}^1 = 0$ and $\eta_1{}^i = 1$ for $i > 1$; and for each i there are positive numbers $\alpha_1{}^i$, $\alpha_2{}^i$, α^i_{a3}, α^i_{b3} such that $u_1{}^i(\cdot) = \alpha_1{}^i \ln(\cdot)$, etc.$\}$; that is, L is the set of economies in which the first agent is the only initially uninformed agent and the utilities are all linear in logarithms, i.e., Cobb–Douglas. For $e \in L$, since $\eta_1{}^i = 1$ for each $i > 1$, only the value of $\eta_2{}^1$ influences the equilibria of (e, η_2). Accordingly, notation will be saved by replacing both η_2 and $\eta_2{}^1$ by the (0, 1)-valued variable η. Variables associated with values of η will be preceded by a superscript.

Let $(p_1, y_1) \in P \times Y$. In Section 4.4 it will be described how there can be chosen, for each i, positive numbers $(\bar{c}^i_{a1}, \bar{c}^i_{b1}, \bar{c}^i_{a2}, \bar{c}^i_{b2}; \overline{m}^i; \alpha_1{}^i, \alpha_2{}^i, \alpha^i_{a3}, \alpha^i_{b3})$, with $\bar{c}^1_{at} = \bar{c}^1_{bt}$, $t = 1, 2$, such that

(A) The three-commodity static exchange environment with endowments $(\bar{c}^i_{a1}, \bar{c}^i_{a2}, \overline{m}^i)$ and utility functions $v_a{}^i(c_1, c_2, c_3) = \alpha_1{}^i \ln c_1 + \alpha_2{}^i \ln c_2 + \alpha^i_{a3} \ln c_3$ has a (unique) equilibrium $({}^1p_{a1}, {}^1y_{a1}, {}^1p_{a2}, {}^1y_{a2})$ with $({}^1p_{a1}, {}^1y_{a1}) = (p_1, y_1)$;

(B) The three-commodity static exchange environment with endowments $(\bar{c}^i_{b1}, \bar{c}^i_{b2}, \bar{m}^i)$ and utility functions $v_b{}^i(c_1, c_2, c_3) = \alpha_1{}^i \ln c_1 + \alpha_2{}^i \ln c_2 + \alpha^i_{b3} \ln c_3$ has a (unique) equilibrium $({}^1p_{b1}, {}^1y_{b1}, {}^1p_{b2}, {}^1y_{b2})$ with $({}^1p_{b1}, {}^1y_{b1}) = (p_1, y_1)$.

(C) The three-commodity static exchange environment with endowments $(\bar{c}^i_{b1}, \bar{c}^i_{b2}, \bar{m}^i)$ and utility functions $v_a{}^1$ and $v_b{}^i$ for $i > 1$ has a (unique) equilibrium $({}^0p_{b1}, {}^0y_{b1}, {}^0p_{b2}, {}^0y_{b2})$ with ${}^0c^1_{b3} \neq {}^1c^1_{s3}$ for each s, where ${}^0c^1_{b3} = \bar{m}^1 - {}^0p_{b1}{}^0y^1_{b1} - {}^0p_{b2}{}^0y^1_{b2}$, and ${}^1c^1_{s3}$ is similarly defined. (These are just consumptions of money. Recall that the price normalization gives $p_m = 1$.)

We now use these characteristics to construct a subset of E. For each i, let $u_1{}^i(\cdot) = \alpha_1{}^i \ln(\cdot)$, and similarly for $u_2{}^i$, u^i_{s3}. For each $\pi \in (0, 1)$ let $e(\pi)$ denote the environment in E determined by the characteristics $(\bar{c}^i_{a1}, \bar{c}^i_{b1}, \bar{c}^i_{b2}, \bar{c}^i_{b2}; \bar{m}^i; u_1{}^i, u_2{}^i, u^i_{a3}, u^i_{b3})$ for each i and the probability π of state a.

Claim. Since ${}^1p_{a1} = {}^1p_{b1} = p_1$ and ${}^1y_{a1} = {}^1y_{b1} = y_1$, then for each π, $({}^1p_{a1}, {}^1y_{a1}, {}^1p_{a2}, {}^1y_{a2}; {}^1p_{b1}, {}^1y_{b1}, {}^1p_{b2}, {}^1y_{b2})$ is an equilibrium $(e(\pi), 1)$.

Proof of claim. $(m_1{}^1, y_1{}^1)$ maximizes

$$\pi u_1{}^1(\bar{c}_1{}^1 + y_1{}^1) + \pi \max_{c,m} \{u_2{}^1(c) + u^1_{a3}(m): \\ p_{a2}c + p_m m = p_{a2}\bar{c}_2{}^1 + p_m m_1{}^1\}$$

and

$$(1 - \pi)\, u_1{}^1(\bar{c}_1{}^1 + y_1{}^1) + (1 - \pi) \max_{c,m} \{u_2{}^1(c) + u^1_{b3}(m): \\ p_{b2}c + p_m m = p_{b2}\bar{c}_2{}^1 + p_m m_1{}^1\}$$

subject to $p_1 y_1{}^1 + p_m m_1{}^1 = p_m \bar{m}^1$, since $y_1{}^1$ is the equilibrium first-period net trade for agent 1 in both (A) and (B). Therefore $(m_1{}^1, y_1{}^1)$

maximizes

$$u_1{}^1(\bar{c}_1{}^1 + y_1{}^1) + \pi \max_{c,m} \{u_2{}^1(c) + u^1_{a3}(m): p_{a2}c + p_m m = p_{a2}\bar{c}_2{}^1 + p_m m_1{}^1\} \\ + (1 - \pi) \max_{c,m} \{u_2{}^1(c) + u^1_{a3}(m): p_{a2}c + p_m m = p_{a3}\bar{c}_2{}^1 + p_m m_1{}^1\}$$

subject to $p_1{}^1 y_1{}^1 + p_m m_1{}^1 = p_m \bar{m}^1$.

Thus $y_1{}^1$ is the equilibrium first-period net trade for agent 1 in the environment $(e(\pi), 1)$ in both states. The trades ${}^1y^1_{a2}$, ${}^1y^1_{b2}$ are chosen to maximize $u_2{}^1(\bar{c}_2{}^1 + {}^1y^1_{s2}) + u^1_{s3}(m)$ subject to $p^1_{s2} y^1_{s2} + p_m m = p_m m_1{}^1$ for $s = a, b$, respectively. Thus according to Section 3.2 $y_1{}^1$, ${}^1y^1_{a2}$, and ${}^1y^1_{b2}$ are the equilibrium net trades for agent 1.

The other agents are initially informed and so face the same decision problems in states a and b as in the static situations (A) and (B). Thus

$({}^1p_{a1}, {}^1y_{a1}, {}^1p_{a2}, {}^1y_{a2}; {}^1p_{b1}, {}^1y_{b1}, {}^1p_{b2}, {}^1y_{b2})$ is an equilibrium for $(e(\pi), 1)$ for each π. In Section 4.4 it will be described how (A) and (B) can be chosen so that this is a unique equilibrium for $(e(\pi), 1)$ for each π. This completes the proof of the claim.

Since ${}^0c^1_{b3} \neq {}^1c^1_{s3}$ for each s, $\alpha^1_{a3} \ln(\cdot)$ can be replaced by a different u^1_{a3} satisfying (i) and (ii) of 2.1, which differs from $\alpha^1_{a3} \ln(\cdot)$ only near ${}^0c^1_{b3}$ so that (A), (B), and (C) remain true and in addition ${}^0c^i_{bt} \neq {}^1c^i_{st}$ for each i, s, t. For each π, let $e'(\pi)$ denote the resulting economy in E. It will be shown in Section 4.4 that such a perturbation can be made and still preserve the uniqueness of the equilibrium $({}^1p_{a1}, {}^1y_{a1}, {}^1p_{a2}, {}^1y_{a2}, {}^1p_{b1}, {}^1y_{b1}, {}^1p_{b2}, {}^1y_{b2})$ of $(e'(\pi), 1)$ for each π.

Since $({}^1p_{a1}, {}^1y_{a1}) = ({}^1p_{b1}, {}^1y_{b1}) = (p_1, y_1)$, the hypothesis of Lemma 4.2 is satisfied so that for each π, $(e'(\pi), 0)$ must have an equilibrium

$$({}^0p_{a1}, {}^0y_{a1}, {}^0p_{a2}, {}^0y_{a2}; {}^0p_{b1}, {}^0y_{b1}, {}^0p_{b2}, {}^0y_{b2})(\pi)$$

with

$$f^1[({}^0p_{a1}, {}^0y_{a1})(\pi)] = f^1[({}^0p_{a2}, {}^0y_{a2})(\pi)].$$

However, as $\pi \to 1$ $({}^0p_{a1}, {}^0y_{a1})(\pi) \to (p_1, y_1)$ and $({}^0p_{b1}, {}^0y_{b1})(\pi) \to ({}^0p_{b1}, {}^0y_{b1})$. This is because as $\pi \to 1$, the behavior of the first agent in state b approaches his behavior in a static economy with utility $v_a{}^1$, while the informed agents, realizing that state b has occurred use their utilities $v_b{}^i$. The continuity of f^1 then implies that

$$f^1(p_1, y_1) = f^1({}^0p_{b1}, {}^0y_{b1}).$$

Since ${}^0c^i_{bt} \neq {}^1c^i_{st}$ for each i, s, t there is a neighborhood V of (p_1, y_1) such that for each $(p_1', y_1') \in V$, there exist utilities $u_1^{i'}$, $u_2^{i'}$, $u_{a3}^{i'}$, $u_{b3}^{i'}$, respectively differing from $u_1{}^i$ only near ${}^1c^i_{a1}$ and ${}^1c^i_{b1}$, from $u_2{}^i$ only near ${}^1c^i_{a2}$ and ${}^1c^i_{b2}$, from u^i_{a3} only near ${}^1c^i_{a3}$, and from u^i_{b3} only near ${}^1c_{b3}$, so that if the new utility functions are substituted into statements (A), (B) and (C), statement (C) remains true with $({}^0p_{b1}, {}^0y_{b1})$ unchanged and statements (A) and (B) remain true if (p_1, y_1) is replaced by (p_1', y_1'). In addition the utilities can be chosen so that if $e''(\pi)$ denotes the new economy for each π, then the equilibria for $(e''(\pi), 1)$ are unique. Lemma 4.2 then implies f^1 is constant on V.

Since (p_1, y_1) was arbitrary in $P \times Y$, which is connected, it follows that f^1 is constant on all of $P \times Y$. The proposition then follows.

4.4. *Uniqueness of the Equilibria*

The construction of the economies (A), (B), and (C) in Proposition 4.3 can be accomplished as follows. Normalize prices so that $p_m = 1$. Let (p_1, y_1) be as specified. Choose an arbitrary $({}^1p_{b2}, {}^1y_{b2}) \in P \times Y$. Since the economies are three commodity Cobb-Doublas exchange economies the conditions for equilibrium are well known. For the economy (B), they are

$$\frac{\alpha_1^i}{\alpha_1^i + \alpha_2^i + \alpha_{b3}^i} = \frac{p_1(y_1^i + \bar{c}_{b1}^i)}{p_1\bar{c}_{b1}^i + {}^1p_{b2}\bar{c}_{b2}^i + \bar{m}^i},$$

$$\frac{\alpha_2^i}{\alpha_1^i + \alpha_2^i + \alpha_{b3}^i} = \frac{{}^1p_{b2}({}^1y_{b2}^i + \bar{c}_{b2}^i)}{p_1\bar{c}_{b1}^i + {}^1p_{b2}\bar{c}_{b2}^i + \bar{m}^i},$$

$$\frac{\alpha_{b3}^i}{\alpha_1^i + \alpha_2^i + \alpha_{b3}^i} = \frac{{}^1y_{b3}^i + \bar{m}^i}{p_1\bar{c}_{b1}^i + {}^1p_{b2}\bar{c}_{b2}^i + \bar{m}^i},$$

$$p_1y_1^i + {}^1p_{b2}{}^1y_{b2}^i + {}^1y_{b3}^i = 0$$

for all i. The budget constraints determine ${}^1y_{b3}$ from (p_1, y_1) and $({}^1p_2, {}^1y_{b2})$. The $\bar{c}_{b1}^i, \bar{c}_{b2}^i, \bar{m}^i$ are chosen large enough that $y_1^i + \bar{c}_{b1}^i$, ${}^1y_{b2}^i + \bar{c}_{b2}^i$, and ${}^1y_{b3}^i + \bar{m}^i$ are positive for each i. This determines the α_1^i, α_2^i, α_{b3}^i up to scalar multiple. Note that the endowments can be made arbitrarily large for any given group of agents by altering the parameters of their utilities without changing the characteristics of the other agents.

The conditions for equilibrium in economy (C) are

$$\frac{\alpha_1^1}{\alpha_1^1 + \alpha_2^1 + \alpha_{a3}^1} = \frac{{}^0p_{b1}({}^0y_{b1}^1 + \bar{c}_{b1}^1)}{{}^0p_{b1}\bar{c}_{b1}^1 + {}^0p_{b2}\bar{c}_{b2}^1 + \bar{m}^1},$$

$$\frac{\alpha_1^i}{\alpha_1^i + \alpha_2^i + \alpha_{b3}^i} = \frac{{}^0p_{b1}({}^0y_{b1}^i + \bar{c}_{b1}^i)}{{}^0p_{b1}\bar{c}_{b1}^i + {}^0p_{b2}\bar{c}_{b2}^i + \bar{m}^i}, \qquad i > 1,$$

$$\frac{\alpha_2^1}{\alpha_1^1 + \alpha_2^1 + \alpha_{a3}^1} = \frac{{}^0p_{b2}({}^0y_{b2}^1 + \bar{c}_{b2}^1)}{{}^0p_{b1}\bar{c}_{b1}^1 + {}^0p_{b2}\bar{c}_{b2}^1 + \bar{m}^1},$$

$$\frac{\alpha_2^i}{\alpha_1^i + \alpha_2^i + \alpha_{b3}^i} = \frac{{}^0p_{b2}({}^0y_{b2}^i + \bar{c}_{b2}^i)}{{}^0p_{b1}\bar{c}_{b1}^i + {}^0p_{b2}\bar{c}_{b2}^i + \bar{m}^i}, \qquad i > 1,$$

$$\frac{\alpha_{a3}^1}{\alpha_1^1 + \alpha_2^1 + \alpha_{a3}^1} = \frac{{}^0y_{b3}^1 + \bar{m}^1}{{}^0p_{b1}\bar{c}_{b1}^1 + {}^0p_{b2}\bar{c}_{b2}^1 + \bar{m}^1}, \tag{*}$$

$$\frac{\alpha_{b3}^i}{\alpha_1^i + \alpha_2^i + \alpha_{b3}^i} = \frac{{}^0y_{b3}^i + \bar{m}^i}{{}^0p_{b1}\bar{c}_{b1}^i + {}^0p_{b2}\bar{c}_{b2}^i + \bar{m}^i}, \qquad i > 1,$$

$${}^0p_{b1}{}^0y_{b1}^i + {}^0p_{b2}{}^0y_{b2}^i + {}^0y_{b3}^i = 0, \qquad i \geqslant 1.$$

The equilibria for Cobb-Douglas economies are unique since Cobb-Douglas utilities give rise to excess demand functions satisfying gross substitutability and indecomposability [6, pp. 305–6] which are sufficient conditions for uniqueness [6, p. 335]. Thus each choice of α_{a3}^1 determines a unique equilibrium for economy (C), given the parameters chosen for (B). If α_{a3}^1 is small enough then in order for the first agent to satisfy (*) with ${}^0y_{b3}^1 = {}^1y_{b3}^1$ it must be the case that either ${}^0p_{b1}$ or ${}^0p_{b2}$ becomes arbitrarily large. In this case the income of the other agents becomes arbitrarily large and their excess demand for money becomes arbitrarily large and cannot be satisfied by the first agent. Thus for α_{a3}^1 small enough, ${}^0y_{b3}^1 \neq {}^1y_{b3}^1$. Note that once the value of α_{a3}^1 is fixed, the endowments of the other agents can be increased without changing this value of α_{a3}^1 and an equilibrium will still obtain.

For economy (A), since $\bar{c}_{a1}^1 = \bar{c}_{b1}^1$ and $\bar{c}_{a2}^1 = \bar{c}_{b2}^1$ the equilibrium conditions for the first agent are

$$\frac{\alpha_1{}^1}{\alpha_1{}^1 + \alpha_2{}^1 + \alpha_{a3}^1} = \frac{p_1(y_1{}^1 + \bar{c}_{b1}^1)}{p_1\bar{c}_{b1}^1 + {}^1p_{a2}\bar{c}_{b2}^1 + \overline{m}^1},$$

$$\frac{\alpha_2{}^1}{\alpha_1{}^1 + \alpha_2{}^1 + \alpha_{a3}^1} = \frac{{}^1p_{a2}({}^1y_{a2}^1 + \bar{c}_{b2}^1)}{p_1\bar{c}_{b1}^1 + {}^1p_{a2}\bar{c}_{b2}^1 + \overline{m}^1},$$

$$\frac{\alpha_{a3}^1}{\alpha_1{}^1 + \alpha_2{}^1 + \alpha_{a3}^1} = \frac{{}^1y_{a3}^1 + \overline{m}^1}{p_1\bar{c}_{b1}^1 + {}^1p_{a2}\bar{c}_{b2}^1 + \overline{m}^1},$$

$$p_1 y_1{}^1 + {}^1p_{a2}{}^1y_{a2}^1 + {}^1y_{a3}^1 = 0.$$

Given the parameters for economy (B) and α_{a3}^1, the first equation determines ${}^1p_{a2}$.

The third equation then determines ${}^1y_{a3}^1$. The parameter α_{a3}^1 can be chosen so as to be small enough for the construction of (C) to be satisfied and so that ${}^1y_{a3}^1 \neq {}^0y_{b3}^1$. The parameters α_{a3}^i, net trades, and endowments of the remaining agents can then be chosen to satisfy the remaining equilibrium conditions. The endowments of these agents can be chosen arbitrarily large.

If $(\bar{c}_{a1}^i, \bar{c}_{a2}^i, \bar{c}_{b1}^i, \bar{c}_{b2}^i, \overline{m}^i)$ is chosen large enough for $i > 1$ then the equilibria for $(e(\pi), 1)$ and $(e(\pi), 0)$ will also be unique. This is because the excess demand function for the initially informed agents $(i > 1)$ is easily seen to satisfy gross subsitutability and indecomposability, coming from a Cobb–Douglas utility in each state. As the endowments of the informed agents increase, the absolute values of the partial derivative of the aggregate excess demand function of the informed agents become arbitrarily large. The first agent's demand will also be smooth, for when the behavior of initially informed agents (Section 3.2) is written in terms of the indirect utility of current consumption and money savings, the objective function is a convex combination of log—linear functions subjected to a change of scale and origin. The derivatives of the excess demand function for the first agent will depend smoothly on π. Thus in any compact neighborhood of the equilibrium price the derivatives of the excess demand functions will be uniformly bounded in π and η_2^1. Thus if the endowments of the informed agents are large enough relative to the endowment of the initially uninformed agent the aggregate excess demand function for all agents will exhibit gross substitutibility in a neighborhood of equilibrium. Outside that neighborhood the excess demand for the uninformed agents will be large in absolute value (being the integral of the large partial derivatives). The endowments of the informed agents can be chosen large enough (without altering the first agent's characteristics) so that outside this neighborhood their excess demand is larger than the uninformed agent's final consumption. The uninformed agent will not be able to satisfy this excess demand so no price outside the neighborhood can be an equilibrium price. The gross substitutability then guarantees a unique equilibrium price in this neighborhood. Thus the equilibrium price is unique.

The economies $(e''(\pi), 1)$ will also have unique equilibria for small enough perturbations of the utilities, for the derivatives of the excess demand functions depand continuously on the first and second derivatives of the utilities. Thus small changes in the utilities will not void the above argument.

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JOURNAL OF ECONOMIC THEORY 22, 407–422 (1980)

Liberal Paradox and the Voluntary Exchange of Rights-Exercising

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Received February 28, 1978; revised March 26, 1979

1. INTRODUCTION

Among many recent contributions on the logical (in)compatibility of the Paretian ethics and the libertarian claims, initiated by Sen's [11, Chap. 6*] theorem on the *impossibility of a Paretian liberal*, Gibbard's [3] analysis, which culminates in the edifice of the alienable rights system, deserves particular scrutiny. It tries, among other things, to call due attention to "a strong libertarian tradition of free contract," according to which "a person's rights are his to use or bargain away as he sees fit."[1] Searching examinations of the Gibbard's system have already been put forward by Karni [7], Kelly [8; 9, Chap. 9], Sen [12, Sect. IV] and Suzumura [14], but it seems to us that there remain many important points to be made on this interesting contribution. The purpose of this paper is to point out that Gibbard's system of alienable rights in a revised version proposed by Kelly [8; 9, Chap. 9] represents a standard for individual liberty which cannot be met by any universal collective choice rule. That is to say, it is logically impossible to construct a collective choice rule with unrestricted domain, which realizes the Gibbard–Kelly system of alienable rights. This clearly contradicts Kelly's assertion to the effect that "[the revision] causes no significant changes in the theorems that make up Gibbard's libertarian claim."[2] This is unfortunate, since Kelly's revision seems to be rather persuasive. To the extent that Kelly's proposed revision is acceptable, therefore, the workability and reasonableness of Gibbard's scheme seem to be in serious doubt. At the very least, the edifice of the alienable rights system should be evaluated with this subtlety in mind. In passing, we will examine the possibility and limitation of resolving the liberal paradox via the metarational exercising of rights a la Howard [4–6] in view of the similarity between the

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[1] Gibbard [3, p. 397]. See also Barry [1, p. 166].

[2] Kelly [8, p. 144; 9, p. 148].

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prisoners' dilemma and the Paretian liberal paradox, which was pointed out by Fine [2].

2. GIBBARD'S CONSISTENT LIBERTARIAN CLAIM AND KELLY'S REVISION THEREOF

2.1. Let $N = \{1, 2, \dots, n\}$ denote the finite set of individuals ($n \geqq 2$) and let X stand for the set of all conceivable social states. What we call a social state is a list of impersonal and personal features of the world. Letting X_0 and X_i stand respectively for the set of all impersonal features of the world and the set of all personal features of the individual $i \in N$, the set of all social states, X, is now given by $X = X_0 \times (\prod_{i \in N} X_i)$. It is assumed that X_0 and X_i are finite with at least two elements each. R_i denotes a weak preference (at least as good as) relation of the individual $i \in N$. We assume that R_i is an ordering on X, being *connected* [for all x and y, $(x, y) \in R_i$ and/or $(y, x) \in R_i$] and *transitive* [for all x, y, and z, $(x, y) \in R_i$ and $(y, z) \in R_i$ imply $(x, z) \in R_i$]. The strict preference relation $P(R_i)$ is defined as usual by $(x, y) \in P(R_i) \leftrightarrow [(x, y) \in R_i \;\&\; (y, x) \notin R_i]$. The indifference relation $I(R_i)$ is defined by $(x, y) \in I(R_i) \leftrightarrow [(x, y) \in R_i \;\&\; (y, x) \in R_i]$. A list $R = (R_1, R_2, \dots, R_n)$ of individual weak preference orderings will be called a *profile*. A *collective choice rule* (CCR) is a function F which represents a method of amalgamating each profile $R = (R_1, R_2, \dots, R_n)$ into a *social choice function* C on the family $\mathscr{S}$ of all finite nonempty subsets of X: $C = F(R)$. When an $S \in \mathscr{S}$ is specified as a set of implementable states, $C(S)$ denotes the nonempty set of socially chosen states. In what follows, we will be concerned with constructing a CCR which may amalgamate *every* logically possible profiles.

Condition U (*Unrestricted domain*). The domain of our CCR consists of all logically possible profiles.

2.2. As a matter of notational convention, we let $X_{)i(} = X_0 \times X_1 \times \cdots \times X_{i-1} \times X_{i+1} \times \cdots \times X_n$ and, for each $i \in N$ and each $x = (x_0, x_1, \dots, x_n) \in X$, $x_{)i(} = (x_0, x_1, \dots, x_{i-1}, x_{i+1}, \dots, x_n)$. Furthermore, if $x_i \in X_i$ and $z = (z_0, z_1, \dots, z_{i-1}, z_{i+1}, \dots, z_n) \in X_{)i(}$, then $(x_i\,;\, z) = (z_0, z_1, \dots, z_{i-1}, x_i, z_{i+1}, \dots, z_n)$. We define D_i by

$$D_i = \{(x, y) \in X \times X \mid x_{)i(} = y_{)i(}\} \qquad (i \in N).$$

Therefore if $(x, y) \in D_i$, x and y may possibly differ only in the specification of i's personal feature. Call $D = (D_1, D_2, \dots, D_n)$ the *Gibbardian rights system* or the *decomposable rights system*.

2.3. The CCR should be so designed, a naive libertarian might claim, that it implements the special say of the ith individual over each and every pair $(x, y) \in D_i$ in the following sense:

Condition GL(1) (*Gibbard's first libertarian claim*). For every profile $R = (R_1, R_2, \ldots, R_n)$, every $i \in N$, and every $x, y \in X$, if $(x, y) \in D_i \cap P(R_i)$, then $[x \in S \rightarrow y \notin C(S)]$ for all $S \in \mathscr{S}$, where $C = F(R)$.

An unfortunate fact is that *there exists no* CCR *which satisfies this naive libertarian claim together with Condition* U. This is Gibbard's first impossibility theorem [3, Theorem 1] on the libertarianism.

2.4. It is now time we formulate two versions of the Pareto rule.

Condition EP (*Exclusion Pareto*). For every profile $R = (R_1, R_2, \ldots, R_n)$ and every $x, y \in X$, if $(x, y) \in \bigcap_{i \in N} P(R_i)$, then $[x \in S \rightarrow y \notin C(S)]$ for all $S \in \mathscr{S}$, where $C = F(R)$.

Condition IP (*Inclusion Pareto*). For every profile $R = (R_1, R_2, \ldots, R_n)$ and every $x, y \in X$, if $(x, y) \in \bigcap_{i \in N} P(R_i)$, then $[\{x \in S \;\&\; y \in C(S)\} \rightarrow x \in C(S)]$ for all $S \in \mathscr{S}$, where $C = F(R)$.

Let x and y be such that everyone in the society strictly prefers x to y. Then Condition EP requires that the CCR prohibit y from being chosen from every envirionment where x is available, while Condition IP requires that the CCR be such that x is chosen from every environment where x is available and y is chosen. Clearly EP, which is a quite common formulation of the Pareto rule, is a stronger requirement on CCR than IP.[3]

2.5. Turn now to Gibbard's second libertarian claim, which goes as follows.

Condition GL(2) (*Gibbard's second libertarian claim*). For every profile $R = (R_1, R_2, \ldots, R_n)$, every $S \in \mathscr{S}$, every $i \in N$, and every $x, y \in X$, if $(x, y) \in D_i \cap P(R_i)$ and $((x_i\,; z_{)i(}), (y_i\,; z_{)i(})) \in P(R_i)$ for all $z_{)i(}$ such that $(x_i\,; z_{)i(}), (y_i\,; z_{)i(}) \in S$, then $[x \in S \rightarrow y \notin C(S)]$, where $C = F(R)$.

In Condition GL(1), it is required that the CCR allow each and every individual to exercise his right $(x, y) \in D_i$ whenever he happens to prefer x to y. In contrast, Condition GL(2) does not necessarily allow the de facto individual preferences to rule the roost: It is required in GL(2) that the CCR protect the right $(x, y) \in D_i$ only when the ith individual prefers the distinguishing feature x_i of x *unconditionally* to the corresponding feature

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[3] It might be of some interest to present a concrete CCR which satisfies the Condition IP but does not satisfy the Condition EP. The simplest example is a rule which assigns to each and every profile $R = (R_1, R_2, \ldots, R_n)$ a choice function C such that $C(S) = S$ for all $S \in \mathscr{S}$. An intrinsically more interesting example is the *majority closure method* (Sen [13, pp. 56, 74]). Let M_R be the simple majority relation corresponding to a given profile R. Take any $S \in \mathscr{S}$ and let $T(M_R \mid S)$ be the transitive closure of M_R on S. A choice set $C_R^*(S)$ represents a subset of S consisting of the $T(M_R \mid S)$-greatest points in S. The majority closure method is a CCR which assigns to each R the choice function C_R^*. It is easy to see that this CCR satisfies IP but not EP.

y_i of y. Clearly GL(2) represents a milder libertarian claim than GL(1). Nevertheless, *there exists no* CCR *which satisfies* GL(2), EP, *and* U. This is Gibbard's second impossibility theorem [3, Theorem 2] on the libertarianism. A slight generalization thereof is the following:

THEOREM 1. *There exists no* CCR *which satisfies* GL(2), IP, *and* U.

Proof. Suppose that there exists such a CCR. Take any $a_0 \in X_0$ and $a_i \in X_i$ $(i \in N\setminus\{1, 2\})$ and fix them for the rest of this proof. Take $x_i, x_i' \in X_i$, $x_i \neq x_i'$, where $i = 1, 2$ and define

$$x^1 = (a_0, x_1, x_2, a_3, \ldots, a_n),$$

$$x^2 = (a_0, x_1, x_2', a_3, \ldots, a_n),$$

$$x^3 = (a_0, x_1', x_2, a_3, \ldots, a_n),$$

and

$$x^4 = (a_0, x_1', x_2', a_3, \ldots, a_n).$$

Let $S = \{x^1, x^2, x^3, x^4\} \in \mathscr{S}$ and let a profile $R = (R_1, R_2, \ldots, R_n)$ be such that

$$R_1(S)\colon x^1, x^3, x^2, x^4,$$

$$R_2(S)\colon x^4, x^3, x^2, x^1,$$

and, for all $i \in N\setminus\{1, 2\}$,

$$R_i(\{x^2, x^3\})\colon x^3, x^2,$$

where $R_i(S)$ denotes the restriction of R_i on the set S: $R_i(S) = R_i \cap (S \times S)$.[4] Clearly, then, $(x^1, x^3) \in D_1$, $(x^2, x^4) \in D_1$ and $(x^2, x^1) \in D_2$. No other individual has right over these states. Since individual 1 prefers x_1 to x_1' unconditionally and individual 2 prefers x_2' to x_2 unconditionally, Condition GL(2) implies that $x^3 \notin C(S)$, $x^4 \notin C(S)$, and $x^1 \notin C(S)$ for a choise set $C(S)$ which corresponds to the specified R and S. Suppose now $x^2 \in C(S)$. Then we have $(x^3, x^2) \in \bigcap_{i \in N} P(R_i)$, $x^3 \in S$ and $x^2 \in C(S)$, so that Condition IP requires that $x^3 \in C(S)$, a contradiction. Therefore we must have $C(S) = \varnothing$, which negates the existence of a CCR satisfying GL(2), IP, and U. ∎

2.6. A salient common feature of GL(1) and GL(2) deserves particular mention: It is supposed that the *i*th individual's right is exercised in complete negligence of any repercussion from the rest of the society, guided solely by the individual rational calculus. From this viewpoint, the gist of the Gibbardian impossibility theorems mentioned so far may be interpreted as the *failure of the isolated rational rights-exercising*. An ingenious proposal crystallized in Gibbard's third libertarian claim is to make individual's libertarian rights *alienable* in cases where the exercise of one's libertarian rights brings him into a situation he likes no better than the situation that would otherwise have been brought about. Gibbard observes that, given an $R = (R_1, R_2, ..., R_n)$ and an $S \in \mathscr{S}$, an individual $i \in N$ has the will as well as the right to exclude y from $C(S)$ if $x \in S$ and $(x, y) \in D_i \cap P(R_i)$, but his right for the pair (x, y) had better be *waived* if there exists a sequence $\{y_1, y_2, ..., y_\lambda\}$ in S such that

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[4] Preference orderings are written horizontally with the less preferred states appearing to the right of the more preferred states, indifferent states, if any, being put together by square brackets.

$$y_\lambda = x, \ (y, y_1) \in R_i \,\&\, y \neq y_1, \tag{1}$$

and

$$(\forall t \in \{1, 2, ..., \lambda - 1\}): (y_t, y_{t+1}) \in \Big(\bigcap_{j \in N} P(R_j)\Big) \cup \Big(\bigcup_{j \in N \setminus \{i\}} [D_j \cap P(R_j)]\Big). \tag{2}$$

Let us define a subset $W_i(R \mid S)$ of D_i, to be called the *waiver set*, by $(x, y) \in W_i(R \mid S)$ iff (1) and (2) are true for some sequence $\{y_\mu\}_{\mu=1}^{\lambda}$ in S.

Condition GL(3) (*Gibbard's third libertarian claim*). For every profile $R = (R_1, R_2, ..., R_n)$, every $S \in \mathscr{S}$, every $i \in N$, and every $x, y \in X$, if $(x, y) \in D_i \cap P(R_i)$ and $(x, y) \notin W_i(R \mid S)$, then $[x \in S \rightarrow y \notin C(S)]$, where $C = F(R)$.

Clearly this is a claim which differs essentially from GL(1) and GL(2) in that it is explicitly recognized that an individual's rights-exercising may induce unfavorable responses of the others which might well nullify the benefit for which the initial exercising was intended. Gibbard has shown that GL(3) represents a Pareto-consistent libertarian claim; i.e., *there exists a* CCR *which satisfies* GL(3), EP, *and* U.[5]

2.7. Kelly [8; 9, Chap. 9] claims to have found some "flaws" in Gibbard's definition of the rights-waiving rule and proposes two revisions thereof, the first of which goes as follows[6]. Let a profile $R = (R_1, R_2, ..., R_n)$ and a set of implementable states $S \in \mathscr{S}$ be given. An individual i waives his right for $(x, y) \in D_i$, namely that $(x, y) \in W_i^*(R \mid S)$, iff there exists a sequence $\{y_1, y_2, ..., y_\lambda\}$ in S satisfying

$$y_\lambda = x \,\&\, (y, y_1) \in P(R_i), \tag{3}$$

and (2).

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[5] Gibbard [3, Theorem 4].

[6] As a matter of fact, Kelly proposes the third revised version. For our purpose it is not necessary to get into this complicated proposal, however.

Condition KL(1) (*Kelly's first libertarian claim*). For every profile $R = (R_1, R_2, \dots, R_n)$, every $S \in \mathscr{S}$, every $i \in N$, and every $x, y \in X$, if $(x, y) \in D_i \cap P(R_i)$ and $(x, y) \notin W_i^*(R \mid S)$, then $[x \in S \rightarrow y \notin C(S)]$, where $C = F(R)$.

The only difference between GL(3) and KL(1) lies in the contrast between (1) and (3). The reason behind this revision is that "in forcing the move from y to x by exercising $[(x, y) \in D_i$, the individual $i]$ does not seem to have gotten into trouble if he is forced in the end to take a y_1 where he is indifferent between y_1 and y. Waiving might be appropriate for a cautious exerciser if $[(y, y_1) \in P(R_i)]$ for some [sequence $\{y_1, y_2, \dots, y_\lambda\}$], but not if only $[(y, y_1) \in R_i$ as in (1)]" (Kelly [8, p. 141; 9, 146–147]).

Going one step further, Kelly proposes his second revised libertarian claim. Suppose that a profile $R = (R_1, R_2, \dots, R_n)$ and an $S \in \mathscr{S}$ are given. This time, an individual i is supposed to waive his right for $(x, y) \in D_i$, namely that $(x, y) \in W_i^{**}(R \mid S)$ iff:

(a) There exists a sequence $\{y_1, y_2, \dots, y_\lambda\}$ in S such that

$$y_\lambda = x \ \& \ (y, y_1) \in P(R_i), \tag{4}$$

and

$$(\forall t \in \{1, 2, \dots, \lambda - 1\}): (y_t, y_{t+1}) \in \Big(\bigcap_{j \in N} P(R_j)\Big) \cup \Big(\bigcup_{j \in N \setminus \{i\}} [D_j \cap P(R_j)]\Big); \tag{5}$$

(b) For any sequence $\{z_1, z_2, \dots, z_{\lambda^*}\}$ in S such that

$$z_{\lambda^*} = y_1 \ \& \ (z_1, y) \in P(R_i),$$

and

$$(\forall t \in \{1, 2, \dots, \lambda^* - 1\}): (z_t, z_{t+1}) \in \Big(\bigcap_{j \in N} P(R_j)\Big) \cup \Big(\bigcup_{j \in N} [D_j \cap P(R_j)]\Big), \tag{7}$$

there exists correspondingly a sequence $\{w_1, w_2, \dots, w_{\lambda^{**}}\}$ in S such that

$$w_{\lambda^{**}} = z_1 \ \& \ (y, w_1) \in P(R_i), \tag{8}$$

and

$$(\forall t \in \{1, 2, \dots, \lambda^{**} - 1\}): (w_t, w_{t+1}) \in \Big(\bigcap_{j \in N} P(R_j)\Big) \cup \Big(\bigcup_{j \in N \setminus \{i\}} [D_j \cap P(R_j)]\Big). \tag{9}$$

Condition KL(2) (*Kelly's second libertarian claim*). For every profile $R = (R_1, R_2, \dots, R_n)$, every $S \in \mathscr{S}$, every $i \in N$, and every $x, y \in X$, if $(x, y) \in D_i \cap P(R_i)$ and $(x, y) \notin W_i^{**}(R \mid S)$, then $[x \in S \rightarrow y \notin C(S)]$, where $C = F(R)$.

The difference between KL(1) and KL(2) is the addition of (b) in the definition of $W_i^{**}(R \mid S)$, which says basically that any sequence $\{z_1, z_2, ..., z_{\lambda *}\}$ which seems to repair in the eyes of the individual i the damage caused upon him by a sequence $\{y_1, y_2, ..., y_\lambda\}$ will be made ineffective by some other, out-of-control sequence $\{w_1, w_2, ..., w_{\lambda **}\}$.

3. Impossibility Theorems

3.1. Taken by themselves, these proposed revisions may seem to be fairly persuasive and, according to Kelly [8, p. 144; 9, p. 148], "it causes no significant changes in the theorems that make up Gibbard's libertarian claim." The truth is, however, that Kelly's reasonable-looking revisions to Gibbard's libertarian claim change it into a standard for individual liberty which cannot possibly be met, as the following impossibility theorems show.

Theorem 2. *There exists no* CCR *which satisfies* KL(1) *and* U.

Theorem 3. *There exists no* CCR *which satisfies* KL(2) *and* U.

To prove these theorems, note first that, for every profile $R = (R_1, R_2, ..., R_n)$ and every $S \in \mathscr{S}$, the following set-inclusions are true.

$$W_i^{**}(R \mid S) \subset W_i^*(R \mid S) \subset W_i(R \mid S) \tag{10}$$

for all $i \in N$. Clearly, then, KL(2) is a *stronger* libertarian claim than KL(1), so that we have only to prove Theorem 2, Theorem 3 being a corollary thereof.

Proof of Theorem 2. Suppose that F is a CCR which satisfies KL(1) and U. Let $S = \{x^1, x^2, x^3, x^4\} \in \mathscr{S}$ be defined as in the proof of the Theorem 1 and let a profile $R = (R_1, R_2, ..., R_n)$ be such that

$$R_1(S)\colon x^1, x^4, [x^2, x^3],$$
$$R_2(S)\colon x^3, x^2, [x^1, x^4].$$

There is no restriction on R_i for $i \in N \backslash \{1, 2\}$ whatsoever. Clearly $(x^1, x^3) \in D_1$, $(x^4, x^2) \in D_1$, $(x^3, x^4) \in D_2$ and $(x^2, x^1) \in D_2$. No other individual has right over these pairs of states. Consider the pair of states $(x^1, x^3) \in D_1 \cap P(R_1)$. The worst which could happen to individual 1 after his exercise of $(x^1, x^3) \in D_1$ is the counterexercise by 2 of $(x^2, x^1) \in D_2$ in view of $(x^2, x^1) \in P(R_2)$. (Note that there is no state in S which strictly Pareto-dominates x^1.) Since x^2 and x^3 are indifferent to individual 1 and $x^2 \neq x^3$, GL(3) would let 1 waive his right over (x^1, x^3), but KL(1) does allow 1 to exercise his right over (x^1, x^3):

$(x^1, x^3) \in W_1(R \mid S)\backslash W_1^*(R \mid S)$. Similar reasoning leads us to $(x^4, x^2) \in W_1(R \mid S)\backslash W_1^*(R \mid S)$, $(x^3, x^4) \in W_2(R \mid S)\backslash W_2^*(R \mid S)$ and $(x^2, x^1) \in W_2(R \mid S)\backslash W_2^*(R \mid S)$. By virtue of the Condition KL(1) it then follows that $C(S) = \varnothing$, a contradition. ∎

3.2. Kelly's first revised libertarian claim thus brings back an impossibility. A fortiori, his second (and stronger) revised libertarian claim is inconsistent with the existence of a universal CCR. One may thereby be tempted to conclude that the system of alienable rights is something like a fragile glasswork which may be easily smashed to pieces while giving the last finish to it. To be fair, however, one should not forget to examine whether the finishing touch was an appropriate one.

Back, then, to the contrast between GL(3) and KL(1), i.e., the contrast between (1) and (3). Stipulation (3) was recommended in place of (1),because, in forcing the move from y to x by exercising $(x, y) \in D_i$, individual i does not *lose* anything even if he is forced in the end to take a y_1 such that $(y, y_1) \in I(R_i)$ and $y \neq y_1$. Note, however, that he does not *gain* anything either. Note also that the rights-exercising in Gibbard's system places very heavy demands on the information gathering and processing[7], so that the rights-exercising would be unwise unless it yielded a positive gain. This argument, if accepted, would favor (1) rather than (3) and would necessitate the following modification of KL(2). Given a profile $R = (R_1, R_2, \ldots, R_n)$ and an $S \in \mathcal{S}$, define the waiver set $W_i^0(R \mid S)$ by $(x, y) \in W_i^0(R \mid S)$ iff:

(a) There exists a sequence $\{y_1, y_2, \ldots, y_\lambda\}$ in S such that

$$y_\lambda = x, \ (y, y_1) \in R_i \ \& \ y \neq y_1, \tag{11}$$

and

$$(\forall t \in \{1, 2, \ldots, \lambda - 1\}): (y_t, y_{t+1}) \in \Big(\bigcap_{j \in N} P(R_j)\Big) \cup \Big(\bigcup_{j \in N\backslash\{i\}} [D_j \cap P(R_j)]\Big); \tag{12}$$

(b) For any sequence $\{z_1, z_2, \ldots, z_{\lambda^*}\}$ in S such that

$$z_{\lambda^*} = y_1 \ \& \ [(z_1, y) \in P(R_i) \vee z_1 = y] \tag{13}$$

and

$$(\forall t \in \{1, 2, \ldots, \lambda^* - 1\}): (z_t, z_{t+1}) \in \Big(\bigcap_{j \in N} P(R_j)\Big) \cup \Big(\bigcup_{j \in N} [D_j \cap P(R_j)]\Big), \tag{14}$$

there exists correspondingly a sequence $\{w_1, w_2, \ldots, w_{\lambda^{**}}\}$ in S such that

$$w_{\lambda^{**}} = z_1, \ (y, w_1) \in R_i \ \& \ y \neq w_1, \tag{15}$$

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[7] Kelly [8, p. 141; 9, p. 146].

and

$$(\forall t \in \{1, 2, \ldots, \lambda^{**} - 1\})\colon (w_t, w_{t+1}) \in \Big(\bigcap_{j \in N} P(R_j)\Big) \cup \Big(\bigcup_{j \in N \setminus \{i\}} [D_j \cap P(R_j)]\Big), \tag{16}$$

for all $i \in N$. Utilizing this modified definition of the waiver set, we now put forward the following:

Condition GKL (*Gibbard-Kelly libertarian claim*). For every profile $R = (R_1, R_2, \ldots, R_n)$, every $S \in \mathscr{S}$, every $i \in N$, and every $x, y \in X$, if $(x, y) \in D_i \cap P(R_i)$ and $(x, y) \notin W_i^0(R \mid S)$, then $[x \in S \rightarrow y \notin C(S)]$, where $C = F(R)$.

How does GKL fare in the context of universal CCRs? That it fares no better than KL(1) and KL(2) is the thrust of the next theorem.

THEOREM 4. *There exists no* CCR *which satisfies* GKL *and* U.

Proof. Suppose that F is an eligible CCR. Let $S = \{x^1, x^2, x^3, x^4\} \in \mathscr{S}$ and $R = (R_1, R_2, \ldots, R_n)$ be the same as in the proof of the Theorem 2. Consider the pair of states $(x^2, x^3) \in D_1 \cap P(R_1)$. The worst situation which individual 1's exercise of $(x^1, x^3) \in D_1$ may induce is the counterexercise by 2 of $(x^2, x^1) \in D_2$ in view of $(x^2, x^1) \in P(R_2)$. Individual 1 may then exercise $(x^4, x^2) \in D_1 \cap P(R_1)$ to secure x^4, which 1 prefers to x^3. Is there any nullifying sequence? The worst which could happen to 1 is the exercise by 2 of $(x^3, x^4) \in D_2$, which does not require 1 to waive his right $(x^1, x^3) \in D_1$ according to the definition of $W_1^0(R \mid S)$.[8] Therefore GKL ensures that $x^3 \notin C(S)$. By the same token we may verify that

$$[(x^4, x^2) \in D_1 \cap P(R_1) \;\&\; (x^4, x^2) \notin W_1^0(R \mid S)] \rightarrow x^2 \notin C(S),$$

$$[(x^2, x^1) \in D_2 \cap P(R_2) \;\&\; (x^2, x^1) \notin W_2^0(R \mid S)] \rightarrow x^1 \notin C(S),$$

---

[8] Two clarifications might be in order here. First, from x^2, something else may happen (besides 1 exercising his right $(x^1, x^2) \in D_1$) *if* the Pareto-dominance relation is weakened from $\bigcap_{i \in N} P(R_i)$ to $P(\bigcap_{i \in N} R_i)$. That is to say, *if* $(x^3, x^2) \in R_i$ for all $i \in N/\{1, 2\}$ and *if* in (12), (14) and (16) all instances of $\bigcap_{i \in N} P(R_i)$ are replaced by $P(\bigcap_{i \in N} R_i)$, x^3 might be picked over x^2 by a Pareto-dominance. It is clear, however, that this possibility does not affect our conclusion that $(x^1, x^3) \notin W_1^0(R \mid S)$. Second, in arriving at the conclusion that $(x^1, x^3) \notin W_1^0(R \mid S)$, we have followed the "path" $x^3 \rightarrow x^1 \rightarrow x^2 \rightarrow x^4 \rightarrow x^3$ generated by the successive rights-exercising of 1 and 2. Namely, we started from x^3 and came back to x^3 again! It might be asked, why don't we let 1 waive his right $(x^1, x^3) \in D_1$ in this case? Put differently, why do we stipulate the condition $y \neq y_1$ in (11)? The reason is that there exists an important difference between (i) the travel from x^3 (via a sequence of states) to x^3 back again, and (ii) the travel from x^3 to an $x^* \in S$ such that $(x^3, x^*) \in I(R_1)$. In the former case, individual 1 comes back to x^3 *without losing his right over* x^3, while in the latter case, 1 may well be stuck at x^* *without having any right over* x^*. This is also the reason why we modified (6) into (13).

and

$$[(x^3, x^4) \in D_2 \cap P(R_2) \,\&\, (x^3, x^4) \notin W_2^0(R \mid S)] \rightarrow x^4 \notin C(S),$$

so that we obtain $C(S) = \emptyset$, a contradiction. ∎

3.3. If we examine the profile which we utilized in proving Theorems 2, 3, and 4, it turns out that *both* 1 and 2 are expressing preferences which are conditional on the other's selection of his personal feature: 1 prefers x_1 to x_1' if 2 has x_2, while he prefers x_1' to x_1 if 2's choice is x_2' and *vice versa*. Probably it is too much to ask for the existence of a universal CCR which protects individual's mere conditional preferences. On reflection, we need only require the existence of a CCR which protects individual's libertarian rights so far as the relevant individual expresses unconditional preference for his personal features. Therefore, let $N(R \mid S)$ be the set of individuals having unconditional preferences, given a profile R and an available set $S \in \mathscr{S}$, i.e., $i \in N(R \mid S)$ iff $(x, y) \in D_i \cap (S \times S) \cap P(R_i)$ always implies that $((x_i ; z_{)i(}), (y_i ; z_{)i(})) \in P(R_i)$ for all $z_{)i(}$ such that $(x_i ; z_{)i(}) \in S$ and $(y_i ; z_{)i(}) \in S$. The relevant waiver set will then have to be specified thus: $(x, y) \in D_i$ will be waived, i.e., $(x, y) \in W_i^{00}(R \mid S)$ iff *either* (i) $i \in N \backslash N(R \mid S)$, *or* (ii) the following conditions hold true:

(a) There exists a sequence $\{y_1, y_2, \dots, y_\lambda\}$ in S such that

$$y_\lambda = x,\ (y, y_1) \in R_i \,\&\, y \neq y_1,$$

and

$$(\forall t \in \{1, 2, \dots, \lambda - 1\}): (y_t, y_{t+1}) \in \Big(\bigcap_{j \in N} P(R_j)\Big) \cup \Big(\bigcup_{j \in N(R \mid S) \backslash \{i\}} [D_j \cap P(R_j)]\Big);$$

(b) For any sequence $\{z_1, z_2, \dots, z_{\lambda^*}\}$ in S such that

$$z_{\lambda^*} = y_1 \,\&\, [(z_1, y) \in P(R_i) \vee z_1 = y]$$

and

$$(\forall t \in \{1, 2, \dots, \lambda^* - 1\}): (z_t, z_{t+1}) \in \Big(\bigcap_{j \in N} P(R_j)\Big) \cup \Big(\bigcup_{j \in N(R \mid S)} [D_j \cap P(R_j)]\Big),$$

there exists correspondingly a sequence $\{w_1, w_2, \dots, w_{\lambda^{**}}\}$ in S such that

$$w_{\lambda^{**}} = z_1,\ (y, w_1) \in R_i \,\&\, y \neq w_1,$$

and

$(\forall t \in \{1, 2, \dots, \lambda^{**} - 1\})$:

$$(w_t, w_{t+1}) \in \Big(\bigcap_{j \in N} P(R_j)\Big) \cup \Big(\bigcup_{j \in N(R \mid S) \backslash \{i\}} [D_j \cap P(R_j)]\Big).$$

Our final version of the libertarian claim in the spirit of Gibbard and Kelly goes as follows.

Condition GKL*. For every profile $R = (R_1, R_2, \dots, R_n)$, every $S \in \mathcal{S}$, every $i \in N$, and every $x, y \in X$, if $(x, y) \in D_i \cap P(R_i)$ and $(x, y) \notin W_i^{00}(R \mid S)$, then $[x \in S \rightarrow y \notin C(S)]$, where $C = F(R)$.

Clearly GKL* is *weaker* than GL(2). Unfortunately this modest version of the libertarian claim still may not break the impasse if there are at least three individuals in the society.

THEOREM 5. *Suppose that* $n \geqq 3$. *Then there exists no* CCR *which satisfies* GKL*, IP, *and* U.

Proof. Suppose that an eligible CCR F does exist. Take any $a_0 \in X_0$ and $a_i \in X_i$ $(i \in N \backslash \{1, 2, 3\})$ and fix them for the rest of this proof. Take x_i, $x_i' \in X_i$, $x_i \neq x_i'$ for $i \in \{1, 2, 3\}$ and define

$$x^0 = (a_0, x_1, x_2, x_3', a_4, \dots, a_n),$$

$$x^1 = (a_0, x_1, x_2, x_3, a_4, \dots, a_n),$$

$$x^2 = (a_0, x_1, x_2', x_3, a_4, \dots, a_n),$$

$$x^3 = (a_0, x_1', x_2, x_3, a_4, \dots, a_n),$$

and

$$x^4 = (a_0, x_1', x_2', x_3, a_4, \dots, a_n).$$

Let $S = \{x^0, x^1, x^2, x^3, x^4\} \in \mathcal{S}$ and let a profile $R = (R_1, R_2, \dots, R_n)$ be such that:

$$R_1(S)\colon x^1, x^3, x^2, x^4, x^0,$$

$$R_2(S)\colon [x^3, x^4], x^2, x^1, x^0,$$

$$R_3(S)\colon x^3, x^2, x^0, x^4, x^1,$$

and, for all $i \in N \backslash \{1, 2, 3\}$:

$$R_i(\{x^0, x^2, x^3\})\colon x^3, x^2, x^0.$$

By definition we see that $(x^1, x^3) \in D_1$, $(x^2, x^4) \in D_1$, $(x^3, x^4) \in D_2$, $(x^2, x^1) \in D_2$ and $(x^0, x^1) \in D_3$. Note that individual 1 prefers x_1 to x_1' unconditionally, and individual 3 prefers x_3' to x_3 unconditionally. On the other hand, individual 2 prefers x_2' to x_2 if 1 chooses x_1, while he is indifferent between x_2 and x_2' if 1 chooses x_1' instead. In view of this conditional nature of 2's preferences, his rights-exercising will be made ineffective by a CCR subject to GKL*. Consider now the pair of states $(x^1, x^3) \in D_1 \cap P(R_1)$.

The worst contingency which could be induced by the exercise of $(x^1, x^3) \in D_1$ is the counterexercise of $(x^0, x^1) \in D_3 \cap P(R_3)$. Since $(x^3, x^0) \in \bigcap_{i \in N} P(R_i)$ holds true, and there exists no nullifying sequence in S, we may conclude from GKL* that

$$[(x^1, x^3) \in D_1 \cap P(R_1) \; \& \; (x^1, x^3) \notin W_1^{(00)}(R \mid S)] \to x^3 \notin C(S).$$

Similarly we may verify that

$$[(x^2, x^1) \in D_1 \cap P(R_1) \; \& \; (x^2, x^1) \notin W_1^{(00)}(R \mid S)] \to x^4 \notin C(S)$$

and

$$[(x^0, x^1) \in D_3 \cap P(R_3) \; \& \; (x^0, x^1) \notin W_3^{(00)}(R \mid S)] \to x^1 \notin C(S).$$

We may then invoke Condition IP (as we did in the proof of Theorem 1) to conclude that $x^2 \notin C(S)$ and $x^0 \notin C(S)$. It then follows that $C(S) = \emptyset$, a contradiction. ▮

4. METARATIONAL EXERCISING OF RIGHTS

4.1. Among the libertarian claims we have examined in the above, GL(3), KL(1), KL(2), GKL, and GKL* differ substantially from GL(1) and GL(2) in that due attention is paid in the former category of claims to the fact that, in deciding whether to exercise one's right or not, one should take into considerations the others' response via their rights-exercising and/or Pareto-dominance. Notice, however, that it is commonly assumed in these claims that each individual, in predicting the others' response to his rights-exercising, presumes that *the others follow the naive rights-exercising rule without making any effort of prediction on their part.*[9] It might be asked, what can we make of the libertarian claims if we assume instead the complete mutual prediction?

4.2. The situation of complete mutual prediction may be modeled after the metagame theory of Howard [4–6]. For simplicity let there be only two individuals in the society and let an impersonal feature vector $s_0 \in X_0$ be fixed for the rest of this paper. S_1 and S_2 denote the set of all available personal feature alternatives of individual 1 and that of individual 2, respectively. Given a profile $R = (R_1, R_2)$, we may construe the 4-tuple $\Gamma = (S_1, S_2; R_1, R_2)$ as a *basic game* played by 1 and 2. Each individual

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[9] This point was duly stressed by Kelly. He has shown that a serious difficulty of *correctable miscalculation* emerges from this peculiarity of the Gibbardian rights-exercising rule. See Kelly [8; 9, Chap. 9].

$k \in \{1, 2\}$ wishes to choose such an $s_k \in S_k$ as to yield, coupled with the fixed s_0 and the choice $s_{\sim k} \in S_{\sim k}$ of the other individual ${}^{\sim}k$, a state $(s_0, s_k, s_{\sim k}) \in \{s_0\} \times S_1 \times S_2 =: S$ which "optimizes" his preference R_k over S.[10] In a liberal society k obviously lacks the power to regulate the choice by ${}^{\sim}k$ of his personal feature, so that the best k can do is to predict the choice by ${}^{\sim}k$ and to form a *metastrategy* which is a function g^k from $S_{\sim k}$ into S_k such that, for each $s_{\sim k} \in S_{\sim k}$, $(s_0, g^k(s_{\sim k}), s_{\sim k}) \in S$ is the best state in S with respect to R_k. In effect, then, k is thinking in terms of the first-level metagame $k\Gamma := (G^k, S_{\sim k}; R_k, R_{\sim k})$, where G^k denotes the set of all functions from $S_{\sim k}$ into S_k. In doing this, however, k should be aware that ${}^{\sim}k$ may be able to, and should rationally try to, predict k's choice of his metastrategy. The subjective game of k in the situation of complete mutual prediction is the second-level metagame $({}^{\sim}k)\, k\Gamma = (G^k, F^{\sim k}; R_k, R_{\sim k})$, where $F^{\sim k}$ denotes the set of all functions from G^k into $S_{\sim k}$. We repeat for emphasis: The metagame $({}^{\sim}k)\, k\Gamma$ represents a model in which ${}^{\sim}k$ is able to predict k's choice in the metagame $k\Gamma$, while the metagame $k\Gamma$ represents a model in which k is able to predict ${}^{\sim}k$'s choice in the basic game Γ. Thus in the metagame $({}^{\sim}k)\, k\Gamma$ there is complete mutual prediction.

4.3. Let us say, following Howard, that an outcome $(g^k_*, f^{\sim k}_*)$ of the metagame $({}^{\sim}k)\, k\Gamma$ is *metarational for k via* $({}^{\sim}k)\, k\Gamma$ iff:

$$[(g^k_*(f^{\sim k}_*(g^k_*)), f^{\sim k}_*(g^k_*)), (g^k(f^{\sim k}_*(g^k)), f^{\sim k}_*(g^k))] \in R_k \tag{17}$$

for all $g^k \in G^k$. In this case the corresponding basic outcome, which is given by $(s_0, g^k_*(f^{\sim k}_*(g^k_*)), f^{\sim k}_*(g^k_*)) \in S$, will be called hereafter the *metarational basic outcome for k via* $({}^{\sim}k)\, k\Gamma$. Let $M_k[({}^{\sim}k)\, k\Gamma]$ denote the set of all metarational basic outcomes for k via $({}^{\sim}k)\, k\Gamma$. We may similarly define $M_{\sim k}[({}^{\sim}k)\, k\Gamma]$, $M_k[k({}^{\sim}k)\, \Gamma]$, and $M_{\sim k}[k({}^{\sim}k)\, \Gamma]$.

It is now time we introduce the concept of metasolutions. In our present context, two possibilities suggest themselves. First we have

$$M^*(\Gamma) = M_k[({}^{\sim}k)\, k\Gamma] \cap M_{\sim k}[(k({}^{\sim}k)\, \Gamma], \tag{18}$$

which is appealing for the following reason. In general, "there will be a strong tendency for player i's subjective metagame to be one in which he follows all the others last. For such a metagame is precisely one in which he can predict the others' basic strategies, they cannot predict his, and they are possibly able to predict his [metastrategy]. Indeed, if we add the condition that i believes there is complete mutual prediction..., then this precisely describes the complete metagames in which he follows all others last."[11]

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[10] It is assumed here that the set of implementable states S satisfies Kelly's [8; 9, Chap. 9] *agenda-closedness* and Seidl's [10] *technological separability*.

[11] Howard [6, p. 106].

*It then follows that, if $s^* \in M^*(\Gamma)$, then s^* is a metarational basic outcome for **both** individuals via their **respective** "natural" subjective metagames with complete mutual prediction. The second metasolution concept of our concern is*

$$M_*(\Gamma) = \{M_k[(\sim k)\, k\Gamma] \cap M_{\sim k}[(\sim k)\, k\Gamma]\} \cup \{M_k[k(\sim k)\, \Gamma] \cap M_{\sim k}[k(\sim k)\, \Gamma]\}. \tag{19}$$

By definition, $s^* \in M_*(\Gamma)$ holds true iff s^* is a metarational basic outcome for *both* individuals via a *common* metagame with complete mutual prediction. Remember that an $s^* \in M^*(\Gamma)$, however "natural" it may be, cannot be an equilibrium via voluntary bargaining or negotiation except as a result of some kind of misunderstanding: Each individual behaves as if he can predict the other's basic strategy while the other can at best predict his metastrategy choice, but they cannot both be correct. In contrast, a metasolution $s^* \in M_*(\Gamma)$ is the basic outcome which constitutes an equilibrium when both individuals have agreed on the same subjective metagame.

4.4. We now examine these metasolution concepts in the context of the liberal paradoxes.

EXAMPLE 1. Let $\Gamma = (S_1, S_2; R_1, R_2)$ be such that $S_1 = \{s_1, s_1'\}$, $S_2 = \{s_2, s_2'\}$ and

$$R_1(S)\colon s^1, s^3, s^2, s^4,$$
$$R_2(S)\colon s^4, s^3, s^2, s^1,$$

where $s^1 = (s_0, s_1, s_2)$, $s^2 = (s_0, s_1, s_2')$, $s^3 = (s_0, s_1', s_2)$ and $s^4 = (s_0, s_1', s_2')$. Note that this profile is the same as the one used in the proof of Theorem 1. In the game-theoretic context, this situation is known as the *prisoners' dilemma*.[12] It is easy, if tedious, to verify for this game Γ that $M_1[21\Gamma] = M_2[21\Gamma] = M_1[12\Gamma] = M_2[12\Gamma] = \{s^2, s^3\}$, so that we have:

$$M^*(\Gamma) = M_*(\Gamma) = \{s^2, s^3\}.$$

Let $O_R(S)$ be the set of Pareto-optimal states in S when the profile R prevails. Note, then, that $\{s^3\} = O_R(S) \cap M^*(\Gamma) = O_R(S) \cap M_*(\Gamma)$. Therefore it may duly be said that the metarational exercising of rights brings about a Pareto-optimal state in the case of prisoners' dilemma profile, thereby resolving the *Paretian liberal paradox*. ▌

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[12] Fine [2] pointed out the similarity between the prisoners' dilemma and the liberal paradox.

An important feature of this scheme should be stressed: Each individual need not know the other's preferences in this scheme, since $M_k[(\sim k)\,k\Gamma]$ as well as $M_k[k(\sim k)\Gamma]$ are definable solely in terms of k's own preferences. This is certainly a nice feature, but the real force of this resolution scheme is revealed if we examine the next example which relates to the internal inconsistency of rights.[13]

EXAMPLE 2. Let Γ^* be the same as Γ in the Example 1 save for the following specification of the profile $R^* = (R_1^*, R_2^*)$:

$$R_1^*(S)\colon [s^1, s^4],\ [s^2, s^3],$$

$$R_2^*(S)\colon [s^2, s^3],\ [s^1, s^4].$$

Consider the metagames $(\sim k)\,k\Gamma^*$ and $k(\sim k)\,\Gamma^*$. We may verify that $M_1[21\Gamma^*] = M_1[12\Gamma^*] = \{s^1, s^4\}$ and $M_2[21\Gamma^*] = M_2[12\Gamma^*] = \{s^2, s^3\}$.[14] It then follows that $M^*(\Gamma^*) = M_1[21\Gamma^*] \cap M_2[12\Gamma^*] = \varnothing$, and $M_*(\Gamma^*) = \{M_1[21\Gamma^*] \cap M_2[21\Gamma^*]\} \cup \{M_1[12\Gamma^*] \cap M_2[12\Gamma^*]\} = \varnothing$. Therefore both metasolution concepts fail to resolve the liberal paradox embodied in the game Γ^*. ∎

4.5. There are two classes of liberal paradox which we should cope with: the *Paretian liberal paradox*, and the *paradox of the internal inconsistency of rights*. We have shown in the above that the metarational exercising of rights in the situation of complete mutual prediction may systematically resolve the former class of paradoxes, while the latter difficulty may not be resolved in this manner.

5. CONCLUDING REMARKS

In reviewing a number of contributions to the Paretian liberal paradox, Sen [12, p. 224] argued that "[Kelly] identifies a number of difficulties with Gibbard's system [which are] essentially arising from problems in deciding *when* a right is useful for a person. ...Some of these difficulties are eliminated by modifications of the Gibbard system proposed by Kelly...." We have examined in this paper Gibbard's system of alienable rights in the light of Kelly's proposed modifications thereof. It was shown that the difficulties identified by Kelly are more serious than Kelly and Sen have thought them

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[13] See Gibbard's Wall-Colour Case [3, p. 389] and Sen's Zubeida–Rehana Case [12, p. 234]. In the game-theoretic literature, the game Γ^* in Example 2 is called the game of Matching Pennies. See Howard [4, p. 182].

[14] Howard's *Characterization Theorem for Metarational Outcomes* [6, pp. 89–96] is useful in verifying these results.

to be, and they are not eliminated by Kelly's proposed modifications. We have also examined the possibility and limitation of resolving the liberal paradox via the metarational rights-exercising. In conclusion, it is hoped that the negative results reported in this paper will serve to clarify the nature and stubbornness of the paradox of a Paretian liberal.

ACKNOWLEDGMENTS

I am grateful to Professors Kiyoshi Kuga and Jerry S. Kelly, whose incisive comments on Suzumura [14] led me to reexamine Gibbard's system of alienable rights. Thanks are also due to an anonymous referee of this Journal for his helpful suggestions.

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JOURNAL OF ECONOMIC THEORY 22, 423–438 (1980)

On the Routewise Application of Choice

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Received May 1, 1978; revised March 19, 1979

1. INTRODUCTION

The attention directed by Arrow to the relation of choice and the sequence in which alternatives are considered, as in his dictum that "the final choice should be independent from the path to it" [2, p. 120], has been pursued with growing interest, notably in the works of Sen [16, p. 48; 18], Fishburn [9], and, since the neat formulation given to the "path-independence" of choice by Plott [13], in a literature to which Blair [3, 4], Blair *et al.* [5], Ferejohn and Grether [8], and Parks [12] have contributed. Of these, [5] characterized Plott's path-independence condition and [12] strengthened it. In [24] we gave 14 other characterizations of this condition, which we call "sequential fidelity" for reasons fully developed in [24]; some of these characterizations occurring earlier in [19, 21, 22], and some of them strengthening Parks' characterization or occupying the strongest position in other classes of characterizations of sequential fidelity.

Specializing Plott, here we import from [23] the notion of the "routewise application" of a choice, according to which choice proceeds on a finite set of alternatives by first considering one element, eliminating or keeping it, then eliminating what it will from the union of what it kept with a second singleton, then choosing from the union of this with a next singleton, and so on, until all the alternatives have been considered and the routewise application of the choice comes up with a final result. With notions indigenous to this approach, after Section 2 of preliminaries and Section 3, which formally introduces the routewise application of choice, in Section 4 we reconsider Plott's sequential fidelity (or path-independence) and, for the case where

0022-0531/80/030423-16$02.00/0

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sets of alternatives are all finite, we offer for it a characterization stronger than Parks' and even our previous strengthening of that characterization. In the same section we also strengthen by a couple of "degrees" Plott's result that a sequentially faithful decisive choice reveals a quasi-transitive comparison on finite sets of alternatives. In Section 5, first we look at binary choices, relating them to choices satisfying other conditions, then we relate a binary choice to the choice obtained as what is common to all its routewise applications (its "folding"). The section ends with a list of distinct conditions a choice may satisfy, all of which degenerate into one and the same condition for binary choices. Finally, in Section 6 we consider the matter of which properties of a choice are inherited by its routewise applications, and vice versa. The paper ends with a few closing remarks (Section 7) and an Appendix collecting nine (counter-)examples referred to in the text.

2. PRELIMINARIES

Throughout, X is a nonempty set of *alternatives* and for any $A \subset X$, $[A]$ denotes the power set of A, $\mathfrak{F}[A]$ the family of finite subsets of A. $\mathfrak{A}$ is any family of sets (of alternatives) satisfying (as in [23])

$$\mathfrak{F}[X] \subset \mathfrak{A} \subset [X], \tag{0.1}$$

$$A \in \mathfrak{A} \Rightarrow [A] \subset \mathfrak{A}, \tag{0.2}$$

$$A,\ B \in \mathfrak{A} \Rightarrow (A \cup B) \in \mathfrak{A}. \tag{0.3}$$

Throughout, A and B are always to be understood as members of the system $\mathfrak{A}$ of *sets of alternatives*, and s is always to be understood as a choice on $\mathfrak{A}$, i.e., a mapping s: $\mathfrak{A} \to \mathfrak{A}$ with $sA \subset A$ (for every $A \in \mathfrak{A}$). We say that s is *decisive* iff $sA \neq \varnothing$ whenever $A \neq \varnothing$.

Every choice s *reveals* (determines) a *comparison* (binary relation) $S \subset X \times X$ through $xSy \Leftrightarrow y \in s\{x, y\}$, and we write ASy to mean that xSy for every $x \in A$. Every choice s also reveals a *strict comparison* $\hat{S} \subset X \times X$ through $x\hat{S}y \Leftrightarrow \{y\} = s\{x, y\}$. The comparison S revealed by a choice s is said to be *complete* iff xSy or ySx whenever x, $y \in X$; *acyclic* iff $x_1\hat{S}x_2\hat{S} \cdots \hat{S}x_{n-1}\hat{S}x_n$ implies that $x_n\hat{S}x_1$ fails ($x_n \not\hat{S} x_1$) for every finite sequence $\{x_i\}_{i=1}^{n}$ of points $x_i \in X$; *quasi-transitive* iff $\hat{S}$ is transitive. (We use this terminology to facilitate comparability to our other papers in this area [19–25]. The reader will note that the comparison S revealed by a choice s is Arrow's [1] "relation generated by a choice function," or Herzberger's [11] "base relation" of a choice.)

Finally, every choice s determines a binary operation $\overset{s}{\cup}$: $\mathfrak{A} \times \mathfrak{A} \to \mathfrak{A}$, as defined by $A \overset{s}{\cup} B = s(sA \cup B)$.

3. THE ROUTEWISE APPLICATION OF CHOICE

Here we construct a piece of apparatus crucial for the investigation to follow. To that end, we first recall that every set may be well-ordered and let $\leqslant$ be a well-order of X (i.e., a total order according to which every non-empty subset $C \subset X$ has a first element, i.e., an element x such that $x \leqslant y$ for every $y \in C$). We denote the set of all permutations of X (one-to-one mappings ω: $X \to X$ of X onto X) by Ω, letting δ stand for the identity permutation of X ($\delta(x) = x$). Now every permutation ω of X defines a well-ordering $\leqslant_\omega$ of X through

$$x \leqslant_\omega y \Leftrightarrow \omega(x) \leqslant \omega(y)$$

and, for any finite subset $A \subset X$, we define $1_\omega A$, $2_\omega A, \ldots, (\#A)_\omega A$ as the first, second,..., $(\#A)$th element of A according to $\leqslant_\omega$. Thus, it is not entirely awkward to refer to elements $\omega \in \Omega$ as "*routes*".[1] For any choice s on $\mathfrak{A}$ and any permutation $\omega \in \Omega$ of X, we may now define the routewise application s_ω: $\mathfrak{F}[X] \to \mathfrak{F}[X]$ of s to ω recursively as follows:

$$s_\omega{}^1 A := s\{1_\omega A\},$$

$$s_\omega{}^k A := s((s_\omega^{k-1} A) \cup \{k_\omega A\}) \qquad (k = 2, \ldots, \#A),$$

$$s_\omega A := s_\omega^{\#A} A.$$

(For example, if $A = \{x, y, z\} \in \mathfrak{A}$ and, for $\omega \in \Omega$, $x <_\omega y <_\omega z$, then for any choice s on $\mathfrak{A}$, the routewise application s_ω of s to ω gives $s_\omega A = s(s(s\{x\} \cup \{y\}) \cup \{z\})$.)

By the *routewise application of choice* we mean the function ρ determining the choice $\rho(s, \omega) = s_\omega$ on $\mathfrak{F}[X]$ for every choice s on $\mathfrak{A}$ and permutation ω of X. We note that every choice s on $\mathfrak{A}$ also determines a choice $\underset{\wedge}{s}$ on $\mathfrak{F}[X]$ through

$$\underset{\wedge}{s} = \bigcap_{\Omega} s_\omega\,;$$

we term such a choice $\underset{\wedge}{s}$ a *folded choice* or the *folding* of s.

---

[1] We are avoiding the term "path" for at least two reasons: (i) because "path" usually means a continous mapping of [0, 1] into a topological space, and (ii) in the literature on choice, a "path" on a set A has come to mean an ordered finite cover of A with subsets of A (see condition (1), below, and [13, 21, 22, 24]).

4. SEQUENTIAL FIDELITY (OR PATH-INDEPENDENCE) OF CHOICE

A condition which has received much deserved attention in the literature of collective choice is that of the *sequential fidelity* of choice whose first neat formulation through the functional equation

$$A \mathring{\cup} B = s(A \cup B) \tag{1}$$

is due to Plott [13] who called it the "*path-independence*" of choice. (We beg the reader's pardon for our renaming of certain conditions, hoping that the consistency of our nomenclature and its suggestiveness regarding the logical relations between the things named will help spare us from at least the harshest of lamentations on this account. The usefulness of this nomenclature is distinctively shown in [24].)

Sequential fidelity (or path-independence) obviously implies the commutativity of $\mathring{\cup}$, and we have shown [24, Theorem 2] that the commutativity and associativity of $\mathring{\cup}$ are equivalent (but do not imply (1) [24, Example 5]).[2] Thus, a choice s is sequentially faithful (1) iff, for every $A \in \mathfrak{A}$, sA may always be computed faithfully (without error) in piecemeal and sequential fashion, by use of any "path" $\langle A_1, \ldots, A_n \rangle$ on A (i.e., any ordered finite cover of A with its own subsets), as $A_1 \mathring{\cup} \cdots \mathring{\cup} A_n$. One part of sequential fidelity (1) is obviously *sequential upper semifidelity*

$$A \mathring{\cup} B \supset s(A \cup B), \tag{$\bar{1}$}$$

which says that the error in the computation described above will be, if anything, on the "upper" side. It is well known (see, e.g., [13, p. 1088; 12; 18; 23–25]) and in any case easy to check that $(\bar{1})$ is equivalent to[3]

$$sA \supset s(A \cup B) \cap A. \tag{$\bar{1}'$}$$

It is perhaps not surprising that, with a few exceptions [24, Theorems 17, 21, 26], characterizations of sequential fidelity (1) to date have expressed it as the conjunction of $(\bar{1})$ with one or another condition. The strongest characterization of (1) in this genre so far is (1) $\Leftrightarrow$ $(\bar{1})$ and (2) [24, Theorems 5, 6], where

$$\text{if} \quad x \in A \backslash sA, \qquad \text{then} \quad s(sA \cup \{x\}) \neq sA \cup \{x\}. \tag{2}$$

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[2] Plott [13] showed that (1) implies the commutativity and associativity of $\mathring{\cup}$.

[3] Condition $(\bar{1})$ (or, equivalently, $(\bar{1}')$) has come to be known as "Sen's condition α" [16, 17] in the theory of collective choice, but has also been called "consistency" [21–23], "coherence" [10], as well as other names [1; 20; 25, p. 125]. The condition $A \mathring{\cup} B \subset s(A \cup B)$, condition $(\underline{1})$, of (*sequential*) *lower semifidelity*, sometimes called "weak path-independence" ([8]), is obviously the part of (1) left over from $(\bar{1})$.

use of a direct specialization (($\bar{1}$r), below) of ($\bar{1}$) to routewise applicaons, in Theorem 3 we will strengthen this characterization further when $= \mathfrak{F}[X]$.

The general idea of sequential fidelity or path-independence (1) of a choice s lends itself naturally to routewise thinking: we say that s is *routewise faithful* iff

$$s_\omega A = sA \qquad (A \in \mathfrak{F}[X]) \tag{1r}$$

for every route $\omega \in \Omega$. Similarly, we say that s is *routewise upper semifaithful* iff

$$s_\omega A \supset sA \qquad (A \in \mathfrak{F}[X]) \tag{$\bar{1}$r}$$

for every route $\omega \in \Omega$. A consequence of (1r) is that $\rho(s_{\cdot})$ is independent of ω; i.e.,

$$s_{\omega'} = s_\omega \tag{3}$$

for any ω, $\omega' \in \Omega$, which we term the *route-independence* of s. The reader should note from Example 1, however, that (1r) and (3) do not coincide.[4]

While we could actually prove Theorem 3 right away by use of basic facts established in [23], to keep the present study self-sufficient it is best to approach the matter through a number of lemmas, a course which in any case offers the advantage of allowing us to develop the material of current interest in a natural order. The first notion we regard in this spirit is that of *routewise computability*,[5] expressed by the functional equation

$$\underset{\wedge}{s}A = sA \qquad (A \in \mathfrak{F}[X]). \tag{4}$$

One should avoid thinking that a folded choice $\underset{\wedge}{s}$ is necessarily route-independent (3) or even that routewise computability (4) guarantees route-independence (3), since Example 2 shows both conjectures to be false. In this connection, neither does route-independence (3) imply routewise computability (4), as Example 1 shows. What routewise computability (4) for a choice s means is that, regardless of whether s is routewise faithful (1r), and *even* if it fails to be route-independent (3), at the expense of extra computational effort, i.e., employing the procedure of computing $s_\omega A$ for each route $\omega \in \Omega$ and listing the elements common to each of these computations, one can correctly list the elements of sA whenever $A \subset X$ is finite.

Now routewise upper semifidelity ($\bar{1}$r), which is obviously a consequence of sequential upper semi-fidelity ($\bar{1}$) and clearly equivalent to the half of equation (4) stating $\underset{\wedge}{s}A \supset sA$ $(A \in \mathfrak{F}[X])$, fails to imply sequential upper semifidelity $(\bar{1})$, as we see from Example 3. In view of the fact, established by Example 1, that sequential upper semifidelity $(\bar{1})$ does not entail routewise computability (4), the following proposition shows that (4) is strictly stronger than $(\bar{1})$ on $\mathfrak{F}[X]$ and actually at least "two degrees" stronger than $(\bar{1}\mathrm{r})$.

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[4] Of course, "path-independence" (defined analogously to (3); cf. the discussion of (1), above) and (1) do coincide, since $\langle A \rangle$ is always a path on A.

[5] For a different but interesting discussion of computational questions relating to choice, see Campbell [6, 7].

1. LEMMA. *Routewise computable* (4) *choices are sequentially upper semifaithful* $(\bar{1})$ *on* $\mathfrak{F}[X]$.

Proof. Assume s routewise computable (4) and let $A, B \in \mathfrak{F}[X]$. Take any $x \in s(A \cup B) \cap A$. Suppose $x \notin sA$. As $sA \supset \underset{\wedge}{s}A$, there is then a route ω such that $x \notin s_\omega A$. Thus, for any $\omega' \in \Omega$ agreeing with ω on A and setting $a <_{\omega'} b$ whenever $a \in A$ and $b \in A\backslash B$, it follows that $x \notin s_{\omega'}(A \cup B)$, hence that $x \notin \underset{\wedge}{s}(A \cup B) \supset s(A \cup B)$, a contradiction. Therefore, $x \in sA$, showing that $(\bar{1}')$ holds whenever $A, B \in \mathfrak{F}[X]$. ▮

2. LEMMA. *Routewise upper semifaithful* $(\bar{1}\mathrm{r})$ *choices which satisfy* (2) *on* $\mathfrak{F}[X]$ *are also routewise computable* (4).

Proof. Assume s satisfies $(\bar{1}\mathrm{r})$ and (2) on $\mathfrak{F}[X]$, and take any $A \in \mathfrak{F}[X]$. As $(\bar{1}\mathrm{r}) \Rightarrow \underset{\wedge}{s}A \supset sA$, we need only show $\underset{\wedge}{s}A \subset sA$. Suppose $a \in \underset{\wedge}{s}A \backslash sA$. Consider any $\omega \in \Omega$ such that $x <_\omega a \leqslant_\omega y$ whenever $x \in sA$ and $y \in A \backslash sA$. As $s_\omega A \supset sA$, we have $s_\omega(sA) = sA \subset s_\omega(sA \cup \{a\}) = s(sA \cup \{a\})$, while (2) demands $s(sA \cup \{a\}) \subsetneq sA \cup \{a\}$. Hence, $s(sA \cup \{a\}) = sA$, so $a \notin s_\omega A \supset \underset{\wedge}{s}A$, a contradiction. Thus, $\underset{\wedge}{s}A \backslash sA = \varnothing$; i.e., $\underset{\wedge}{s}A \subset sA$, as was to be shown. ▮

3. THEOREM (Characterization of sequential fidelity (or path-independence) (1) on $\mathfrak{F}[X]$). *The choices sequentially faithful* (1) *on* $\mathfrak{F}[X]$ *are precisely the routewise upper semifaithful* $(\bar{1}\mathrm{r})$ *choices which satisfy* (2) *on* $\mathfrak{F}[X]$.[6]

Proof. Obviously, $(1) \Rightarrow (\bar{1}\mathrm{r})$. Also, for x as in (2), (1) gives $A \overset{s}{\cup} \{x\} = \{x\} \overset{s}{\cup} A = s(s\{x\} \cup A) = sA \neq sA \cup \{x\}$, showing $(1) \Rightarrow (2)$. For the rest, let s be a routewise upper semifaithful $(\bar{1}\mathrm{r})$ choice satisfying (2) on $\mathfrak{F}[X]$. Thus, by Lemma 2, s is routewise computable (4). Now Lemma 1 guarantees that s is sequentially upper semifaithful $(\bar{1})$ on $\mathfrak{F}[X]$.[7] Taking any $A, B \in \mathfrak{F}[X]$, $(\bar{1})$ gives $A \overset{s}{\cup} B \supset s(A \cup B)$ and, thus, also $A \cup B \supset sA \cup B \supset s(A \cup B)$. Considering any $\omega \in \Omega$ with $x <_\omega y$ whenever $x \in s(A \cup B)$ and $y \in (A \cup B)\backslash s(A \cup B)$, we see that the conjunction of $(\bar{1}\mathrm{r})$ and (2) on $\mathfrak{F}[X]$ now gives $s_\omega(sA \cup B) = s(A \cup B)$. Thus, by (4), $A \overset{s}{\cup} B = s(sA \cup B) = \underset{\wedge}{s}(sA \cup B) \subset s(A \cup B)$, so $A \overset{s}{\cup} B = s(A \cup B)$ and s satisfies (1) on $\mathfrak{F}[X]$. ▮

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[6] Theorem 3 is an improvement of Theorem 9 in [23].

[7] In view of the characterization $(1) \Leftrightarrow (\bar{1})$ and (2) of [24, Theorem 5), we could actually end the proof here, but refrain from doing so only in the interest of self-containment.

The strongest characterization of (1) in terms of a conjunction involving $(\bar{1})$ remains the characterization $(1) \Leftrightarrow (\bar{1})$ and (2) of [24, Theorems 5, 6]. However, when we restrict attention to the behavior of a choice just on $\mathfrak{F}[X]$, i.e., we take $\mathfrak{A} = \mathfrak{F}[X]$ or regard only the restriction of a choice to $\mathfrak{F}[X]$, theorem 3 gives us a yet stronger characterization of (1) (on $\mathfrak{F}[X]$), since it characterizes (1) on $\mathfrak{F}[X]$ as the conjunction of (2) with a condition strictly weaker than $(\bar{1})$.

Note. As Theorem 3 is concerned only with the behavior of a choice on $\mathfrak{F}[X]$, instead of (2) it uses the weaker condition that states (2) to hold whenever A is a finite set of alternatives.

Finally, a straightforward fact to be demonstrated in closing this section concerns the quasi transitivity of the comparison S revealed by a choice s (or, in Herzberger's [11] terminology, the quasi transitivity of the base relation of s). The interest of the matter is that Plott [13], assuming decisiveness, showed sequentially faithful (or path-independent) (1) choices to reveal quasi-transitive comparisons and *binary* choices, i.e., choices satisfying

$$sA = \{x \in A \mid ASx\}, \tag{5}$$

to be sequentially faithful (1) on $\mathfrak{F}[X]$ whenever they reveal a quasi-transitive comparison. (In Sen's [18] terminology, a choice satisfying (5) is called "basic binary.")

4. PROPOSITION. *Route-independent* (3) *choices decisive on the singletons reveal quasi-transitive comparisons.*[8]

Proof. Assume that s is decisive on the singletons and satisfies (3), and that $x\hat{S}y\hat{S}z$. Consider any $\omega, \omega' \in \Omega$ such that $x <_{\omega} y <_{\omega} z$ and $y <_{\omega'} z <_{\omega'} x$. Then $\{z\} = s_{\omega}A = s_{\omega'}A = s(s(s\{y\} \cup \{z\}) \cup \{x\}) = s\{z, x\}$, so $x\hat{S}z$ and S is quasi-transitive. ▮

The point in recording Proposition 4 is that $(1) \Rightarrow (1r) \Rightarrow (3)$, while neither of these implications is reversible (as we know from Examples 4 and 1, respectively), so that Proposition 4, despite its simplicity, extends Plott's result ($(1) \Rightarrow S$ is quasi-transitive) for decisive choices s by a couple of "degrees."

Postscriptum. While we are not explicitly concerned here with "collective rationality" and the aggregation of comparisons or choices, we have had the good fortune to be awakened by Kotaro Suzumura to a consequence, in the mentioned realm of interest, following directly from the conjunction of

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[8] Or, to use Herzberger's terminology [11]: Route-independent (3) choices decisive on the singleton sets have a quasi-transitive base relation.

Proposition 4 with Theorem 4 of [5]. Using the terminology of [5], we state this as the following.

4′. COROLLARY (K. Suzumura). *When there are at least three voters, there is no collective choice rule F satisfying all of*:

(i) *the choices in the range of F are all decisive on the singletons and route-independent* (3),

(ii) *the Pareto condition*,

(iii) *independence of irrelevant alternatives*, and

(iv) *non-weak-dictatorship.*

(Theorem 4 of [5] reads like Corollary 4′ above with the exception that in place of (i) one substitutes the condition that the choices in the range of F all reveal quasi-transitive comparisons. Hence, it is evident how Corollary 4′ follows from Proposition 4 and Theorem 4 of [5].) In entirely similar fashion the reader may also obtain a direct corollary of Proposition 4 and Theorem 5 of [5], the statement of which need not be detailed here.

Corollary 4′, as an impossibility theorem concerning collective choice theory proper, seems to indicate that route-independence (3)—as well as path-independence (1)—fails to provide a successful way out from Arrow's dilemma [2].

5. BINARY CHOICES AND FOLDED CHOICES

This section deals with some results concerning binary (5) choices, or choices "rationalized" by a (preference) relation in the sense of Richter [14, 15], or "basic binary" choices in the sense of Sen [18]. Plott [13] showed that binariness (5) of a choice is neither necessary for the sequential fidelity or path-independence (1) of a choice nor—in the absence of strong demands on the comparison revealed, viz., quasi transitivity—sufficient, even when $\mathfrak{A} = \mathfrak{F}[X]$. A binary (5) choice need not be route-independent (3) (see Example 2) and, hence, need not be routewise faithful (1r); neither need a routewise faithful (1r) choice be binary (5) (see Example 4). Furthermore, binariness (5) does not imply condition (2) (see Example 2) and, hence, fails to imply the stronger condition

$$(A \cup \{x\}) \overset{s}{\cup} \{x\} = s(A \cup \{x\}), \tag{6}$$

called *point-absorbance*;[9] neither does point-absorbance of a choice imply binariness (5) (see Example 5).

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[9] Introduced in [23–25], this condition, in connection with the stronger condition (7), below, receives the attention of Lemma 7 and Proposition 8.

Turning to binary (5) choices and foldings, we first see that a binary (5) choice coincides with its folding:

5. THEOREM. *Choices binary* (5) *on* $\mathfrak{F}[X]$ *are routewise computable* (4) (*but the converse fails*).[10]

Proof. Setting $\mathfrak{A} = \mathfrak{F}[X]$, let s be a binary (5) choice. First recall that s then satisfies ($\bar{\mathrm{I}}$) (or, equivalently, ($\bar{\mathrm{I}}'$)), and thus s is also routewise upper semifaithful ($\bar{\mathrm{I}}$r). This leaves to show only that $\underset{\wedge}{s} \subset s$. Take any $A \in \mathfrak{F}[X]$ and any $x \in \underset{\wedge}{s}A$. Considering all the routes $\omega \in \Omega$ according to which x is first element in A, every $y \in A$ distinct from x is second element for at least one of these routes. Thus, for every $y \in A$, $x \in s(s\{x\} \cup \{y\})$, hence $x \in s\{x\}$ and $x \in s\{x, y\}$, i.e., ySx. Therefore, ASx, and the binariness of s gives $x \in sA$, showing $\underset{\wedge}{s} \subset s$. For the converse failing, see Example 4. ▌

The next proposition extends the well-known fact (see, e.g., [16, Chap. 1*]) that a binary (5) choice is decisive on $\mathfrak{F}[X]$ iff it reveals a complete and acyclic comparison S.

6. PROPOSITION. *If the folding* $\underset{\wedge}{s}$ *of a choice* s *is decisive, then* s *reveals a complete and acyclic comparison* S (*but not conversely*).[11]

Proof. Let s be a choice whose folding $\underset{\wedge}{s}$ is decisive. Clearly, s is then decisive on the singletons. Thus, taking any x, $y \in X$, $s\{x, y\} = s(s\{x\} \cup \{y\}) \supset \underset{\wedge}{s}\{x, y\} \neq \varnothing$, so S is complete. To see the acyclicity of S, let $A = \{x_1, \ldots, x_n\} \in \mathfrak{F}[X]$ with $x_i < x_{i+1}$ and $x_i \hat{S} x_{i+1}$ for $i = 1, \ldots, n-1$. Thus $s_\delta A = \{x_n\}$. Now suppose $x_n \hat{S} x_1$. Let ω be any route with $\omega(x_n) = x_1$ and $\omega(x_i) = x_{i+1}$ for $i = 1, \ldots, n-1$. Then $x_n \notin s_\omega A$, so $s_\delta A \cap s_\omega A = \varnothing$, contradicting that $\underset{\wedge}{s}$ is decisive. Hence, $x_n \hat{S} x_1$ fails, showing that S is acyclic. For the converse failing, see Example 6. ▌

For decisive choices which are binary (5) on $\mathfrak{F}[X]$, many of the conditions formulated so far become indistinguishable from the quasi transitivity of the comparison revealed by s (i.e., of the base relation of s). To properly state this result, we reintroduce a condition also studied in [21–25], viz., the condition

$$(A \cup B) \overset{s}{\cup} A = s(A \cup B) \tag{7}$$

termed *absorbance*, clearly implying point-absorbance (6). Although in general point-absorbance (6) is strictly weaker than absorbance (7)[12] (see Example 5), we note the following.

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[10] See Proposition 6.1 in [23].

[11] See Proposition 3 in [23].

[12] Regardless of whether $\mathfrak{A} = \mathfrak{F}[X]$ or not; see also Examples 9 and 10 in [24].

7. LEMMA [23]. *For sequentially upper semifaithful* $(\bar{1})$ *choices, point-absorbance* (6) *and absorbance* (7) *coincide.*

Proof. That (7) $\Rightarrow$ (6) is obvious. For the rest, let s be a choice satisfying $(\bar{1})$ (or, equivalently, $(\bar{1}')$) and assume s to be point-absorbent (6). Take any $A, B \in \mathfrak{A}$. Then $s(A \cup B) \cup A \subset A \cup B$, so $(\bar{1}')$ gives $s(A \cup B) = s(A \cup B) \cap (s(A \cup B) \cup A) \subset s(s(A \cup B) \cup A)$. For the reverse containment, taking any $x \in s(s(A \cup B) \cup A)$, using $(\bar{1}')$ and (6), we have $x \in s(s(A \cup B) \cup A) \cap (s(A \cup B) \cup \{x\}) \subset s(s(A \cup B) \cup \{x\}) = s(A \cup B)$, so that $s(s(A \cup B) \cup A) \subset s(A \cup B)$. Thus, (7) holds. ∎

8. THEOREM. *Assuming* $\mathfrak{A} = \mathfrak{F}[X]$, *let* s *be a decisive binary* (5) *choice. Now all the following are equivalent:*[13]

(i) s is absorbent (7),
(ii) s is point-absorbent (6),
(iii) s satisfies (2),
(iv) s is sequentially faithful or path-independent (1),
(v) s is routewise faithful (1r),
(vi) s is route-independent (3),
(vii) s reveals a quasi-transitive comparison S.

Proof. That (*i*) $\Rightarrow$ (ii) $\Rightarrow$ (iii) is plain. From [24, Proposition 5] we know that $(\bar{1})$ and (2) $\Rightarrow$ (1); since s is binary (5) and (5) $\Rightarrow$ $(\bar{1})$, it follows that (iii) $\Rightarrow$ (iv). Now (iv) $\Rightarrow$ (v) $\Rightarrow$ (vi) is again plain. As s is decisive, Proposition 4 gives us (vi) $\Rightarrow$ (vii), leaving to show[14] only (vii) $\Rightarrow$ (i). In view of Lemma 7, to prove this it actually suffices to show (vii) $\Rightarrow$ (ii). Let $x_0 \in B \in \mathfrak{A}$. Now (5) implies that $s(sB \cup \{x_0\}) \supset sB$. If $x_0 \in sB$, then also $s(sB \cup \{x_0\}) = s(sB) \subset sB$, leaving nothing to show. So assume $x_0 \in B \backslash sB$. As s is decisive, S is complete; s being binary (5), there is then a (finite) sequence of distinct points $x_1, \ldots, x_n \in B$ with $x_0 \hat{S} x_1 \hat{S} \cdots \hat{S} x_n$ and $x_n \in sB$. Assuming (vii), the transitivity of $\hat{S}$ then gives $x_0 \hat{S} x_n$, so $x_0 \notin s(sB \cup \{x_0\})$, i.e., $s(sB \cup \{x_0\}) \subset sB$, whereby (ii) obtains, as desired. ∎

Recently, Parks proved another list of equivalents when $\mathfrak{A} = \mathfrak{F}[X]$ and s is binary (5) [12, Theorem 5]; Parks' list can be combined with Theorem 8 to yield a list of altogether 11 equivalents. Thus, the potential richness of the structure of "piecemeal" choice vanishes considerably when we restrict attention to decisive binary (5) choices.

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[13] This improves upon Theorem 10 in [23].

[14] Actually, Plott [13, Theorem 4] showed that a decisive binary (5) choice on $\mathfrak{F}[X]$ revealing a quasi-transitive comparison is sequentially faithful (1), whereby it is obviously absorbent (7). For the sake of self-containment, however, we indicate a brief proof here as well.

6. GENETIC ASPECTS OF THE ROUTEWISE APPLICATION OF CHOICE

In this section we are concerned with the relations between the properties enjoyed by a choice and those enjoyed by its routewise applications.

It seems interesting that properties of a choice are not generally "inherited" by its routewise applications. In particular,

9. *Note*. The routewise applications s_ω of a binary (5) choice s need not satisfy $(\bar{1}')$, let alone (5). For let $X = \{x, y, z\}$ with all elements distinct, let $x < y < z$, and consider the binary (5) choice s determined by the comparison S with $z\hat{S}x\hat{S}ySzSy$. Then $s_\delta X = \{y, z\}$, but $s_\delta\{x, z\} = \{x\} \not\supset \{z\} = s_\delta X \cap \{x, z\}$.

Therefore the question arises to what extent binary (5) or sequentially upper semifaithful $(\bar{1})$ choices may degenerate in their routewise applications.

First note that a binary (5) choice is always *completely optimizing* in the sense that it chooses—by definition—precisely the set of "best" elements according to the comparison S it reveals (i.e., according to its base relation). In fact, the binary (5) choices are just the completely optimizing choices. As to the choices satisfying $(\bar{1})$, we know (e.g., from [20]) that they are "optimizing" in the sense that they choose only best elements according to their own revealed comparisons. What can we say for the routewise applications of choices satisfying (5) or $(\bar{1})$? Let us agree to call a choice s *satisfying* iff, for every $A \in \mathfrak{A}$, sA owns only elements x such that ySx for every $y \in A$ which is best (i.e., ASy) in A according to the comparison S revealed by s. Let us also agree to say that a choice s is *routewise satisfying* iff, for every $\omega \in \Omega$ and every $A \in \mathfrak{F}[X]$, $s_\omega A$ owns only elements x such that ySx for every $y \in A$ with ASy. Finally, let us say that a choice s is *routewise tamed* iff, for every $\omega \in \Omega$ and every $A \in \mathfrak{F}[X]$, $x \in s_\omega A$ implies $(sA)Sx$.

10. LEMMA. *Let s be a choice. If s is decisive on the singletons, then $S = S_\omega$* (i.e., s and s_ω reveal the same comparison) *for every $\omega \in \Omega$.*

Proof. Assume that s is decisive on the singletons and take any $\omega \in \Omega$. Considering any $x, y \in X$, say $x \leqslant_\omega y$, we see that $s_\omega\{x, y\} = s(s\{x\} \cup \{y\}) = s\{x, y\}$, so evidently $S = S_\omega$. ▮

11. THEOREM. *Let s be a choice.*

1. *If s is sequentially upper semifaithful* $(\bar{1})$, *then s is routewise tamed.*
2. *If s is binary* (5), *then s is routewise satisfying.*
3. *If s is binary* (5) *and decisive on the singletons, then every routewise application s_ω of s is satisfying.*

Proof. ad 1. Assume s satisfies $(\bar{1})$, and take any route $\omega \in \Omega$ and any $A \in \mathfrak{F}[X]$. Since $(\bar{1}) \Rightarrow (\bar{1}\mathrm{r})$, $sA \subset s_\omega A = s((s_\omega^{n-1}A) \cup \{n_\omega A\})$, where $n = \#A$, so $sA \subset (s_\omega^{n-1}A) \cup \{n_\omega A\}$. Now, for every $x \in s_\omega A$, by ($\bar{1}$), we have $((s_\omega^{n\ 1}A) \cup \{n_\omega A\})\, Sx$, so $(sA)\, Sx$.

ad 2. Assuming s binary (5), for any $A \in \mathfrak{A}$, sA is the set of "best" elements of A according to S. Apply Theorem 11.1.

ad 3. Assume the hypothesis of Theorem 11.3, and take ω, A as in *ad* 11.1, so that, by Theorem 11.1, for every $x \in s_\omega A$, $(sA)\, Sx$. But Lemma 10 gives $S = S_\omega$, and sA is the set of best elements according to S_ω, so s_ω is satisfying. ▌

As we saw at the end of the last section, again a number of otherwise distinct conditions become equivalent for binary (5) choices. For, according to Theorems 11.1 and 11.2, being routewise tamed and being routewise satisfying coincide for binary (5) choices. In fact, *the conditions of being routewise tamed and being routewise satisfying for a choice are independent of each other.* Check Example 7 to see that a routewise tamed choice s need not be routewise satisfying, even if s is decisive and sequentially upper semifaithful ($\bar{1}$) and reveals a (complete) quasi-transitive comparison. On the other hand, check Example 8 to see that a routewise satisfying choice s need not be routewise tamed, even if s is decisive and reveals a complete acyclic comparison.

While sequential upper semifidelity ($\bar{1}$) or binariness (5) of a choice s need not be preserved by its routewise applications s_ω (see Note 9), Theorem 11 shows that choices satisfying ($\bar{1}$) or (5) do not entirely degenerate when routewise applied. We should now note, furthermore, that the properties enjoyed by a choice s may be far less attractive than those enjoyed by its routewise applications s_ω. For instance, Example 9 shows that the routewise applications s_ω of a choice s failing even ($\bar{1}$) may all be binary (5) as well as sequentially faithful or path-independent (1).[15]

7. Summary and Concluding Remarks

Whenever something has to be chosen from a set of alternatives we may proceed in a "divide-and-conquer" manner: the set of alternatives is split up one way or another, and we choose by a step-by-step or piecemeal procedure. Plott [13], in a stimulating paper, developed the notion of a "path" on a set (of alternatives) to discuss conditions we might want to see satisfied by such a piecemeal procedure; here, we present the notion of a "route" as an alternative way of systematic thinking about divide-et-impera choice.

This approach enabled us to give the strongest characterization of Plott's path-independence—or sequential fidelity (1), in our terminology—known so far, when the set of alternatives is finite (see Theorem 3): i.e., (1) is equivalent to the conjunction of conditions (2) and ($\bar{1}$r), the latter being a condition specifically developed for routewise applications of choices. Also, we strengthened Plott's result that quasi transitivity of the comparison revealed by a choice (i.e., of its base relation) is necessary for sequential fidelity (1) of the choice, since it is necessary even for the route-independence (3) of the choice (see Proposition 4), a condition implied by routewise fidelity (1r) which in turn is implied by sequential fidelity (1) (whereas the converses fail).

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[15] In fact, the choice s in this example even fails *idempotence*, i.e., the condition that $sA = s(sA)$ ($A \in \mathfrak{A}$). For discussions of this condition, see [20, 22, 24].

Talking about properties of the comparison revealed by a choice (i.e., the base relation of a choice), we might note in passing that its acyclicity is necessary for routewise applications of a choice to agree on at least some element(s) common to all routewise results, i.e., for a decisive "folding" of the choice (see Proposition 6). On the other hand, even full transitivity of the comparison revealed by a choice is not sufficient for the decisiveness of its folding (Example 6), and neither is it sufficient for routewise computability (4), sequential upper semifidelity ($\bar{1}$), point-absorbance (6), or the condition of being routewise tamed, even together with certain other conditions (see Examples 1, 3, 4, and 5, respectively).

Furthermore, we checked the relation between binary (5) choices and their routewise applications, first seeing that binary (5) choices coincide with their foldings (see Theorem 5; compare Example 1 to see that this is generally not true for nonbinary choices). Also, the conditions of being routewise tamed and routewise satisfying for a choice become indistinguishable when dealing with binary (5) choices (see Theorem 11); and, whenever the comparison revealed by a decisive binary (5) choice is quasi-transitive, we get a list of altogether seven equivalent conditions, among which there are two conditions for routewise applications of choices (viz., routewise fidelity (1r) and route independence (3); see Theorem 8).

Finally, we asked the question to what extent choices may "degenerate" in their routewise applications, i.e., to what extent conditions enjoyed by a choice may be violated by its routewise applications (cf. Note 9). We found sequentially upper semifaithful ($\bar{1}$) choices guaranteed to be at least "routewise tamed," and binary (5) choices to be at least "routewise satisfying" (see Theorem 11), so that these choices cannot lose all their attractiveness when routewise applied. On the other hand, we also noted that the routewise applications s_ω of a choice s may satisfy attractive conditions which the "original" choice s violates (see Example 9).

In summary, then, we hope to have demonstrated that the notion of "routes" in piecemeal choice yields new insights into the nature of "paths" in the sense of Plott [13], as well as being of interest of its own, opening up a new avenue of research in quest of the relation of choice to the sequence in which alternatives are considered.

APPENDIX: (COUNTER-) EXAMPLES

(Throughout, $T = \{x, y, z\} \in \mathfrak{A}$ and $Q = \{w, x, y, z\} \in \mathfrak{A}$ and all elements of Q (hence, of T) are distinct.)

EXAMPLE 1. (A decisive, sequentially upper semifaithful $(\bar{1})$ and route-independent (3) choice revealing a complete preorder as a comparison but failing routewise computability (4) (and, hence, routewise fidelity (1r)).

Take $X = Q$ and consider the decisive choice s with $sQ = \{z\}$, $w\hat{S}x\hat{S}ySzSy$ and S transitive, and s behaves as a binary (5) choice on all proper subsets $A \subsetneq Q$.

Check: For all $\omega \in \Omega$, $s_\omega Q = \{y, z\} \supsetneq \{z\} = sQ$.

EXAMPLE 2 (A decisive, binary (5) (hence, routewise computable (4)) choice revealing a complete acyclic comparison but failing route-independence (3) and condition (2)).

Take $X = T$ and consider the decisive choice s with $sT = \{z\}$, $x\hat{S}y\hat{S}zSxSz$.

Check (failure of (3)): For $x <_\omega y <_\omega z$ and $y <_{\omega'} z <_{\omega'} x$, $s_\omega T = \{z\} \neq \{x, z\} = s_{\omega'} T$.

(Failure of (2)): $s\{x, z\} = s(sT \cup \{x\}) = sT \cup \{x\}$.

EXAMPLE 3 [25] (A decisive, routewise upper semifaithful $(\bar{1}r)$ choice revealing a complete preorder as a comparison but failing sequential upper semifidelity $(\bar{1})$).

Take $X = Q \cup \{v\}$ with $\{v\}$ disjoint from Q and consider the decisive choice s with $x\hat{S}y\hat{S}z$, S complete and transitive on all of X as well as symmetric on $\{z, w, v\}$; s behaves as a binary (5) choice on all $A \subset X$ when $A \neq Q$; and $sQ = \{z\}$.

Check (failure of $(\bar{1}')$): $Q \cap sX = \{z, w\} \not\subset \{z\} = sQ$.

EXAMPLE 4 (A decisive, routewise faithful (1r) (hence, by Lemma 1, also sequentially upper semifaithful $(\bar{1})$) choice revealing a complete preorder as a comparison but failing sequential fidelity (1), binariness (5), and point-absorbance (6)).

Take $X = T$ and consider the decisive choice s with $sT = \{z\}$ and $sA = A$ for all $A \subsetneq T$.

Check (failure of (1)): $T \overset{\circ}{\cup} \{x\} = \{x, z\} \supsetneq \{z\} = s(T \cup \{x\})$.

(Failure of (5)): $sT = \{z\} \neq T = \{t \in T \mid TSt\}$.

(Failure of (6)): $(T \cup \{x\}) \overset{\circ}{\cup} \{x\} = \{x, z\} \neq \{z\} = s(T \cup \{x\})$.

EXAMPLE 5 [24] (A decisive, point-absorbent (6) choice revealing a complete preorder as a comparison but failing to be routewise tamed (hence, by Theorem 11.1, also failing $(\bar{1})$) and failing to be absorbent (7)).

Take $X = Q$ and consider the decisive choice s with $Q\hat{S}w$, S complete and transitive on all of Q as well as symmetric on T, $s\{w, x, y\} = \{x, y\}$, $sQ = \{w\}$; $sA = A$ for all subsets $A \subset Q$ so far unlisted.

Check (failure of being routewise tamed): For $x <_\omega y <_\omega w <_\omega z$, $s_\omega Q = \{x, y, z\}$, but $(sQ)\, Sx$ fails.

(Failure of (7)): $(\{x, y\} \cup Q) \overset{s}{\cup} \{x, y\} = \{x, y\} \neq \{w\} = s(\{x, y\} \cup Q)$.

EXAMPLE 6 (A decisive choice s revealing a complete preorder as a comparison but yielding a nondecisive folding $\underset{\circ}{s}$).

Take $X = Q$ and consider the choice s with the comparison S as given in Example 5, $sQ = \{x\}$, and s behaving as a (decisive) binary (5) choice on all $A \subsetneq Q$.

Check: For $x <_\omega y <_\omega z <_\omega w$ and $w <_{\omega'} x <_{\omega'} y <_{\omega'} z$, $(s_\omega Q = \{x\}) \cap (s_{\omega'} Q = \{w\}) = \varnothing$.

EXAMPLE 7 (A decisive, sequentially upper semifaithful ($\bar{\text{I}}$) (hence, routewise tamed) choice revealing a complete quasi-transitive comparison but failing to be routewise satisfying).

Take $X = Q$ and consider the decisive choice s with $w\hat{S}z\hat{S}y$ and S quasi-transitive, $sD = D$ for every doubleton $D \subset Q$ with $x \in D$, $sQ = \{x\}$, and s behaving as a binary (5) choice on every tripleton contained in Q, except that $s\{w, x, y\} = \{x\}$.

Check: $s_8 Q = \{x, z\}$, but $ASy\not{S}z$.

EXAMPLE 8 (A decisive, routewise satisfying choice revealing a complete acyclic comparison but failing to be routewise tamed).

Take $X = Q$ and consider the decisive choice s with $x\hat{S}z\hat{S}ySxSy$ and $w\hat{S}a$ for all $a \in Q$, $sQ \supset \{z\}$, and s behaving as a binary (5) choice on all proper subsets $A \subsetneq Q$.

Check: For $z <_\omega y <_\omega w <_\omega x$, $s_\omega Q = \{x, y\}$, but $z \in sQ$ and $z\not{S}x$.

EXAMPLE 9 (A decisive choice that fails sequential upper semifidelity ($\bar{\text{I}}$) but whose routewise applications are all binary (5) as well as sequentially faithful (I)).

Take $X = T$ and consider the decisive choice s with $sX = \{x, y\}$ and $x\hat{S}y\hat{S}z$ with the comparison S transitive and complete. Then $s_\omega X = \{z\}$ for all $\omega \in \Omega$ and s_ω behaves as a binary (5) choice; and, since the comparison $S_\omega = S$ (cf. Lemma 10) revealed by s_ω is a complete preorder, s_ω is also sequentially faithful (I) (cf. Theorem 8).

ACKNOWLEDGMENTS

The authors are indebted to the referees of this Journal for their comments on the terminology of our paper, i.e., its comparability to the previous literature. Thanks are also due to Kotaro Suzumura for his observations with respect to our Proposition 4.

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JOURNAL OF ECONOMIC THEORY 22, 439–450 (1980)

Economies as Distributions: Implications for Aggregation and Stability*

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Received November 20, 1978

I. INTRODUCTION

Aumann [2] pointed out that a mathematical model appropriate to the intuitive notion of perfect competition must contain infinitely many participants. He then proposed that the most useful model contains a continuum of traders, since this allows us access to the tools of functional analysis. There followed an elegant but slender literature which has been comprehensively surveyed by Hildenbrand [7].

The Aumann–Hildenbrand methodology is as follows: An economy is defined by a probability measure on the space of consumers' characteristics – i.e., by the actual distribution of tastes and initial endowments in the economy – and the space of economies is topologized. Mean behavior in the economy may then be found by integration over the space of consumers' characteristics, and is continuous in the underlying parameters defining the economy.

Hildenbrand notes that this modeling technique has not been applied to stability analysis. The present paper addresses this void. Theorems in probability theory are exploited to derive some interesting results for aggregation and stability in competitive economies. Moreover, by considering the space of economies as the space of probability measures on tastes and endowments, it is possible to show structural stability for these results.

It is important to note, however, that the strength of this formulation lies in considering demand, income, etc., as random elements, not in a stochastic sense, but in the sense that each has a distribution function; the approach is equally valid whether the variables are distributed finitely or continuously.

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* The problem of showing stability in an economy with a continuum of traders was posed to me by Yoshihiko Otani. This work has benefited from discussions with and comments by Joseph Sicilian, Robert Meyer, John McDonald, Allen DeSerpa, David Spencer, and Andreu Mas-Colell. An anonymous reviewer also provided many useful comments. All remaining errors are my own.

0022-0531/80/030439-12$02.00/0

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The present model is formulated in a way sufficiently general to allow either a finite set or a continuum of agents. If the set of agents is finite, averages replace integrals and the measure is a counting measure. It follows from Aumann [3] that the assumption of convex preferences for all agents may be dropped when the space of agents is atomless.

In Section II the formal model is constructed. Tastes are characterized by well-behaved continuously differentiable demand functions and economies are defined by a joint probability measure on tastes and initial endowments.

In Section III the question of stability in economies of nearly identical consumers is addressed. It is known that if the Jacobian of excess demand with respect to prices is negative definite at an equilibrium price vector, that the equilibrium is locally stable. It is further known that in this respect the matrix of mean substitution effects is well behaved and that instability is caused only by "large" badly behaved income effects. Theorem 1 shows that, for economies of nearly identical consumers, income effects are small. In particular, it is shown that the matrix of income effects depends on the variances of demand and endowments, both of which are continuous with respect to the probability measure which defines the economy.

In Section IV economics of homothetic consumers are considered. Under the assumption of independent distribution of tastes and endowments, it is shown that at equilibrium the matrix of mean income terms is positive semidefinite and hence always stabilizing regardless of size. Moreover, it is shown that mean demand in the economy is generated by a homothetic utility function and hence behaves like the demand of a single homothetic consumer.

II. The Model

We begin with notation:

$X = E_+^N$ is the space of bundles of N perfectly divisible *consumption goods.*

$W = \dot{E}_+^N$ is the space of *initial endowment vectors.*

$P = \{p \in \dot{E}_+^N \mid \sum_i p_i = 1\}$ is the space of normalized strictly positive *price vectors.*

$M = \dot{E}_+^1$ is the space of potential *consumer incomes.*

If $\mathscr{H}$ is a set of continuously differentiable functions $h\colon M \times P \to X$, denote

$$h_{ij}(m, p) = \partial h_i(m, p)/\partial p_j\,;$$

$$h_{im}(m, p) = \partial h_i(m, p)/\partial m.$$

Then define the following maps as shown:

$$H\colon \mathscr{H} \times M \times P \to X$$
$$\text{by} \quad H(h, m, p) = h(m, p).$$
$$J\colon \mathscr{H} \times M \times P \to E^{N^2}$$
$$\text{by} \quad J(h, m, p) = [h_{ij}(m, p)].$$
$$H_m\colon \mathscr{H} \times M \times P \to E^{N}$$
$$\text{by} \quad H_m(h, m, p) = (h_{1m}(m, p), h_{2m}(m, p), \ldots, h_{Nm}(m, p)).$$
$$I\colon \mathscr{H} \times M \times P \to E^{N^2}$$
$$\text{by} \quad I(h, m, p) = H_m(h, m, p) \cdot (H(h, m, p) - \partial m/\partial p)^T.$$
$$S\colon \mathscr{H} \times M \times P \to E^{N^2}$$
$$\text{by} \quad S(h, m, p) = J(h, m, p) + I(h, m, p).$$

It is immediate from construction that each of the above maps is continuous everywhere on $M \times P$.

$\bar{J}$, $\bar{I}$, and $\bar{S}$ will be used to denote any $N - 1$ principal minor matrix of J, I, S, respectively.[1]

We will call a continuously differentiable map $h\colon M \times P \to X$ *regular* at $(m, p) \in M \times P$ if every $N - 1$ principal minor matrix $\bar{S}(h, m, p)$ of $S(h, m, p)$ is strictly negative definite.

We shall represent *preferences* by continuously differentiable demand functions $h\colon M \times P \to X$, which satisfy the following conditions:

(1) $h(m, p)$ satisfies the budget constraint as an equality.

(2) h is homogeneous of degree zero.

(3) h satisfies Richter's Congruence Axiom.[2]

(4) If $\{(m^n, p^n)\}$ is any sequence in $M \times P$ such that $(m^n, p^n) \to (m, p) \in M \times \bar{P}\backslash P$, then $\lim_{n\to\infty} \| h(m^n, p^n)\| = \infty$.

(5) h is regular everywhere on $M \times P$.

Mas-Colell [9] has shown that the set of regular continuously differentiable demand functions is dense in the space of convex valued, upper hemicontinuous demand correspondences.

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[1] For any $N \times N$ matrix V, the $N - 1$ principal minor matrices are the set of matrices consisting of the first $N - 1$ rows and columns of $P'VP$, where P is an $N \times N$ matrix for which each row and each column contains a 1, all other elements being 0.

[2] Define the binary relation $W_h \subset X \times X$ by $W_h = \{(x, y) \in X \times X \mid \text{for some } (m, p) \in M \times P, x = h(m, p) \text{ and } py \leqslant m\}$. Let $\hat{W}_h$ be the transitive closure of W_h (i.e., the smallest transitive relation containing W_h). Then for every $(m, p) \in M \times P$, $x = h(m, p)$ implies that $y \in \hat{W}_h$ for every $y \in X$ such that $p \cdot y \leqslant m$.

$\mathscr{H}$, *the space of preferences*, is the set of continuously differentiable demand functions that satisfy conditions (1)–(5) above, endowed with the Whitney C^1 topology.[3] Then $I(h, m, p)$ is the matrix of income effects and $S(h, m, p)$ is the Slutsky (substitution) matrix of h at (m, p).

$C = \mathscr{H} \times W$ is *the space of consumers' characteristics*. Let $\mathscr{B}$ be the Borel σ-algebra on C relative to the product topology. Then $(C, \mathscr{B})$ is a measurable space.

Define the following maps:

(i) $\hat{W}: C \to W$
by $\hat{W}(h, w) = w$.

(ii) $\hat{H}: C \times P \to X$
by $\hat{H}(h, w, p) = H(h, p^T w, p)$.

(iii) $\hat{H}_m: C \times P \to E^N$
by $\hat{H}_m(h, w, p) = H_m(h, p^T w, p)$.

(iv) $\hat{J}: C \times P \to E^{N^2}$
by $\hat{J}(h, w, p) = J(h, p^T w, p)$.

(v) $\hat{I}: C \times P \to E^{N^2}$
by $\hat{I}(h, w, p) = \hat{H}_m(h, w, p) \cdot (\hat{H}(h, w, p) - \hat{W}(h, w))^T$.

(vi) $\hat{S}: C \times P \to E^{N^2}$
by $\hat{S}(h, w, p) = \hat{J}(h, w, p) + \hat{I}(h, w, p)$.

(vii) $\hat{M}: C \times P \to M$
by $\hat{M}(h, w, p) = p^T w$.

For a consumer characterized by tastes h and initial endowments w, and facing prices p, the above functions indicate (i) endowments, (ii) demand, (iii) the marginal propensity to consume relative to income, (iv) the Jacobian of demand with respect to prices, (v) the matrix of income effects, (vi) the Slutsky matrix, and (vii) income.

The continuity of each of these maps is immediate and the differential of $\hat{M}(h, w, p)$ with respect to p is $\hat{W}(h, w)$.

A *space of agents* is any measure space (A, Ω, μ) where Ω is a μ-additive σ-algebra on a class of subsets of A, and μ is a probability measure.

A *pure exchange economy* with N goods is a measurable compact-valued function

$$\xi: A \to C.$$

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[3] The topology of uniform convergence of the function and its first derivatives on compacta.

The map ξ induces a probability measure $f_\xi\colon \mathscr{B} \to [0, 1]$ by

$$f_\xi(B) = \mu\{a \in A \mid \xi(a) \in B\},\ B \in \mathscr{B};$$

i.e., f_ξ gives the taste-wealth distribution in economy ξ. Define $\operatorname{supp}(f)$ as the smallest closed set B in $\mathscr{B}$ such that $f(B) = 1$. Let F be the set of probability measures $f\colon \mathscr{B} \to [0, 1]$ such that $\operatorname{supp}(f)$ is compact. We may (without severely abusing notation) call F *the set of pure exchange economies* with N goods whose agents have demand functions in $\mathscr{H}$ and initial endowments in W.

The functions $\hat{W}$, $\hat{H}(\cdot, p)$, $\hat{H}_m(\cdot, p)$, $\hat{J}(\cdot, p)$, $\hat{I}(\cdot, p)$, $\hat{S}(\cdot, p)$, and $\hat{M}(\cdot, p)$ are continuous in C and hence Borel measurable and bounded on $\operatorname{supp}(f)$ for each $f \in F$, $p \in P$. Thus we may define for $p \in P$, $f \in F$:

$$w(f) = \int_C \hat{W}(h, w)\, df$$

= the mean endowment vector.

$$m(f, p) = \int_C \hat{M}(h, w, p)\, df$$

= mean income at prices p.

$$D(f, p) = \int_C \hat{H}(h, w, p)\, df$$

= the mean demand vector at prices p.

$$J(f, p) = \int_C \hat{J}(h, w, p)\, df$$

= mean Jacobian matrix of $D(f, p)$ evaluated at p.

$$I(f, p) = \int_C \hat{I}(h, w, p)\, df$$

= the matrix of mean income terms.

$$S(f, p) = \int_C \hat{S}(h, w, p)\, df$$

= the mean Slutsky matrix.

$$H_m(f, p) = \int_C \hat{H}_m(h, w, p)\, df$$

= the vector of mean partial derivatives of demand with respect to income.

It follows from assumption (5) on $\mathscr{H}$ that each $\bar{S}(f, p)$ is strictly negative definite for $f \in F$, $p \in P$, where $\bar{S}(f, p)$ represents an arbitrary $N - 1$ principal minor matrix of $S(f, p)$.

Let $\pi(f) = \{p \in P \mid D(f, p) = w(f)\}$ = the set of equilibrium price vectors.

Topologizing F. We endow F with the topology of weak convergence of measures, with the additional restriction that for each sequence $\{f^n\}$, there exists a compact subset $K \subset C$ such that $\{\operatorname{supp}(f^n)\} \subset K$. Under the topology of weak convergence, for a sequence $\{f^n\}$ in $F, f \in F, f^n \to f$ if and only if [5, Theorem 2].

$$\int_C g\, df^n \to \int_C g\, df, \qquad \text{for every bounded and continuous real-valued function } g \text{ defined on } C.$$

Then for a sequence $\{f^n\}$ whose family of supports is contained in a compact set K, if g is a continuous real-valued function defined on C, g is bounded on K, and $f^n(C \backslash K) = f(C \backslash K) = 0$. So, by the condition on sequential convergence, $g(f^n) \to g(f)$. We have immediately that the mean functions $w\colon F \to E_+^1$, $D\colon F \times P \to E_+^N$, $m\colon F \times P \to E_+^1$, $J\colon F \times P \to E^{N^2}$, $I\colon F \times P \to E^{N^2}$, $H_m\colon F \times P \to E^N$, $S\colon F \times P \to E^{N^2}$ are continuous everywhere on $F \times P$.

Hildenbrand [7, Sect. 2.3] has shown that

(R.1) For every $f \in F$, $\pi(f)$ is nonempty and compact-valued and the correspondence $\pi\colon F \to P$ given by $\pi(f) = \{p \in P \mid D(f, p) = w(f)\}$ is upper hemicontinuous in f.

(R.2) The set of economies in F in which $\pi(f)$ is finite and $\#\pi(f)$ (the number of elements in $\pi(f)$) is locally constant is dense in F; furthermore, a sufficient condition for $\#\pi(f)$ locally constant at $f \in F$ is that for all $p \in \pi(f)$, $\bar{J}(f, p)$ is strictly negative definite, where $\bar{J}$ is any principal minor matrix of J.

III. IDENTICAL AND "NEARLY IDENTICAL" CONSUMERS

Arrow and Hurwicz [1] showed that in the case of an economy of identical consumers, equilibrium is unique and globally stable. The basis of their proof is that with identical consumers, there is no trade at equilibrium, and hence the matrix of income terms vanishes. To prove these properties for economies of nearly identical consumers, one wants some sort of structural stability result. A referee has pointed out to me two such results. Sattinger [12] has shown that when initial holdings are close to equilibrium holdings, income effects are small and dominated by substitution effects. Balasko [4, Proposition 6], shows that the Jacobian is negative definite in a neighborhood of the set of economies for which the aggregate excess demand functions obey the weak axiom of revealed preference. Both of these results rely on continuity of the derivative matrices—the substitution and income effects in the first case, the Jacobian in the second—with respect to the parameters defining the economy. Our first result tells exactly why this happens.

The proof of Theorem I below shows that the matrix of income terms depends directly on the standard deviations of demand and endowments and hence approaches zero as the economy becomes composed of more nearly identical consumers. We can conclude (Corollary 1) that for economies of nearly identical consumers, equilibrium is unique and locally stable.

Let $F(I) = \{f \in F \mid \text{there exists } (h, w) \in C \text{ such that } \operatorname{supp}(f) = \{(h, w)\}\}$.

$F(I)$ is *the space of economies in which all consumers are identical.* $F(I)$ is well defined since C is metrizable and hence Hausdorff, so that each one-point set in C is closed [14, p. 86]. It follows immediately that each one-point set in C is an element of $\mathscr{B}$. $F(I)$ is the set of economies of identical consumers in the sense of identical tastes and endowments.

THEOREM I. *If $\{f^n\}$ is a convergent sequence in F, $f^n \to f \in F(I)$, $\{p^n\}$ is a sequence in P such that $p^n \in \pi(f^n)$ and $p^n \to p \in \pi(f)$, then the corresponding sequence of income effects $\{I(f^n, p^n)\}$ converges to zero.*

Before proving Theorem I, we need some additional machinery. If f is a measure on $(C, \mathscr{B})$ [11, pp. 250–251], define

$$\mathscr{L}^2(f) = \left\{ g\colon C \to E^1 \mid g \text{ is measurable and } \int_C g^2\, df < \infty \right\}.$$

Since F is a set of probability measures, to show $\int_C g^2\, df < \infty$ it suffices to show that g is bounded on $\operatorname{supp}(f)$. This follows immediately for any continuous real-valued function defined for C, for all $f \in F$. In particular, it holds for $\hat{H}_m(\cdot, p)$, $\hat{H}(\cdot, p)$, and $\hat{W}(\cdot)$. Also, for given $p \in P, f \in F$, $D(f, p)$ and $w(f)$ are constant on C, and hence continuous and bounded. It follows [11, Theorem 10.36] that

$$\text{(R.3)}\quad \begin{aligned} &\text{a.}\quad (\hat{H}(\cdot, p) - \hat{W}(\cdot)) \in \mathscr{L}^2(f),\\ &\text{b.}\quad (\hat{H}(\cdot, p) - D(f, p)) \in \mathscr{L}^2(f),\\ &\text{c.}\quad (w(f) - \hat{W}(\cdot)) \in \mathscr{L}^2(f). \end{aligned}$$

Let

$$\sigma_f^2(\hat{H}(\cdot, p)) = \int_C (\hat{H}(h, w, p) - D(f, p))^2\, df,$$

$$\sigma_f^2(\hat{W}) = \int_C (w(f) - \hat{W}(h, w))^2\, df.$$

Then by R.3b and c, $\sigma_f^2(\hat{H}(\cdot, p)$, $\sigma_f^2(\hat{W})$ are well defined and it is straightforward to prove:

Lemma 1. (a) *$\sigma_f{}^2(\hat W)\colon F \to E_+{}^N$ is continuous everywhere on F.* (b) *For given $p \in P$, $\sigma_f{}^2(\hat H(\cdot, p))\colon F \to E_+{}^N$ is continuous everywhere on F.*

Proof of Theorem I.

By Rudin [11, Theorem 10.35], for each $i, j = 1,\ldots, N$, and all $f^n \in F$, $p \in P$,

$$\hat I_{ij}(\cdot, p) = \hat H_{m,i}(\cdot, p)(\hat H_j(\cdot, p) - \hat W_j(\cdot)) \in \mathscr{L}^2(f^n), \tag{1}$$

$$\begin{aligned} \{I_{ij}(f^n, p^n)\}^2 &= \left\{\int_C \hat I_{ij}(h, w, p^n)\, df^n\right\}^2 \\ &\leq \left\{\int_C \hat H^2_{m,i}(h, w, p^n\, df^n)\right\}\left\{\int_C (\hat H_j(h, w, p^n) - \hat W_j(h, w))^2\, df^n\right\}. \end{aligned} \tag{2}$$

$\hat H_m{}^2(\cdot, p)$ is bounded on supp (f) for any p. Moreover, $\hat H_m{}^2$ is continuous on C and by R.2, π is upper-hemicontinuous in F. Hence

$$\int_C \hat H_m{}^2(h, w, p^n)\, df^n \to \int_C \hat H_m{}^2(h, w, p)\, df.$$

We have shown that the first term is eventually bounded; we shall show that the second is eventually zero.

We note that $p^n \in \pi(f^n)$ implies that $D(f, p^n) = w(f^n)$. Hence,

$$\begin{aligned} &\int_C (\hat H_j(h, w, p^n) - \hat W_j(h, w))^2\, df^n \\ &\quad = \int_C (\hat H_j(h, w, p^n) - D_j(f^n, p^n) + w_j(f^n) - \hat W_j(h, w))^2\, df^n. \end{aligned} \tag{3}$$

Then by the Minkowsky inequality [11, Theorem 10.36],

$$\begin{aligned} &\int_C (\hat H_j(h, w, p^n) - \hat W_j(h, w))^2\, df^n \\ &\quad \leq \left\{\left(\int_C (\hat H_j(h, w, p^n) - D_j(f^n, p^n))^2\, df^n\right)^{1/2} \right. \\ &\quad\quad \left. + \left(\int_C (w_j(f^n) - \hat W_j(h, w))^2\, df^n\right)^{1/2}\right\}^2 \\ &\quad = \{\sigma_{f^n}(\hat H_j(\cdot, p^n)) + \sigma_{f^n}(\hat W_j)\}^2 \quad \text{by definition.} \end{aligned} \tag{4}$$

The last term is clearly zero for $f \in F(I)$, since identical consumers (in the sense of tastes and resources), also behave identically. Hence $\{I_{ij}(f, p)\}^2 = 0$. By Lemma 1 and (R.2)

$$\{\sigma_{f^n}(\hat H_j(\cdot, p^n) + \sigma_{f^n}(\hat W_j)\}^2 \to \{\sigma_f(\hat H_j(\cdot, p) + \sigma_f(\hat W_j)\}^2$$

and hence $| I_{ij}(f^n, p^n)| \to_n 0$. Q.E.D.

COROLLARY 1. *If $f \in F(I)$, equilibrium is unique and locally stable for every f' in a neighborhood of f in F.*

Proof. It is known that equilibrium is unique in the case of identical consumers [1]. From Theorem 1, if $\hat{p} = \pi(f)$, $I(f, \hat{p}) = 0$. That $\bar{J}(f, \hat{p})$ is strictly negative definite follows immediately from $J(f, p) = S(f, p) - I(f, p)$ and the strict negative definiteness of $\bar{S}(f, p)$. Then from (R.2), for n sufficiently large $\pi(f^n)$ is a singleton.

$S: F \times P \to E^{N^2}$ is continuous everywhere on $F \times P$. Then by (R.2),

$$S(f^n, p^n) \to S(f, p);$$

i.e., for n sufficiently large, $\bar{S}(f^n, p^n)$ is bounded away from 0 near the negative definite matrix $\bar{S}(f, \hat{p})$, while by Theorem 1, $|I(f^n, p^n)| \to 0$. Thus $\bar{J}(f^n, p^n)$ is eventually quasi-negative definite; i.e., for n sufficiently large, p^n is locally stable. Q.E.D.

IV. ECONOMIES OF HOMOTHETIC CONSUMERS

Chipman [6] considered conditions under which aggregate demand may be represented by a single community demand function. Under the basic assumption that consumers' preferences are homothetic, Chipman showed that aggregate demand behaves like that of a single homothetic consumer if preferences are identical or if incomes are proportional.

Our Theorem 3 shows that in an economy of homothetic consumers, independent distribution of tastes and endowments is a sufficient condition for mean demand to behave like that of one homothetic consumer. Clearly an economy for which all consumers have identical tastes (and/or incomes) would have stochastic independence of tastes and endowments. Requiring only proportional incomes does not rule out correlation between tastes and endowments; hence there are economies satisfying Chipman's hypothesis which serve as counterexamples to the proposition that stochastic independence is necessary for aggregation.

The proof of Theorem 3 shows that for an economy of homothetic consumers with independent distribution of tastes and endowments, the Jacobian is strictly negative definite. It follows (Corollary 2) that the unique equilibrium is globally stable. Corollary 3 extends this result to economies of "nearly homothetic" consumers with "nearly independent" distribution of tastes and endowments.

The first theorem in this section, Theorem 2, shows a surprising result. The Jacobian at equilibrium is strictly negative definite not because income effects are small, but because they are stabilizing. The proof of this theorem is based on the fact that the matrix of income effects is a multiple of the variance/covariance matrix of tastes.

Characterizing Homothetic Consumers

A homothetic consumer is one for whom $h(\lambda m, p) = \lambda h(m, p)$ for all $(m, p) \in M \times P$. Chipman [6] has shown that for such consumers

$$\hat{H}_m(h, w, p) = \hat{H}(h, w, p)/\hat{M}(h, w, p);$$

i.e., expenditure on good i is a constant (positive) fraction of income, prices remaining constant, and hence $\hat{H}_m$ is a function of h, p only. It is convenient in this case to think of H_m as the primary characterization of preferences. Let

$$Z = \{h \in \mathscr{H} \mid \hat{H}_m(h, w, p) = \hat{H}(h, w, p)/\hat{M}(h, w, p), \quad \text{all} \quad (w, p) \in W \times P\}.$$

$$C(Z) = \{(h, w) \in C \mid h \in Z,\ w \in W\}.$$

$\mathscr{B}(Z) = \{B \cap C(Z) \mid B \in \mathscr{B}\}$ is the Borel σ-algebra on $C(Z)$.[4]

Let $F(Z)$ be the set of probability measures $f\colon \mathscr{B}(Z) \to [0, 1]$ with compact support, and endowed with the topology of weak convergence. Clearly $F(Z)$ is *the space of economies of homothetic consumers* and $F(Z) \subset F$. $\hat{H}_m(\cdot, p)$ is a vector of random variables with variances and covariances defined by

$$\sigma_f^2(\hat{H}_{m,i}(\cdot, p)) = \int_C (\hat{H}_{m,i}(h, w, p) - H_{m,i}(f, p))^2\, df;$$

$$\begin{aligned} &\operatorname{cov}_f(\hat{H}_{m,i}(\cdot, p), \hat{H}_{m,j}(\cdot, p)) \\ &\quad = \int_C (\hat{H}_{m,i}(h, w, p)\, \hat{H}_{m,j}(h, w, p)\, df - H_{m,i}(f, p)\, H_{m,j}(f, p). \end{aligned}$$

Let $V(f, p) =$ var/cov matrix of $\hat{H}_m(\cdot, p)$ over supp (f). The variances are clearly finite, hence $V(f, p)$ is positive semidefinite [13, p. 72].

Independent Distribution of Tastes and Endowments

Let $F(ZZ) = \{f \in F(Z) \mid$ the joint random variables $\hat{H}_m(\cdot, p)$, $\hat{W}$ are stochastically independent for all $p \in P\}$. Then $F(ZZ)$ is *the space of economies of homothetic consumers in which preferences and initial endowments are independently distributed.*

THEOREM 2. *If $f \in F(ZZ)$, $\bar{p} \in \pi(f)$, then the matrix of income effects $I(f, \bar{p})$ is positive semidefinite.*

---

[4] Note that Z is closed in $\mathscr{H}$, since if $\{h^n = g^n(p) \cdot m\}$ is a sequence in Z converging to $h \in \mathscr{H}$, then $h = g(p) \cdot m$, where $g = \lim_{n \to \infty} g^n$. Hence $C(Z) = Z \times W$ is closed in C, $\mathscr{B}(Z)$ is well defined, and $(C(Z), \mathscr{B}(Z))$ is a measurable subspace of $(C, \mathscr{B})$.

Proof. For $f \in F(ZZ)$,

$$
\begin{aligned}
I_{ij}(f, \bar{p}) &= \int_C \hat{H}_{m,i}(h, w, \bar{p})[\hat{H}_{m,j}(h, w, \bar{p})\, \hat{M}(h, w, \bar{p}) - \hat{W}(h, w)]\, df \\
&= m(f, \bar{p}) \int_C \hat{H}_{m,i}(h, w, \bar{p})\, \hat{H}_{m,j}(h, w, \bar{p})\, df - w_j(f)\, H_{m,i}(f, \bar{p}). \qquad (5)
\end{aligned}
$$

But $\bar{p} \in \pi(f)$ implies that $w_j(f) = D_j(f, \bar{p}) = \int_C \hat{H}_{m,j}(h, w, \bar{p})\hat{M}(h, w, \bar{p})\, df$. Substituting into (5) yields

$$
I_{ij}(f, \bar{p}) = m(f, \bar{p}) \left\{ \int_C \hat{H}_{m,i}(h, w, \bar{p})\, \hat{H}_{m,j}(h, w, \bar{p})\, df - H_{m,i}(f, \bar{p})\, H_{m,j}(f, \bar{p}) \right\}. \tag{6}
$$

It follows that $I(f, \bar{p}) = m(f, \bar{p}) V(f, \bar{p})$. Q.E.D.

THEOREM 3. *If $f \in F(ZZ)$, there exists a utility function u: $X \to E$ which is homogeneous of degree 1 and continuous in x, such that u generates $D(f, p)$ everywhere on $M \times P$.*

Proof. For $f \in F(ZZ)$, $D(f, p) = H_m(f, p)m(f, p)$ for all $p \in P$ [8, p. 79, Theorem 3]. $D(f, p)$ is continuous in p and clearly satisfies $p \cdot D(f, p) = M(f, p)$ for all $p \in P$; $D(f, p)$ also possesses partial derivatives with respect to p, namely $J(f, p)$, everywhere on P; and $(\partial D(f, p)/\partial m) = H_m(f, p)$.

Hence from Chipman [6], it suffices to show that the Slutsky matrix $\bar{T}(f, p)$ is symmetric and negative semidefinite, where

$$
T_{ij}(f, p) = J_{ij}(f, p) + H_{m,i}(f, p)(D_j(f, p) - w_j(f)).
$$

We have $J(f, p) = S(f, p) - I(f, p)$, where $f \in F(ZZ)$ implies

$$
I_{ij}(f, p) = m(f, p) \int_C \hat{H}_{m,i}(h, w, p)\, \hat{H}_{m,j}(h, w, p)\, df - H_{m,i}(f, p)\, w_j(f). \tag{7}
$$

It follows that

$$
\begin{aligned}
T_{ij}(f, p) &= S_{ij}(f, p) - m(f, p) \int_C \hat{H}_{m,i}(h, w, p)\, \hat{H}_{m,j}(h, w, p)\, df \\
&\quad + H_{m,i}(f, p)\, w_j(f) + H_{m,i}(f, p)(D_j(f, p) - w_j(f)) \\
&= S_{ij}(f, p) - m(f, p) \int_C \hat{H}_{m,i}(h, w, p)\, \hat{H}_{m,j}(h, w, p)\, df \\
&\quad + H_{m,i}(f, p)\, H_{m,j}(f, p)\, m(f) \\
&= S_{ij}(f, p) - \operatorname{cov}_f(\hat{H}_{m,i}(\cdot, p), \hat{H}_{m,j}(\cdot, p))\, m(f, p). \qquad (8)
\end{aligned}
$$

That is, $T(f, p) = S(f, p) - m(f, p)\, V(f, p)$. Q.E.D.

COROLLARY 2. *If* $f \in F(ZZ)$, $\pi(f)$ *is unique and globally stable. Moreover,* $\bar{T}(f, p)$ *is strictly negative definite for all* $p \in P$, *where* $\bar{T}(f, p)$ *is any* $N - 1$ *principal minor matrix of* $T(f, p)$.

COROLLARY 3. *If* $f \in F(ZZ)$, $\{f^n\}$ *a sequence in* F *such that* $f^n \to f$, *then there exists* $\bar{n}$ *such that* $n \geq \bar{n}$ *implies that* $p^n = \pi(f^n)$ *is unique and locally stable.*

Proof. For $\bar{p} \in \pi(f)$, $J(f, \bar{p}) = T(f, \bar{p})$. So by (R.2) and Corollary 2, $\pi(f^n)$ is unique for n sufficiently large. By (R.1) and continuity of J on $F \times P$, $J(f^n, p^n) \to J(f, \bar{p})$.

Then $\bar{J}(f, \bar{p})$ strictly negative definite implies that there exists $\bar{n}$ such that for $n \geq \bar{n}$, $\bar{J}(f^n, p^n)$ is quasi-negative definite. Q.E.D.

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JOURNAL OF ECONOMIC THEORY 22, 451–464 (1980)

Innovation, Rates of Profit, and Uniqueness of von Neumann Prices

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Received November 28, 1978; revised May 21, 1979

1. INTRODUCTION

A classical problem of Marxian economics has been to investigate the effect of technical innovation upon the equilibrium profit rate. Marx (1966) surmised that, if the real wage remained constant, the technical changes which capitalists introduced would have a "tendency" to lower the rate of profit. It has been shown this is not the case: briefly, *viable* technical changes cause the wage–profit rate frontier to move outward, and therefore raise the equilibrium profit rate at constant real wages. This result was rigorously demonstrated by Okishio (1961) in a linear, Leontief model of an economy. More recently, the question has been investigated by this author (Roemer, 1977, 1979).

Formal discussions of the falling (or rising) rate of profit have been limited to simple Leontief models. For a treatment of the question in which the existence of fixed capital, differential turnover times, and joint products are fully taken into account, it is natural to ask what happens to the rate of profit consequent upon technical change in a von Neumann model of an economy, a model capable of handling these more general specifications of production. This is the purpose of the present paper.

Let a von Neumann economy be specified (B, A), where B is the $n \times m$ matrix of outputs and A is the $n \times m$ input matrix. The ith column of B or A specifies the vector of outputs or inputs produced or used from unit operation of the ith process. (There are n goods and m processes.) We abstract away from the question of labor by assuming that labor's requirements are already embodied in the matrix A. (Hence (B, A) is a model of commodities produced entirely by commodities.) Since we shall assume the real wage is fixed, this is an appropriate abstraction. We say the semipositive vector p is a price vector associated with profit factor ρ if

$$p \geqslant 0, \qquad \rho > 0, \qquad pB \leqq \rho pA.$$

Under the assumptions that $A \geqslant 0$, $B \geqslant 0$, $A^i \geqslant 0$, and $B_j \geqslant 0$ (where A^i

0022-0531/80/030451-14$02.00/0

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is the ith column of A and B_j is the jth row of B), it is well known that there exists a minimal positive value $\rho_{\min}$ with respect to which a semipositive price vector exists.[1] (See Gale, 1960.) If we write $\rho_{\min} = 1 + \pi_{\min}$, then $\pi_{\min}$ can be thought of as the minimum possible profit rate which the economy can sustain, or what Morishima (1974) calls the guaranteed profit rate.

Suppose the economy is sustaining equilibrium prices p at $\rho_{\min}$. A technical innovation is a new pair of columns (b', a') which may be appended to the matrices (B, A). The innovation will be called *viable at prices p* if and only if

$$pb' > \rho_{\min} pa'.$$

A viable innovation will immediately be adopted by capitalists who treat prices as given, as they will make super-profits from its operation. If a viable innovation appears, then it is reasonable to append it to the old technology, creating a new technology (B', A'), where $B' = (B \mid b')$, $A' = (A \mid a')$, and ask: what happens to the minimal profit rate in passing from (B, A) to (B', A')? It is easy to see the minimal profit rate cannot fall; it may, however, not rise either.

The central task of the paper is to provide conditions which guarantee that the minimal profit does rise. This turns out to be akin to defining a kind of indecomposability for von Neumann economies. It is known that the same phenomenon occurs in Leontief models: there, a viable innovation in an indecomposable Leontief economy produces an increase in the rate of profit (which in that model is related to the eigenvalue of a matrix), whereas the rate of profit may remain constant in a decomposable economy.

At the mathematical level, then, this study investigates indecomposability in von Neumann models. It turns out that the previous definition of indecomposibility in the von Neumann literature, Gale's *irreducibility* (Gale, p. 314), is not a sufficiently strong condition to provide what is needed here. It shall be seen, also, that the question is equivalent to asking for a condition which guarantees that a unique price ray exists at $\rho_{\min}$. Interest in the unicity of the von Neumann price ray has appeared elsewhere (Balinski and Young, 1974). From the economic point of view, this study shows that the rising-profit-rate story which has been told for the Leontief model generalizes suitably to the general activity analysis of von Neumann. In particular, the existence of fixed capital does not change the effect of technological change on the profit rate from the simpler circulating capital story. (For the view of the von Neumann model as a model of fixed capital, see Morishima (1969).)

The economic importance of the result is this: the innovations that competitive, price-taking capitalists will introduce can only cause the rate of profit to rise. This result is robust for production sets involving fixed capital, differential turnover times, and the other attributes of production which can be caputed by the von Neumann model. Moreover, the profit rate will *strictly* increase if the technology enjoys a type of indecomposability.

---

[1] Convention on vector inequalities: $A \geqq B$ mean $A_i \geqq B_i$ $\forall$ components i; $A > B$ means $A \geqq B$ and $A \neq B$.

2. GEOMETRY OF THE VON NEUMANN MODEL

DEFINITION 2.1. A *von Neumann equilibrium* for the model (B, A) is a triplet (p, x, ρ), where $\rho \in \mathbb{R}$, $\rho > 0$, p an n-row vector, $p \geqslant 0$; x an m-column vector, $x \geqslant 0$, such that

(a) $pB \leqq \rho pA$,

(b) $pBx = \rho pAx$,

(c) $Bx \geqslant \rho Ax$.

(Note that (b) is redundant given (a) and (c)).

We say p is a price vector for ρ, and x is an intensity vector for ρ. In addition, if

(d) $pBx > 0$

then (p, x, ρ) is an *economic* von Neumann equilibrium. (For the study of economic von Neumann equilibria, see Kemeny, Morgenstern, and Thompson (K–M–T) (1956).)

DEFINITION 2.2. For any $\rho > 0$, define

$$\begin{aligned} P(\rho) &= \{p \geqq 0 \mid pB \leqq \rho pA\}, \\ \Gamma(\rho) &= \{B^i - \rho A^i \mid p(B^i - \rho A^i) = 0 \ \forall\, p \in P(\rho)\} \\ &\quad \cup \{-e^i \mid pe^i = 0 \ \forall\, p \in P(\rho)\} \\ &\quad \cup \{0\}. \end{aligned}$$

(e^i is the ith unit vector in $\mathbb{R}^n$.)

$P(\rho)$ is the set of all price vectors for ρ; $\Gamma(\rho)$ is derived from the processes which are profitable, or binding, at profit factor ρ, for all price equilibria. (We may view $-e^i$ as equivalient to a disposal activity $0 - \rho e^i$, this last being written in the form $B^i - \rho A^i$. The outputs of the ith disposal activity, $i = 1, n$, are the zero vector; the inputs are given by the unit vector.) The vector 0 is appended to the set $\Gamma(\rho)$ in case the set is otherwise empty, for notational convenience.

The geometry of the von Neumann model relevant for our inquiry is summarized in this theorem:

THEOREM 2.1. $\forall\rho > 0$, $P^{\perp}(\rho) = \text{Cone } \Gamma(\rho)$.

($P^{\perp}(\rho) = \{v \in \mathbb{R}^n \mid p \cdot v = 0, \forall p \in P(\rho)\}$; Cone $\Gamma(\rho)$ is the *convex cone* generated by the set $\Gamma(\rho)$.)

Proof. It is clear that Cone $\Gamma(\rho) \subseteq P^{\perp}(\rho)$.

To show the converse, let

$$\Gamma(\rho) = \{B^i - \rho A^i \mid \forall i = 1, m\} \cup \{-e^i \mid \forall i = 1, n\}.$$

It is claimed that $P^{\perp}(\rho) \subsetneq \text{Cone } \Gamma(\rho)$ implies $P^{\perp}(\rho) \subsetneq \text{Cone } \Gamma(\rho)$. For suppose not. Then $\exists v \in P^{\perp}(\rho)$, $v \in \text{Cone } \Gamma(\rho) - \text{Cone } \Gamma(\rho)$. Then $v = \sum_1^m \alpha^i(B^i - \rho A^i) + \sum_1^n \beta^i(-e^i)$; $\alpha^i, \beta^i \geq 0$. Since p annihilates v, p annihilates every term $(B^i - \rho A^i)$ or $(-e^i)$ which appears with positive coefficient α^i or β^i in the sum—since every term in the sum $p \cdot v$ is nonpositive. Consequently,

$$(\forall i\colon \alpha^i > 0)(\forall p \in P(\rho))(p(B^i - \rho A^i) = 0),$$
$$(\forall i \beta^i > 0)(\forall p \in P(\rho))(pe^i = 0),$$

which means precisely that $v \in \text{Cone } \Gamma(\rho)$.

It is enough to show, therefore, that $P^{\perp}(\rho) \subsetneq \text{Cone } \Gamma(\rho)$. Suppose not: $\exists v \in P^{\perp}(\rho)$, $v \notin \text{Cone } \Gamma(\rho)$. There exists a hyperplane separating $\{v\}$ from Cone $\Gamma(\rho)$. That is: $\exists q \in \mathbb{R}^n$. $q \cdot v > 0$, $q\Gamma(\rho) \leqq 0$. The latter condition means $q \geq 0$ and $qB \leq \rho qA$; hence q is a price vector, that is, $q \in P(\rho)$. But then $q \cdot v = 0$ since $v \in P^{\perp}(\rho)$. This contradicts the choice of q. Q.E.D.

We define $g_{\max}$ in an analogous way to $\rho_{\min}$:

DEFINITION 2.3. $g_{max} \quad \max\{\rho \mid (\exists x \geq 0)(Bx \geqq \rho Ax)\}$.

Under Gale's assumptions, mentioned above, $g_{\max} \geq \rho_{\min}$. (See Gale, p. 314.].)

We define the dual concepts to $P(\rho)$ and Cone $\Gamma(\rho)$:

DEFINITION 2.4. For $\rho > 0$, let:

$$\begin{aligned} X(\rho) &= \{x \geqq 0 \mid Bx \geqq \rho Ax\}, \\ \Omega(\rho) &= \{B_j - \rho A_j \mid (B_j - \rho A_j)x = 0 \ \forall x \in X(\rho)\} \\ &\quad \cup \{-e_j \mid e_j x = 0 \ \forall x \in X(\rho)\} \\ &\quad \cup \{0\}, \end{aligned}$$

where $\{e_j\}$ are row unit vectors in $\mathbb{R}^m$.

THEOREM 2.2. *For* $\rho > 0$, $X^{\perp}(\rho) = \text{Cone } \Omega(\rho)$.

Proof. Same as for Theorem 2.1.

As a consequence of Theorems 2.1 and 2.2 a geometric proof of a classical result of Kemeny *et al.* (1956) can be given; it is provided in the Appendix.

3. INDECOMPOSABLE VON NEUMANN SYSTEMS AND THE PROFIT RATE

DEFINITION 3.1.[2] Let $p \geqslant 0$ be a price vector associated with factor ρ. An innovation (b', a') is *viable* at prices p and factor $\rho \Leftrightarrow pb' > \rho pa'$.

Assumption. We assume throughout that $B_j \geqslant 0 \; \forall \, j = 1, n$ and $A^i \geqslant 0$ $\forall i = 1, m$.

By Gale (1960, p. 313), this is sufficient to guarantee that $\rho_{\min} \leqslant g_{\max}$ and von Neumann solutions exist for all $\rho \in (\rho_{\min}, g_{\max})$. In particular, for such ρ, (p, x) is a von Neumann solution if and only if $p \in P(\rho)$ and $x \in X(\rho)$.

THEOREM 3.1. *Let p be a price vector for the minimal profit factor,* $\rho_{\min}$, *of system* (B, A). *Let* (b', a') *be a viable innovation at prices p, and let* $\rho'_{\min}$ *be the minimal profit factor for the appended technology* (B', A'). *Then*

(a) $\rho'_{\min} \geqslant \rho_{\min}$,

(b) $\rho'_{\min} > \rho_{\min}$ *for every viable innovation at p if and only if p is the unique price ray at* $\rho_{\min}$.

Proof. Part (a) is immediate, since $p'B' \leqq \rho p'A' \Rightarrow p'B \leqq \rho p'A$.

Part (b): $\Leftarrow$. Let p' be a price vector for (B', A') at $\rho'_{\min}$:

$$p'B' \leqq \rho'_{\min} p'A'. \tag{3.1}$$

In particular:

$$p'B \leqq \rho'_{\min} p'A.$$

If $\rho'_{\min} = \rho_{\min}$ then, by hypothesis, p' is a multiple of p, the unique price vector for (B, A) at $\rho_{\min}$. But $pb' > \rho_{\min} pa'$ by viability, which contradicts (3.1).

$\Rightarrow$. Let q and p be two (independent) price vectors for (B, A) at $\rho_{\min}$. Let v be a vector separating p from q:

$$p \cdot v < 0, \qquad q \cdot v \geqslant 0.$$

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[2] For simplicity of exposition, we shall assume that *one* new process is introduced. However, the argument which shall be presented remains correct, *verbatim*, if the innovation consists of a set of new processes. That is, a technology (B', A') is viable at prices p over technology (B, A) if *every* process (b', a') in the former and not in (B, A) passes $pb' \geqslant \rho_{\min} pa'$, and for at least one such process, the inequality is strict. We shall refer to this general case at appropriate points in footnotes.

Write $v^i = \rho A^i - B^i$, $i = 1, m$. Then

$$p \cdot v^i \geqslant 0, \qquad qv^i \geqslant 0.$$

Choose semipositive vectors a' and b' in such a way that

$$v = \rho_{\min} a' - b'. \tag{3.2}$$

Since $p \cdot v < 0$, the constructed innovation (b', a') is viable for (B, A) at $(p, \rho_{\min})$. However, the minimal profit factor does not increase in passing from (B, A) to (B', A'), since $q \cdot v^i \geqslant 0 \; \forall i = 1, n$ and $q \cdot v \geqslant 0$. That is, q is a price vector for (B', A') at $\rho_{\min}$. Q.E.D.

Part (a) of Theorem 3.1, whose proof is trivial, may appear to imply that the appending of a viable innovation trivially implies the new *equilibrium* profit rate does not decrease. This conclusion, although true, is not trivial, as $\rho_{\min}$ is not a priori a profit rate associated with a full equilibrium, but only with a price vector. More precisely, we have:

COROLLARY 3.1a. (A) *Let $\hat{\rho}_{\min}$ be the minimal profit factor associated with a von Neumann equilibrium (p, x) for technology (B, A). Let $\hat{\rho}'_{\min}$ be the minimal profit factor sustaining a von Neumann equilibrium for the appended technology (B', A'), where (b', a') is viable at prices p. Then $\hat{\rho}'_{\min} \geqslant \hat{\rho}_{\min}$.*

(B) *Let π be the unique profit rate associated with the Leontief technology (I, A), and π' the unique profit rate associated with $(I, \hat{A})$, where $\hat{A}$ differs from A in that one column of A has been replaced by a viable innovation. Then $\pi' \geqslant \pi$.*

Proof of (A). Let q be a price vector associated with $\rho_{\min}$. By Gale's proof of the existence theorem for von Neumann equilibrium, q and $\rho_{\min}$ support a full von Neumann equilibrium. (That is, there is a von Neumann equilibrium $(q, x; \rho_{\min})$.) Hence $\rho_{\min} = \hat{\rho}_{\min}$. Also, $\rho'_{\min} = \hat{\rho}'_{\min}$ by the same reasoning. Hence, by part (a) of Theorem 3.1, Q.E.D.[3]

Proof of (B). To demonstrate this, the trivial adjustment must be made of viewing the new technology $\hat{A}$ in the von Neumann way. Instead of replacing the ith column A_i of A with the viable innovation A'_i, we append A'_i to the original matrix A, forming $A' = (A \,\vdots\, A'_i)$, an $n \times (n + 1)$ matrix. It is easily seen, by the Frobenius–Perron theorem, that $\pi = \rho_{\min} - 1$, where $\rho_{\min}$ is the minimal profit factor for (I, A) as here defined and that $\pi' = \rho'_{\min} - 1$, where $\rho'_{\min}$ is the minimal profit factor for (I, A'). By part (A), Q.E.D.

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[3] It can easily be checked that Theorem 3.1 holds for the more general case that (B', A') is a viable technology with many new processes. (See footnote 2.) It follows that Corollary 3.3 below is true for the general case.

Part (B) in the above corollary has been explicitly stated, as it is the theorem which Marxian writers have proved previously. (See Okishio, 1961; Roemer, 1977.) Part (A) of the corollary shows that in the general von Neumann model, which can include a specification of production involving joint processes, fixed capital, alternative processes, and differential turnover times, viable innovations can only increase the minimal profit rate enjoyed by the economy in (von Neumann) equilibrium.

We proceed to investigate when $\rho_{\min}$ (or $\hat{\rho}_{\min}$) increases strictly.

To guarantee that $\rho_{\min}$ rises, then, under viable innovation, we must guarantee that the price ray at $\rho_{\min}$ is unique. By Theorem 2.1, this is equivalent to the condition that dim Cone $\Gamma(\rho_{\min}) = n - 1$.

We recall Gale's concept of an irreducible von Neumann economy:

Definition 3.2 (Gale, 1960, p. 314). A set of goods is *independent* if it is possible to produce each good in the set without consuming goods outside the set. The model (B, A) is *irreducible* iff it has no proper independent subset. Formally: the set of goods $S \subseteq \{1,..., n\}$ is independent if there is a subset of processes indexed by $T \subseteq \{1,..., m\}$ such that: $(\forall i \in S)(\exists j \in T)$ $(b_{ij} > 0)$ and $(\forall j \in T)(\forall i \notin S)$ $(a_{ij} = 0)$.

We introduce a stronger notion:

Definition 3.3. A model (B, A) is *indecomposable* iff all intensity vectors x at $\rho_{\min}$ use at least n processes. That is: $x \in X(\rho_{\min})$, $x \neq 0$, implies $x_j > 0$ for at least n components j.

Remark. (B, A) is indecomposable if and only if any semipositive intensity vector, for *any* factor $\rho \geq \rho_{\min}$, requires n positive intensity levels. For $\rho \geqslant \rho_{\min} \Rightarrow X(\rho) \subseteq X(\rho_{\min})$. Since von Neumann equilibria exist only for $\rho \in (\rho_{\min}, g_{\max})$, indecomposability means the system (B, A) cannot reproduce itself at any equilibrium unless it operates at least as many processes as there are goods.

Theorem 3.2. *If* (B, A) *is indecomposable then* dim *Cone* $\Gamma(\rho_{\min}) = n - 1$.

Corollary 3.3. *If* (B, A) *is indecomposable, then the minimal profit factor rises with the appending of any viable innovation.*

Proof of Corollary. By Theorem 2.1 and Theorem 3.1, from Theorem 3.2.

Proof of Theorem 3.2. By Theorem 2.1, Cone $\Gamma(\rho_{\min})$ is a proper subspace of $\mathbb{R}^n$. Suppose dim Cone $\Gamma(\rho_{\min}) = r < n - 1$. By Carathéodory's theorem,[4] any point in the convex hull of Γ can be expressed as a convex combination of at most $r + 1$ points of Γ. In particular, since Cone Γ is a subspace, $0 \in$ Hull Γ and 0 may be so expressed:

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[4] Carathéodory's Theorem (Mangasarian, p. 43). Let $\Gamma \subseteq R^n$. If x is in the convex hull of Γ then x is a convex combination of $n + 1$ or fewer points of Γ.

$$0 = \sum \alpha^i(B^i - \rho_{\min} A^i) - \sum \beta^i e^i,$$

where at most $r + 1$ terms occur in the sums together. It follows that

$$\sum \alpha^i B^i \geqq \rho_{\min} \sum \alpha^i A^i,$$

where there are fewer than n terms in the sums, since $r + 1 < n$ by hypothesis. But the vector α comprises a von Neumann intensity vector at $\rho_{\min}$ with fewer than n positive intensity levels, which contradicts indecomposability.
Q.E.D.

We next investigate the relationship between irreducibility and indecomposability.

DEFINITION 3.4. Let H be the class of indecomposable models, and G the class of irreducible models.

Remark. Leontief models (I, A) which are indecomposable in the classical sense are members of $G \cap H$.

THEOREM 3.4. *$H \subset G$ but $G \not\subset H$.*

Proof.

LEMMA 3.5. *Every economy (A, B) can reproduce itself using not more than n processes at $\rho_{\min}$. (That is: $\exists x \in X(\rho_{\min})$ with at most n positive components.)*

Proof of Lemma. This fact is well known; the proof is omitted.

Proof of Theorem 3.4. $G \not\subset H$: Let (B, A) be a model which consists of two independent processes (b', a') and (b'', a''), and several other processes which are positive convex combinations of these two. Let there be three goods in the model. This can easily be constructed to have no proper independent subsets of goods. Then $(B, A) \in G$. Clearly, however, if x is an intensity vector which reproduces the system $(Bx \geqslant \rho Ax)$, the same results can be achieved by operating only the first two processes, and so $(B, A) \notin H$.

$H \subseteq G$: Let $(B, A) \notin G$. Let $S = \{1,..., s\}$ index a proper independent subset of goods. Let the processes T which are used to produce the goods in S, by using goods in S only, be indexed as the first t processes: $T = \{1,..., t\}$ Consider B, and delete those columns j, $m \geqslant j > t$. Delete also the rows of B (corresponding to goods) i, $n \geqslant i > s$. Call the reduced matrix $\hat{B}$. In like manner, define $\hat{A}$. Notice $\hat{A}$ has the property that every one of its processes (i.e., columns) uses some good in S: since every process of A used some good, and the processes left in $\hat{A}$ used no goods outside of S, by hypothesis. Similarly, $\hat{B}$ has the property that each of its rows contains a positive element: because by hypothesis all goods $1,\ldots, s$ are produced by the processes in T.

Therefore, a von Neumann equilibrium $(\hat{p}, \hat{x}, \hat{\rho})$ exists for $(\hat{B}, \hat{A})$, since $\hat{B}_j \geqslant 0$ for $j = 1, s$ and $\hat{A}^i \geqslant 0$ for $i = 1, t$.

Let p be the n-vector which is obtained by appending to $\hat{p}$ a string of zeros which correspond to prices for goods not in S; let x be the m-vector which is obtained by appending to $\hat{x}$ a string of zeros which are intensity levels for processes not in T. It immediately follows that $(p, x, \hat{\rho})$ is a von Neumann equilibrium for (B, A).

By Lemma 3.5, $(\hat{B}, \hat{A})$ possesses an intensity vector $\hat{x}$ which uses not more than s processes. By the present construction, the existence of an intensity vector x for (B, A) has been shown using not more than s processes. Since $s < n$, $(B, A) \notin H$. Q.E.D.

Since it has been demonstrated that the von Neumann price ray is unique for $(B, A) \in H$, and $H \subset G$, we show below that the condition $(B, A) \in G$ is not sufficient to guarantee unicity of the price ray, and hence a rising rate of profit.

EXAMPLE. Let (B, A) be a "replication" economy

$$B^i = \lambda^i A^i, \qquad \lambda_i > 1,\ A^i > 0,\ \forall i.$$

This simple economy is irreducible. It is, however, not indecomposable, as the economy can "reproduce" itself using any one process. Clearly $\rho_{\min} = \max \lambda^i$, and *any* price vector is a von Neumann price ray at $\rho_{\min}$.

We next ask a related question. Suppose the economy is operating at a von Neumann equilibrium and a viable innovation appears and is appended to the technology. At the new equilibrium, will the new process in fact be used? A degenerate situation certainly exists if it is not used. The next two results show that the same type of indecomposability which guarantees that the profit rate rises strictly with an innovation also guarantees that the innovation will in fact be used at the new equilibrium.

THEOREM 3.6. *Let $\rho_{\min}$, $g_{\max}$ be the min profit factor and max growth factor for (B, A). Let $\rho'_{\min}$ be the min profit factor for the economy (B', A') after appending an innovation (b', a') viable at $\rho_{\min}$ for (B, A). If $\rho'_{\min} > g_{\max}$ then all von Neumann equilibria (p', x') for (B', A') use the innovation with positive intensity. Conversely, if all von Neumann intensity vectors for (B', A') use the innovation, then $\rho'_{\min} > g_{\max}$.*

Proof. $\Rightarrow$. Let x' be an intensity vector for (B', A'), at any factor $\rho \leq \rho'_{\min}$:

$$B'x' \geqq \rho A'x'.$$

If x' were zero in its last component, giving an intensity of zero for the new process, then the m-vector x consisting of the first m components of x' would be a semipositive intensity vector for (B, A) at ρ which is impossible, since $\rho > g_{\max}$.

$\Leftarrow$. Conversely, if $g_{\max} \geq \rho'_{\min}$ then there is a semipositive intensity vector for (B, A) at $\rho'_{\min}$. Appending a zero component to this vector produces a semipositive intensity vector for (B', A') at $\rho'_{\min}$. Q.E.D.

COROLLARY 3.7. *If (B, A) is indecomposable, then all viable innovations will be used with positive intensity at all von Neumann equilibria in the appended technology.*[5]

Proof. Gale (1960, p. 315) has shown that $(B, A) \in G$ implies $g_{\max} = \rho_{\min}$. Since $(B, A) \in H \subset G$ (Theorem 3.4), we have $g_{\max} = \rho_{\min}$. Since $(B, A) \in H$, it also follows that $\rho'_{\min} > \rho_{\min} = g_{\max}$, and by Theorem 3.6, Q.E.D.

It should be remarked that the irreducibility of (B, A) is not sufficient to guarantee the conclusion of Corollary 3.7. This follows from the counterexample provided above, after the proof of Theorem 3.4. By that example, there exist irreducible technologies (B, A) to which can be appended viable innovations which leave the new minimal profit factor unchanged. Thus: $\rho'_{\min} = \rho_{\min} = g_{\max}$ for this situation. It follows from the last statement of Theorem 3.6 that there exist von Neumann intensity vectors for the appended technology which will not use the innovation.

Some final comments are warranted on the uniqueness of von Neumann equilibria. If we demand that the von Neumann equilibria be *economic*, then Theorem K–M–T (Appendix) shows that there exists a finite number of values ρ at which solutions exist. If $\rho_{\min} = g_{\max}$ then there exists at most one such solution. (In particular, if $(B, A) \in G$ we need not speak of a minimal profit factor or guaranteed profit rate, since there is only one profit factor capable of sustaining a full equilibrium, in the von Neumann sense.) An examination of Gale's duality theorem (Gale, 1960, p. 315) shows that if $(B, A) \in G$ then, in fact, $Bx > 0$ for any intensity vector at $\rho_{\min}$, and a fortiori, it follows that all von Neumann equilibria are economic. Hence, if $(B, A) \in H$, all von Neumann equilibria are economic, the equilibrium profit factor is unique, and the equilibrium price ray vector is unique.

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[5] The generalization of this theorem to the case of a viable technology (B', A') with several new processes is this: if (B, A) is indecomposable, then at least one of the new processes must be used with positive intensity at any equilibrium of (B', A').

It is, however, not clear that indecomposability or irreducibility are good economic assumptions in modeling fixed capital. The principal joint products with which we are concerned in modeling fixed capital are old capital goods which exit from the production process in a depreciated, but still potentially useful, state. In the von Neumann model, every capital good of every vintage counts as a separate commodity. A natural kind of indecomposability to assume for such an economy would be that the economy can reproduce itself using not more processes than that there are *new* goods. (That is, if there were some process for producing each new good reasonably efficiently using only new goods—a reasonable assumption—such indecomposability would exist.) If such were the case, the economy would not be in the class H, nor even in the class G (as the set of new goods would comprise a proper independent subset).

Hence the strictly rising-rate-of-profit theorem may not apply to real fixed capital economies, due to the decomposable nature of those economies. in the technical sense discussed here.

The reader should note, however, that indecomposability was only a sufficient condition for guaranteeing the strictly rising rate of profit. If one believes that real fixed capital economies, even if decomposable, enjoy a unique price equilibrium, then Theorem 3.1 assures us of a strictly rising profit rate in any case.

APPENDIX: A GEOMETRIC PROOF OF A THEOREM OF KEMENY, MORGENSTERN, AND THOMPSON

As a consequence of Theorems 2.1 and 2.2 a new proof of the following theorem is provided:

THEOREM (K–M–T). *There are a finite number, r, of values ρ for which* economic *von Neumann equilibria exist, and* $r \leqslant \min(m, n)$.

Note the cone $X(\rho)$, of Section 2, decreases as ρ increases, becoming trivial for $\rho > g_{\max}$. Hence the dimension of $X^{\perp}(\rho)$ increases as ρ increases. As ρ increases, starting from zero, the dimension of $P^{\perp}(\rho)$ changes, at most n times, and dim $X^{\perp}(\rho)$ changes at most m times. We shall show that economic equilibria (see Definition 2.1) can only occur at values ρ which are simultaneous jump values of the dimension of these two subspaces, which proves the K–M–T Theorem. A *jump value* of $P^{\perp}(\rho)$ is a value ρ_i such that

$$\dim P^{\perp}(\rho_i) < \dim P^{\perp}(\rho_i - \epsilon) \qquad \text{for small } \epsilon > 0;$$

a *jump value* of $X^{\perp}(\rho)$ is a value $\bar{\rho}_j$ such that

$$\dim X^{\perp}(\bar{\rho}_j) > \dim X^{\perp}(\bar{\rho}_j - \epsilon) \qquad \text{for small } \epsilon > 0.$$

To prove Theorem K–M–T, a topological lemma is needed:

LEMMA A. *Let $x(t)$ be a bounded vector-valued function defined on an interval of real numbers, I. Then there exists an uncountable subset $S \subset I$ with this property*:

$$S = \{s \in I \mid (\exists \text{ sequence } \{t_j^s\}_{j=1,\infty})(t_j^s > s,\ t_j^s \in I,\ t_j^s \to s \text{ and } x(t_j^s) \to x(s)).\}$$

Proof. We shall show that $I - S$ is at most a countable set.

Let $t_0 \in I - S$, and choose t_0 in the interior of I. By hypothesis, there is an $\epsilon > 0$ and a $\delta > 0$ such that

$$t' \in (t_0, t_0 + \delta) \Rightarrow \|x(t') - x(t_0)\| \geqslant \epsilon \tag{*}$$

where we take $\|\ \|$ to be the Euclidean norm. Consider how many points $t_0 \in I$ can have property (*) for the given pair (ϵ, δ). If there were an infinite number of such t_0, then there would be an infinite number in some subinterval I_δ of length δ of I. Let t_1 and t_2 be two such points in I_δ. Then $t_1 \in [t_2, t_2 + \delta]$, with appropriate labeling of t_1 and t_2, and so $\|x(t_1) - x(t_2)\| \geqslant \epsilon$. Hence the spheres of radius $\epsilon/3$ around $x(t_1)$ and $x(t_2)$ are disjoint. By this argument, we can construct an infinite sequence of disjoint spheres of radius $\epsilon/3$ in a bounded region, which is impossible. Hence there can be only a finite number of points $t_0 \in I - S$ with the given property for the pair (ϵ, δ).

Choose, now, a sequence $\epsilon_n \to 0$; for each ϵ_n, choose t_n with the property that

$$(\exists\, \delta_n > 0)\,(t' \in (t_n, t_n + \delta_n) \Rightarrow \|x(t') - x(t_n)\| \geqslant \epsilon_n).$$

If for some ϵ_n we can find no δ_n satisfying this condition, then we have shown the set $I - S$ is finite—since by the argument above, there are at most a finite number of points t_n associated with each ϵ_n. Suppose, then, there is a value δ_n for each ϵ_n. We still have only a countable number of points $\{t_n \mid n = 0, \infty\}$ which satisfy (*). But these points exhaust $I - S$, since $\epsilon_n \to 0$. Hence S is uncountable.

LEMMA B. *Let $\rho_1 \neq \rho_2$ be two values of ρ for which economic solutions exist, (p_1, x_1) and (p_2, x_2), respectively. Then $p_1 \neq p_2$ and $x_1 \neq x_2$.*

Proof. Let $\rho_1 > \rho_2$. Suppose $p_1 = p_2$. Then

$$p_1 B \leqq \rho_2 p_1 A,$$
$$B x_1 \geqq \rho_1 A x_1,$$

and so

$$p_1 B x_1 \leqq \rho_2 p_1 A x_1,$$
$$p_1 B x_1 \geqq \rho_1 p_1 A x_1,$$

and so $\rho_1 p_1 A x_1 \leqslant p_1 B x_1 \leqq \rho_2 p_1 A x_1$. But $p_1 B x_1 > 0$, and so $p_1 A x_1 > 0$, which implies $\rho_1 \leqslant \rho_2$, a contradiction. Hence $p_1 \neq p_2$.

A similar argument shows $x_1 \neq x_2$. Q.E.D.

Lemma C. *There can exist no interval of numbers I such that economic solutions exist for all $\rho \in I$.*

Proof. Suppose I were such an interval. Let $(p(\rho), x(\rho))$ be the economic solution identified with $\rho \in I$.

We must have, for all $\rho \in I$,

$$p(\rho)\, Bx(\rho + \delta) = 0 \qquad \text{for all} \quad \delta > 0.$$

For if $p(\rho)\, Bx(\rho + \delta) > 0$, then $(p(\rho), x(\rho + \delta))$ is also an economic solution at ρ, which is impossible by Lemma B.

Since

$$p(\rho)\, Bx(\rho + \delta) = 0 \qquad \forall \delta > 0,$$
$$p(\rho)\, Bx(\rho) > 0$$

it follows that

$$(\forall \rho \in I)\, (\exists \epsilon, \delta > 0)\, (\rho' \epsilon (\rho, \rho + \delta) \Rightarrow \| x(\rho') - x(\rho) \| \geqslant \epsilon).$$

But this is impossible by Lemma A. Q.E.D.

Proof of Theorem K–M–T. Note:

$$\begin{aligned} \nexists \text{ economic solutions at } \rho &\Leftrightarrow P(\rho) \cdot B \cdot X(\rho) = 0, \\ &\Leftrightarrow B \cdot X(\rho) \subseteq P^{\perp}(\rho), \\ &\Leftrightarrow B \cdot [X(\rho)] \subseteq P^{\perp}(\rho), \end{aligned}$$

where $[X]$ means the subspace generated by X.

As ρ increases, $P(\rho)$ increases as a set so $P^{\perp}(\rho)$ decreases as a set. $P^{\perp}(\rho)$ decreases only at the jump values ρ_i. Consequently $[P(\rho)]$ increases as a set only at jump values ρ_i. Similarly, $[X(\rho)]$ decreases as a set only at values $\bar{\rho}_j$.

Suppose there were an economic solution at value ρ, and $\rho \neq \rho_i$ for any i. Then by definition there is a neighborhood $(\rho - \epsilon, \rho)$ to the left of ρ where $\dim P^{\perp}(\rho)$ does not change. To the left of ρ, $[X(\rho)]$ can only get larger. At ρ, $B \cdot [X(\rho)] \nsubseteq P^{\perp}(\rho)$ since there is an economic solution at ρ: but since $P^{\perp}(\rho)$ stays constant in $(\rho - \epsilon, \rho)$ and $[X(\rho)]$ can only get larger in that interval this set inequality continues to hold $\forall \rho \in (\rho - \epsilon, \rho)$. Hence there exist economic solutions in an interval of ρ's, an impossibility by Lemma C.

A similar argument demonstrates $\rho = \bar{\rho}_j$, some j. Q.E.D.

ACKNOWLEDGMENT

It is a pleasure to acknowledge collaboration, on certain essential mathematical aspects of this paper, with R. E. Howe. Credit goes to Howe for discovering that indecomposability, as here defined, is a sufficient condition for Theorem 3.2; and the proof of Lemma A in the Appendix is due to him.

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JOURNAL OF ECONOMIC THEORY 22, 465–476 (1980)

Efficient Allocations under a General Transaction Technology

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Received December 12, 1978; revised August 10, 1979

1. INTRODUCTION

The Arrow–Debreu[1] and Aumann (1964) characterizations of feasible distribution of economic goods are based on the following law of conservation: the sum of all the amounts consumed in the conomy must equal the total produced plus initial endowment.[2] This implies the nonexistence of transaction costs: i.e., nothing is wasted in the process of reallocation. In addition, consumption is excludable; there are no public goods, either local or nonlocal in nature. In order to extend the descriptive power of the general equilibrium model over these cases, special constructions were required. They were furnished for transaction costs by Foley (1970a), Hahn (1971), Kurz (1974), and for purely public goods by Foley (1970b). Ellickson (1973) outlined how the same technique can be applied to local public goods. All this was accomplished assuming a finite set of agents. Roberts (1973) and Richter (1977) studied public goods in a measure space of agents.

In this study, we propose a general framework that allows for description of allocation possibilities. We present a transaction technology which contains all the possible transactions and explore the possibility of decentralization by using efficiency prices in a measure space of agents. As sometimes happens, generalizations simplify problems; in this case, we can prove Richter's main results more directly.

Section two contains the model; the third section reports the main results. In the fourth section we apply the results obtained for examples of specific interest. We show that the prevalence of Aumann's transaction technology is equivalent to fixed prices for all agents and then discuss the transaction costs case and the public good case. The Samuelson condition on the prices is obtained quite easily.

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* The final version of this paper was prepared while the author was on leave at the University of California, Los Angeles.

[1] As discussed in Arrow and Hahn (1971).

[2] With the conventions of inputs bearing a negative sign, and outputs bearing a positive sign.

0022-0531/80/030465-12$02.00/0

2. THE MODEL

We consider the *economy with production and transaction technology* $\mathcal{E} = \{(T, \mathcal{T}, \lambda), X, \succ, e, Y, Z\}$. All the parameters except the last one have standard definitions and interpretations. $(T, \mathcal{T}, \lambda)$ is a complete measure space. $T = [0, 1]$ is the set of economic agents; the associated σ-field $\mathcal{T}$ represents the set of admissible coalitions and λ is the measure on $\mathcal{T}$. The space of agents T is divided into the space of consumers $H = [0, \frac{1}{2}]$ and of producers $F = (\frac{1}{2}, 1)$. An action $a(t)$ is a vector in the Euclidean space E^l. For $t \in H$ this is an action of consumption, and for $t \in F$ an action of production. $X_t = X(t)$ contains the set of possible consumption plans for $t \in H$, and $Y_t = Y(t)$ contains the possible production plans for $t \in F$. We also denote $A_t = A(t) = X(t)$ for $t \in H$, and $A(t) = -Y(t)$, $t \in F$.[3] It is assumed that $A\colon T \rightarrow E^l$ is a convex-valued correspondence; to X_t there corresponds a measurable preference ordering $\succ_t$, i.e., an irreflexive, transitive, and binary relation on X_t. By $e\colon T \rightarrow E^l$ we designate the initial endowment function $e(t) \in X_t$, $t \in H$, $e(t) = 0$ elsewhere. e is an essentially bounded $\mathcal{T}$-measurable function (or $e \in L_\infty{}^l[0, 1]$). $L = \{1, ..., l\}$ denotes the set of all goods.

The only departure from the usual formulation in general equilibrium theory is Z. Z is a set of essentially bounded[4] measurable functions; $z \in Z$ is a function, $z\colon T \rightarrow E^l$. Z represents all the possible transferences of commodities within the economy. As is standard $z_i(t) = -3$ (or 3) means that trader t gave away (or received) three units of commodity i, so z is an activity of reallocation of commodities.

The function $a\colon T \rightarrow E^l$ is called *an allocation* iff a is measurable, essentially bounded and $a(t) \in A_t$ a.e. in T.

DEFINITION. The allocation a is called *feasible* iff $a = e + z$ for some $z \in Z$.

To show how this notation is related to the standard approach to formalizing feasible allocations, write: $a = x - y$, where $x = a1_H$ and $y = -a1_F$.[5] We will then have $x - y = z + e$ or $x = y + z + e$, so the total consumed x is equal to the total endowment plus the activities of production y and z. Note that formally z is a production activity: it transforms goods of one agent to goods of another agent.

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[3] Note that the sign of the firms' actions has been reversed.

[4] e is essentially bounded if there exists M and $\|e(t)\| \leq M$ a.e. in T. $\|x\| = \sum_{i=1}^{l} |x_i|$, where $x \in E^l$, is the norm we use on E^l. The least M satisfying this property is denoted as ess sup $\|e\|$.

[5] 1_S is the characteristic function of the set S:

$$1_S(t) = 1, \quad t \in S$$
$$\quad\;\; = 0, \quad t \notin S.$$

With feasibility and preference relations we may define Pareto efficiency as usual. a^* is *Pareto efficient* iff there is no Pareto superior and feasible allocation b: **b** is *Pareto superior* to a^* if $b(t) \succ_t a^*(t)$ a.e. in H.

Definition. The integrable function p, $p: T \rightarrow E^l$ is called *efficiency prices for the allocation* a^* iff for every allocation b

(i) $p > 0$,

(ii) $b(t) \succ_t a^*(t)$ implies $p(t) \cdot b(t) \geq p(t) \cdot a^*(t)$ a.e. in H,

(iii) $b(t) \in Y(t)$ implies $p(t) \cdot b(t) \leqq -a^*(t) \cdot p(t)$ a.e. in F,

(iv) $z^* = a^* - e$ satisfies $\int_T p \cdot z^* \, d\lambda = \sup_{z \in Z} \int p \cdot z \, d\lambda$.

Proposition. *Let a^* have efficiency prices p, such that for some set of positive measure of consumers S, $p(t) \cdot a^*(t) > \inf p(t) \cdot X_t$, $t \in S$, and let $\succ_t$ be continuous, then a^* is Pareto efficient.*

Proof. Suppose b is a feasible allocation which is Pareto superior to a. We want to show a contradiction.

$-b(t) \in Y(t)$ a.e. in F, so $b(t) \cdot p(t) \geqq a(t) \cdot p(t)$ a.e. in F.

$b(t) \succ_t a(t)$ a.e. in $H \backslash S$, so $b(t) \cdot p(t) \geqq a(t) \cdot p(t)$ a.e. in $H \backslash S$.

$b(t) \succ_t a(t)$ a.e. in S so $b(t) \cdot p(t) > a(t) \cdot p(t)$ a.e. in S.[6]

So $\int_T b \cdot p \, d\lambda > \int_T a \cdot p \, d\lambda$ and hence $\int (b - e) \cdot p \, d\lambda > \int (a - e) \cdot p \, d\lambda = \int z \, d\lambda$. But $b - e \in Z$, because b is feasible, which contradicts requirement (iv) of efficiency prices.

To ensure existence of such S, it would suffice to assume that $X_t = E_+{}^l$ a.e. in H, and also that there exists a feasible allocation $\bar{a}$ for which $\bar{a}(t) \gg 0$ a.e. in T. We also have to assume that $0 \in Y(f)$ a.e. in F.

3. Existence Theorems

To characterize Pareto-efficient allocation with prices, we will naturally resort to a separation theorem of convex sets. The two sets that will be separated are: the set of feasible allocations, and the set of allocations that are Pareto superior to the prespecified Pareto-efficient allocation. Convexity of these sets involves convexity of the preference relations, of the production and of the transaction technology. We cannot dispense with the first two assumptions as is done in Hildenbrand (1974), because we no longer sum the actions of agents. The convexity assumed on the transaction technology implies that the technology involved in transferring goods among people obeys the same nonincreasing returns to scale and declining marginal productivity[7] as in other types of production. However, since $0 \in Z$ is not required, set-up costs are not ruled out. Analogously to the finite case, we expect the separating continuous linear functional to play the role of prices. However, here the results depend very much on the norm chosen. At least one of the two sets must be open relative to that norm, and the linear functional will be continuous with respect to that norm. If, for example, the norm were $\| x - y \|_{\mathscr{L}_1{}^l} = \int_T \| x - y \| \, d\lambda$, and the feasible set contained an interior point a, then every other allocation a' for which the *sum* of its distance from a is sufficiently small would be feasible. In particular, the technology allows a small subset of agents to get large amounts of all the goods. Summing up quantities of goods for different agents has thus been brought back, through the back door. If this norm is used, the attempted generality will be forsaken. The appropriate norm should take into consideration every single agent, except for a set of measure zero; it is: $\| x - y \|_{\mathscr{L}_\infty{}^l} = \operatorname{ess\,sup} \| x - y \|$. Note that for that reason we have confined ourselves to essentially bounded functions.

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[6] By a well-known argument, if $t \in S$ and $b(t) \cdot p(t) = a(t) \cdot p(t)$, then by taking c for which $c \cdot p(t) < a(t) \cdot p(t)$, we obtain that $b_\lambda = (1 - \lambda) b(t) + \lambda c$ will satisfy $b_\lambda \succ_t a(t)$ for sufficiently small $\lambda > 0$, and $p(t) \cdot b_\lambda < p(t) \cdot a(t)$, which is a contradiction.

But the functional obtained in this way may have no representation by a price vector. So a part of the existence proof will be to find suitable economic conditions ensuring that such representation is indeed possible. We would have been saved this trouble if the norm were $\| \, \|_{\mathscr{L}_1{}^l}$, because $(\mathscr{L}_1{}^l)^*$, the space of functionals on $\mathscr{L}_1{}^l$, is exactly $\mathscr{L}_\infty{}^l$, the space of essentially bounded measurable function on T; if p is the separating functional, $p(t)$ is then the price vector associated with agent t. We will later see that the unrepresentable case may be interpreted as an unbounded price for the goods of an arbitrarily small group of agents. The economic conditions will guarantee that prices will be finite in this sense.

One of the conditions is given by:

DEFINITION. Let Q be a set of goods, $Q \subset L$. $\mathscr{E}$ is *irreducible with respect to Q* ($\mathscr{E}$ is Q-irreducible) iff given a decreasing sequence of sets $\{E_m\}_{m=1}^{\infty}$ in $\mathscr{T}$, $\lambda(E_m) \to 0$ and a feasible allocation $a = e + z$, there exists a sequence $\{\Delta z^m\}_{m=1}^{\infty} \subset \mathscr{L}_\infty{}^l$, satisfying

(a) $\Delta z^m + z \in Z$ for m sufficiently large.

(b) $\liminf \Delta z_i{}^m(t) > 0$, $i \in Q$, $t \in E_m$, $\Delta z_i{}^m(t) \geqq 0$, $i \notin Q$, $t \in T$.

(c) If $D_m = \{t \in T: \Delta z^m(t) \neq 0\}$, then $\lambda(D_m) \to 0$.

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[7] Where this term is meaningful.

This condition says that to a small set of agents E_m it is possible to supply positive amounts of all the goods in Q, using as inputs goods in Q only, from a small set of agents D_m.

THEOREM 1. *Let* $\mathscr{E}$ *be a transaction technology economy where*:

(i) $A(t)$ *is convex for all* t. ($=$ *all the consumption and production sets are convex*).

(ii) Z *is convex.*

(iii) $\succ_t$ *is convex, continuous, and monotonous*[8] *for all* $t \in H$.

(iv) *Free disposal at all* $Y(t)$: $y \in Y(t)$ *and* $x \leqq y$ *implies* $x \in Y(t)$ *for all* $t \in F$. *Also, if* $x \in X(t)$ *and* $x' \geqq x$ *then* $x' \in X(t)$ *for all* $t \in H$.

(v) $\mathscr{E}$ *is irreducible with respect to the set of all goods* L.

Then for every Pareto-efficient allocation efficiency prices $p \neq 0$ *exist and* $p(t) \geqq 0$ *a.e.*

Proof. a^* is the Pareto-efficient allocation. Let B be the set of all allocations b that are Pareto superior to a^*, $b(t) \succ_t a^*(t)$ a.e. in H, $-b(t) \in Y(t)$, $t \in F$. B is convex, due to convexity of $\succ_t$, $X(t)$, and $Y(t)$. $Z + \{e\}$ is also convex, and $(Z + \{e\}) \cap B = \emptyset$, because a^* is Pareto efficient. Our assumptions ensure that B has a nonempty interior relative to the $\mathscr{L}_\infty{}^l$ topology, because for every $b \in B$, the set $\{b' \colon b' \in \mathscr{L}_\infty{}^l, b'(t) \gg b(t), t \in T\}$, is contained in B and it is open. For $t \in H$, $b'(t) \gg b(t)$ implies $b'(t) \succ_t b(t) \succ_t a^*(t)$; for $t \in F$, $b'(t) \gg b(t)$ implies $-b'(t) \ll -b(t)$ and by free disposal $b'(t) \in Y(t)$. So we may apply a standard separation theorem (Dunford and Schwartz, 1958, p. 471) to obtain a continuous linear functional $\pi \neq 0$ separating the sets $Z + \{e\}$ and B. We choose the sign of π such that $\pi \cdot a < \pi \cdot b$ for all $a \in Z + \{e\}$, $b \in B$, where $\pi \cdot a$ denotes the result of the functional π operating on a.

Following Bewley's (1972) line of thought we list the following facts:

Fact 1. $\pi \in ba^l(T, \mathscr{T}, \lambda)$; i.e., π consists of l different bounded additive set functions π_i, absolutely continuous with respect to λ (Yosida and Hewitt, 1956, Theorem 23). For $f \in \mathscr{L}_\infty{}^l$, $\pi \cdot f = \sum_{i=1}^{l} \pi_i f_i = \sum_{i=1}^{l} \int f_i \, d\pi_i$. We say that $\pi \geqq 0$ if $E \in \mathscr{T}$ implies $\pi(E) \geqq 0$.

Fact 2. If $\pi \geqq 0$ then $\pi = \pi^c + \pi^p$, where $\pi^c \geqq 0$ is a countably additive set function, and $\pi^p \geqq 0$ is purely finitely additive[9] (Yosida and Hewitt, 1956, Theorem 1.23, p. 52).

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[8] That is, if $x_i \geq y_i$ all $1 \leqslant i \leqslant l$, and $x, y \in X(t)$, then $x \succ_t y$, $t \in H$.

[9] $\pi^p \geqq 0$ is purely finitely additive iff the only countably additive set function $\phi \geqq 0$, $\pi(E) - \phi(E) \geqq 0 \ \forall E \in \mathscr{T}$ is $\phi = 0$.

Fact 3. There exists a decreasing sequence of measurable sets $\{E_m\}_{m=1}^{\infty}$ $\lambda(E_m) \to 0$ and $\pi^p(T\backslash E_m) = 0$ for all m (by simple application of Yosida and Hewitt (1956, Theorem 1.19, p. 50)).

Fact 4. $\pi^c \cdot x = \int_T p(t) \cdot x(t)\, d\lambda$, where $p \in \mathscr{L}_1{}^l$ is the Radon–Nikodym derivative of π^c (Halmos, 1950, p. 134, Theorem B).

We note first that by monotonicity in H and free disposal in F, $\pi \geqq 0$. p of Fact 4 satisfies conditions (ii) and (iii) for efficiency prices:

(ii) $b(t) \succ_t a^*(t) \Rightarrow p(t) \cdot b(t) \geqq p(t) \cdot a^*(t)$ a.e. in H.

(iii) $b(t) \in Y(t) \Rightarrow p(t) \cdot b(t) \leqq -a^*(t) \cdot p(t)$ a.e. in F.

To contradict (ii), assume that there exist a set of positive measure $S \subset H$ and an allocation b such that $b(t) \succ_t a^*(t)$ but $p(t) \cdot b(t) < p(t) \cdot a^*(t)$, $t \in S$. By Fact 3, S may be chosen to satisfy $\pi^p(S) = 0$. Now let $\{b^n\}$ be a sequence in B converging to a^* in the $\mathscr{L}_\infty{}^l$ norm: $b^n(t) = a^*(t) + (1/n)\, w$, $w \gg 0$, is such a sequence.

$$c^n = b 1_S + b^n 1_{T\backslash S} \text{ is in } B, \qquad \text{so } \pi \cdot c^n \geqq \pi \cdot a^*.$$

$$\pi \cdot (c^n - a^*) = \pi^c \cdot (b - a^*)\, 1_S + \pi \cdot (b^n - a^*)\, 1_{T\backslash S} \geqq 0.$$

As the second term tends to zero, $\pi^c(b - a^*)\, 1_S = \int_S (b - a^*) \cdot p\, d\lambda \geqq 0$. But this contradicts the assumption $p(t) \cdot b(t) < p(t) \cdot a^*(t)$, $t \in S$. A very similar argument proves condition (iii).

We now note that $\pi \cdot z^* \geqq \pi \cdot z$, $z \in Z$. It follows from

$$\lim_{n \to \infty} \pi \cdot b^n \geqq \pi \cdot (e + z) \qquad \text{or} \qquad \pi \cdot (e + z^*) \geqq \pi \cdot (e + z).$$

The analysis, so far, was based only on assumptions (i)–(iv) of the theorem. We shall complete the proof by showing that $\pi^p = 0$ from assumption (v), because then $\pi^c = \pi \neq 0$, and hence $p \neq 0$, and $\pi^c \cdot z^* \geqq \pi^c \cdot z \Leftrightarrow \int_T p \cdot z^*\, d\lambda \geqq \int_T p \cdot z\, d\lambda$, $z \in Z$.

LEMMA. *If* $\pi \geq 0$ *and* $\mathscr{E}$ *is Q-irreducible, then* $\pi_i{}^p = 0$, $i \in Q$.

Take the sequence $\{E_m\}$ associated with π^p in Fact 3, and the sequence $\{\Delta z^m\}$ implied by the Q-irreducibility. We want to show that if $\pi^p \neq 0$, then $\pi \cdot \Delta z^m > 0$ for m large enough, contradicting $\pi \cdot (z^* + \Delta z^m) \leqq \pi \cdot z^*$, as $z^* + \Delta z \in Z$.

$$\pi \cdot \Delta z^m = \pi^c \cdot \Delta z^m + \pi^p \cdot \Delta z^m = \pi^c \cdot \Delta z^m 1_{D_m} + \pi^p \cdot \Delta z^m 1_{E_m}.$$

The first term tends to zero, $\pi^c \cdot \Delta z^m$ being a λ-continuous set function and $\lambda(D_m) \to 0$. By (b) of the Q-irreducibility, for large m, $\Delta z_i{}^m(t) \geq \epsilon > 0$, $i \in Q$.

$$\pi^p \cdot \Delta z^m\, 1_{E_m} \geqq \sum_{i \in Q} \pi_i{}^p(E_m)\, \epsilon = \epsilon \sum_{i \in Q} \pi_i{}^p(T).$$

So, if $\pi_i^p(T) > 0$ for some $i \in Q$, then $\lim_{m\to\infty} \pi \cdot \Delta z^m = 0 + \lim \pi^p \cdot \Delta z^m 1_{E_m} > 0$; hence $\pi \cdot \Delta z^m > 0$ for large m.

This completes the proof of the theorem.

We would like to comment on the economics implied in the lemma: If $\pi_i^p > 0$, then for the arbitrarily small set of traders E_m, commodity i's price is unbounded. But then the firm that runs the transaction technology can make profits by supplying more of commodity i to the traders in E_m, at virtually no cost, because the quantity is small. Hence, if $\pi_i^p > 0$, then excess supply prevails. A slightly different approach was pursued by Bewley (1972) and Richter (1977) and will be used in Theorem 2: to assume that goods that cannot be always supplied (not in Q) are inessential in consumption and production. If something is inessential, its price cannot go beyond limits.

DEFINITION. The set of commodities R, $R \subsetneq L$, is called *inessential* iff:

(i) For every feasible allocation a there exist $\epsilon > 0$; a set E with $\lambda(E) < \epsilon$; and an allocation d such that $d_R(t) = 0$, $t \in E$;[10] and $d(t) \succ_t a(t)$ $t \in E \cap H$.

(ii) Given $a \in Z + \{e\}$ and a declining sequence $\{E_m\}_{m=1}^{\infty}$, $\lambda(E_m) \to 0$, then $a^m = a_{L\setminus R} + a_R 1_{T\setminus E_m} \in Z + \{e\}$, for m sufficiently large.

(iii) There exist an integrable function $a \in Z + \{e\}$ and a set E of positive measure with $\bar{a}_R(t) \gg 0$, $t \in E$.

Condition (i) is the inessentiality in production and consumption (ii) is the inessentiality in effecting transactions. In (iii) note that $\bar{a}$ is not necessarily a feasible transaction, because since $\bar{a}(t)$ is not necessarily an allocation it may not be in $A(t)$ for all t.

THEOREM 2. *The following conditions are sufficient for the existence of efficiency prices*:

(1) *Assumptions* (i)–(iv) *as stated in Theorem* 1.

(2) *There exists a partition* (Q, R) *of the commodity set* L (*i.e.*, $Q \cup R = L$, $Q \cap R = \varnothing$) *such that* Q *is nonempty ,the economy is* Q*-irreducible, and the* R *are inessential commodities.*

Proof. From Theorem 1 we know that $\pi = \pi^c + \pi^p$, and $\pi_Q{}^p = 0$. We want to show that, as before, the efficiency price p is the Radon–Nikodym

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[10] $x_R \in E^l$ is defined by

$$\begin{aligned} x_{R_i} &= x_i, \quad & i \in R \\ &= 0, & i \notin R. \end{aligned}$$

Clearly if $Q \cup R = L$ and $Q \cap R = \phi$ then $x_Q + x_R = x$.

derivative of π^c. We are left to show that $\pi^c \cdot z^* \geqq \pi^c \cdot z$, $z \in Z$, and $\pi^c \neq 0$.

Assume now that $\pi_R{}^p \neq 0$, and let $\{E_m\}_{m=1}^{\infty}$ be fact 3's associated sequence. Let d be the allocation guaranteed by inessentiality, for large m. We define $c^{n,m} = b^n 1_{T\setminus E_m} + d1_{E_m}$, $\{b^n\}$ being the same sequence as in Theorem 1.

$$c^{n,m} \in B \qquad \text{so} \qquad c^{n,m} \geqq \pi \cdot a^*$$

or

$$\begin{aligned} &\pi^c \cdot b^n 1_{T\setminus E_m} + \pi^c \cdot d1_{E_m} + \pi^p d1_{E_m} \\ &\qquad \geqslant \pi^c \cdot a^* 1_{T\setminus E_m} + \pi^c \cdot a^* 1_{E_m} + \pi^p \cdot a^* 1_{E_m}. \end{aligned}$$

Noting that $\pi^p \cdot d = 0$ we obtain

$$\pi^c \cdot (b^n - a^*)\, 1_{T\setminus E_m} + \pi^c \cdot (d - a^*)\, 1_{E_m} \geqq \pi^p \cdot a^* 1_{E_m} = \pi^p \cdot a^*.$$

By arguments already familiar, the left-hand side tends to zero, so $\pi^p \cdot a^* \leqq 0$.

Using now the feasible allocation a^m guaranteed by (ii) of the inessentiality definition, we claim that $\pi^c \cdot a \leqq \pi \cdot a^*$, $a \in Z + \{e\}$, which follows because $a^m \in Z + \{e\}$ so $\pi \cdot a^m = \pi^c \cdot a^m \leqq \pi \cdot a^*$. For $m \to \infty$, $\pi^c \cdot a < \pi \cdot a^*$. Hence, $\pi^c \cdot a \leqq \pi \cdot a^* = \pi^c \cdot a^* + \pi^p \cdot a^* \leqq \pi^c \cdot a^*$, since $\pi^p \cdot a^* \leqslant 0$, and we have shown that $\pi^c \cdot z \leqq \pi^c \cdot z^*$.

It is left to show that $\pi^c \neq 0$. But for $\pi^c = 0$, $\pi \cdot a^* \geqq \pi \cdot a$ reduces to $\pi^p \cdot a^* \geqslant \pi^p \cdot a$, $a \in Z + \{e\}$. But $\pi^p \cdot a^* \leqq 0$, and $\pi^p \cdot \bar{a} = \pi_R{}^P \cdot \bar{a}_R > 0$, by (iii) of inessentiality. The implied contradiction completes the proof.

A Note on the assumptions. Alternative assumptions to monotonicity and free disposal in $Y(t)$ may sometimes be desirable. These may be: free disposal in Z ($z'(t) \leqq z(t)$, $t \in T$, $z \in Z \Rightarrow z' \in Z$) and existence of a sequence $\{b^n\}$. Free disposal in Z assures $\pi \geqq 0$ and int $Z \neq \varnothing$.

Existence of $\{b^n\}$ may be deduced from local-nonsatiation of $\succ_t$, and measurability of the correspondences

$$\succ: (H, \mathscr{H}, \lambda) \to R^l \times R^l \qquad \text{and} \qquad X: (H, \mathscr{H}, \lambda) \to R^l,$$

in the sense that their graphs

$$\{(h, x, y): x \succ_h y\}, \qquad \{(h, x): x \in X(h)\}$$

are in $\mathscr{H} \otimes B^{2l}$ and $\mathscr{H} \otimes B^l$, respectively. ($\mathscr{H} = \mathscr{T} \cap H$, B^l is the Borel σ-field generated by the open balls in R^l.)

Under these assumptions, the correspondence $\psi^n: (H, \mathscr{H}, \lambda) \to R^l$ defined by $\psi^n(h) = \{x \in X(h): x \succ_t a^*(t), \|x - a^*(t)\| < 1/n\}$ is measurable, and by local-nonsatiation it is nonvoid for all t. The Measurable Selection Theorem (Hildenbrand, 1974, p. 54) ensures the existence of a measurable selection which is the desired function b^n, because ess sup $\|b^n(t) - a^*(t)\| < 1/n$. (Of course, for $f \in F$ we define $b^n(f) = a^*(f)$.)

4. ILLUSTRATIVE EXAMPLES

The difference between the public good economies, the transaction costs economy, and the Aumann economy is only in the transaction technology.

EXAMPLE 1. Aumann economy.

$\mathscr{T}$ is the collection of all the Lebesgue measurable sets, and λ is Lebesgue measure.

$Z_A = \{z: \int_T z\, d\lambda = 0\}$ is the technology where all the commodities are purely private: nothing is wasted in the reallocation process, and addition to someone's consumption must be accompanied by abstention from consumption somewhere else in the economy. Z_A is a convex cone, and one can easily verify that assumption (v) of Theorem 1 is satisfied, so existence of p is assured. Note that if we allow free disposal, the resulting set $Z'_A = \{z: \int z\, d\lambda \leq 0\}$ has a nonempty interior relative to $\mathscr{L}_1$, and the existence of p is immediate.

Remark 1. The Aumann efficiency prices satisfy

$$p(t) = p(t') \qquad \text{a.e.}$$

Take any two disjoint sets S, S', $\lambda(S) = \lambda(S') > 0$. Define

$$z(t) = -e_i 1_s + e_i 1_{s'}$$[11]

$z \in Z_A$, so $\int_T p \cdot z\, d\lambda \leqq 0$, because Z_A is a cone, and the p are maximum profits prices in Z_A, so $\int_S -p_i\, d\lambda + \int_{S'} p_i\, d\lambda \leqq 0$ or $\int_{S'} p_i\, d\lambda \leqq \int_S p_i\, d\lambda$. Reversing the roles of S and S' we have $\int_S p_i\, d\lambda = \int_{S'} p_i\, d\lambda$, and this completes the proof. If the assertion were false, there would be sets S, S' and i such that $p_i(t) \neq p_i(t')$ all $t \in S$, $t' \in S'$. But there is no loss of generality in assuming that $\lambda(S) = \lambda(S')$, and that $p_i(t) > p_i(t')$, and this would imply $\int_S p_i\, d\lambda > \int_{S'} p_i\, d\lambda$, a contradiction. Q.E.D.

We have, of course, obtained Hildenbrand's (1974) results that efficiency prices exist for the Aumann economy with production.

EXAMPLE 2. A transaction costs economy.

$Z_{TC} = \{z: \int_T z\, d\lambda < 0\}$. If the transaction technology Z satisfies $Z \subset Z_{TC}$, then no transaction may be effected without some waste of resources. The

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[11] e_i is the unit vector where the ith entry is 1, and the others are zero.

economies studied by Hahn (1971), Foley (1970a), and Kurz (1974) all satisfy this assumption; one could easily construct the corresponding transaction technology for them.

Speaking loosely, it is the ability to transfer goods between agents that ensures the existence of equilibrium for Foley (p. 285) and Kurz ((T.4) on p. 420). This is the essence of assumption (v) in our Theorem 1. Our Theorem 2 has a parallel in Kurz' Theorem 2, where a commodity exists which is always desirable and freely transferable. However, desirability neither implies nor is implied by inessentiality in consumption.

The next remark is a certain converse of Remark 1. It shows that price equality implies nonexistence of transaction costs.

Remark 2. Let z satisfy: $0 \in Z$; for all $z \in Z$, $\int z \, d\lambda \leq 0$, and let a^* be an efficient allocation, whose efficiency prices p satisfy: $p(t) = \bar{p} \gg$ a.e. Then $\int a^* \, d\lambda = \int e \, d\lambda$.

We denote $z^* = a^* - e^*$. $\int z^* \cdot p \, d\lambda = \bar{p} \cdot \int z^* \, d\lambda$. $\bar{p} \cdot \int z^* \, d\lambda \geq 0$, because $0 \in Z$ and z^* maximizes profits. Also, $\int z^* \, d\lambda \leqq 0$. So $\bar{p} \cdot \int z^* \, d\lambda = 0$. For all i, $\bar{p}_i \int z_i^* \, d\lambda \leq 0$, so $\bar{p}_i \int z_i^* \, d\lambda = 0$ for all i, and as $\bar{p}_i > 0$, $\int z_i^* \, d\lambda = 0$ for all i. Q.E.D.

EXAMPLE 3. Purely public goods.

Commodity i will be a pure public good if the set $Z_i = \{z: z_j(h) = 0, j \neq i \text{ a.e.}, z_i(h) + \int_F z_i \, d\lambda = 0 \text{ a.e. in } H\}$ is contained in the recession cone[12] AZ of the transaction technology Z; and given $z \in Z$, its ith component z_i is in Z_i. Public good economies will typically not satisfy the conditions for Theorem 1. By its intrinsic nature it is not possible to supply more of i to an elected small group of consumers. We will have to assume that i is inessential.

Remark 3. The efficiency prices satisfy the Samuelson property:[13] $p_i(t) = \int_H p_i \, d\lambda$ a.e. in F.

Let z_i^* be the ith component of the vector z^* that gives rise to the efficient allocation. First, we note that as $z_i(h) = 0$ is contained in Z_i, and Z_i is a cone containing 0, then, using the arguments of Remark 1, we will have $p_i(t) = \bar{p}_i$ a.e. in F.

Now, $\int_T p_i z_i^* \, d\lambda = 0$, because $0 \in Z_i$, and if it were positive, profits would be unbounded; $\int_T p_i z_i^* \, d\lambda = \int_H p_i z_i^* \, d\lambda + \int_F p_i z_i^* \, d\lambda = \int_H p_i z_i^* \, d\lambda + \bar{p}_i \times \int_F z_i^* \, d\lambda = \int_H p_i [- \int_F z_i^* \, d\lambda] \, d\lambda + \bar{p}_i \int_F z_i^* \, d\lambda$. For the nontrivial case $\int_F z_i^* \, d\lambda \neq 0$, and $\bar{p}_i = \int_H p_i \, d\lambda$. Q.E.D.

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[12] For a convex set Z, $z \in AZ$, if for all $z' \in Z$, $z + z' \in Z$. See Rockafellar (1970) for a complete discussion.

[13] Samuelson (1954).

EXAMPLE 4. W. Richter's conomy[14].

$\mathscr{M}$ denotes a measurable partition of H, i.e., a system of pairwise disjoint sets in $\mathscr{H}$ that cover H, such that if $M \in \mathscr{M}$, either $\lambda(M) > 0$ or $|M|$ is finite. The commodity space E^l is divided into private and nonprivate commodities l_p and l_c, respectively, $E^l = E^{l_p} x E^{l_c}$ and $x \in E^l$ is written as $x = (x^p, x^c)$, where $x^p \in E^{l_p}$, $x^c \in E^{l_c}$. We shall also denote L_p and L_c as the sets of private and nonprivate goods, respectively. Using this notation

$$Z_{WR} = \Big\{ z: \int_H z^p \, d\lambda + \int_F z^p \, d\lambda \leqslant 0, \frac{1}{\lambda(M)} \int_M z^c \, d\lambda + \int_F z^c \, d\lambda \leqslant 0 \quad \text{for} \quad \lambda(M) > 0,$$

$$\sum_{t_i \in M} z^c(t_i) + \int_F z^c \, d\lambda \leqslant 0 \qquad \text{for} \quad |M| < \infty \Big\}.$$

Z_{WR} is a convex cone, possessing a nonempty interior. As before, we immediately observe that $p^p(t) = p^p(t')$ $t, t' \in T$, $p^c(f) = p^c(f')$, $f, f' \in F$. This economy $\mathscr{E}_{WR}$ is irreducible with respect to the private goods. Richter's (1977) assumptions are almost sufficient to ensure that the other goods are inessential: nonprivate goods appear only as outputs, so they are inessential to production; however, the Compensation Assumption is slightly weaker than inessentially in consumption: only for condition (i) of inessentiality. Part (ii) is true for $E \subset H$, because for $h \in H$, $X^p(h) = E^{l_c}_+$ and there is free disposal in Z. We do not need it for $E \subset F$, because $p^c(f) = p^c(f')$ all $f, f' \in F$ imply $\pi^p_{L^c}(F) = 0$. Part (iii) is implied by $\dot{Y} \neq \varnothing$ (part (b) of the theorem, p. 95)[15].

5. CONCLUSION

We have presented a general way to model all kinds of transaction technologies. All other conceivable types of commodities that involve a specific transaction technology may be studied by our model. Locally public goods (Ellickson, 1973), is one immediate example. As we saw, specific transaction structures will involve specific price structures.

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[14] Our presentation contains slight deviations from Richter's (1977) theorem: we also assume free disposal of private goods; thus we do not need (δ) in the efficiency price definition (p. 91). The compensation assumption (p. 95) is a little weaker than the inessentiality assumptions here. Both changes could be incorporated into our framework, at the cost of more cumbersome notation.

[15] It seems that W. Richter overlooked the fact that his own assumptions ensured that $(P, \pi_\sigma) \neq 0$, using his own notation (p. 97): $(f, \bar{y}, \bar{p} := 0, \pi := \pi_g(A) 1_A)$ is *not* a quasi-Samuelson allocation, because for $y \in \dot{Y}$, $y^c \gg 0$, so $(\bar{p}, \int \pi d\lambda) \cdot Y \leqq \pi_g(A) \cdot Y$ is unbounded, contradicting (γ) on p. 91.

The transaction technology economy is a natural framework for further study. Does the economy possess a general equilibrium? What transaction technologies would give rise to the equivalence of core and competitive allocations? An answer to the latter question may, it is hoped, broaden our understanding of the meaning of free competition.

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JOURNAL OF ECONOMIC THEORY 22, 477–498 (1980)

On the Aggregation of Information in Competitive Markets

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Received June 27, 1979

1. INTRODUCTION

This paper analyzes how a competitive market serves to communicate information between the market participants. The communication process in a market is usually described by the phrase that the equilibrium price "reflects all the available information in the market" and communicates it to the participants (Fama, 1970).

If there is only one piece of information to consider, the meaning of this phrase is unambiguous. Thus the simple news of higher firm profits will be "reflected" in a higher share price because the demand for shares has gone up.

Problems arise when there are many agents with *different* pieces of information, and the vector of market prices has a smaller dimension than the vector of information available to different agents. In this case the equilibrium price vector corresponds to some *aggregate* of all the individual pieces of information. Then the question arises how this aggregate is formed.

In two recent papers, Grossman (1976, 1978) proposed a remarkable solution to this problem. In a model of the capital market, he argues that the equilibrium price aggregates the available information perfectly. If agents $i = 1, 2, \ldots, n$ take their information from the market price, the economy achieves the same allocation as if each trader knew the whole vector $(I_1, \ldots, I_n)$ of informations available to individual agents. Any aspect of the information vector $(I_1, \ldots, I_n)$ that is not reflected in the price is not woth communicating because it would merely be treated as noise.

However, this approach involves some conceptual difficulties. In Grossman's model the aggregation of information through the price depends only on the statistical properties of the information vector $(I_1, \ldots, I_n)$ and is independent of agents' preferences. This is quite surprising. Intuitively one would expect that the weight with which agent i's information I_i affects the equilivrium price should depend on the strength of agent i's reaction to this information, which in turn should depend on his preferences. Presumably it should make a difference whether the news of an increase in a firm's profits is passed to somebody who is almost risk neutral and responds by buying a

0022-0531/80/030477-22$02.00/0

large number of shares, or whether this piece of news is passed to a risk averter who hardly responds at all.

In Grossman's model this intuition fails because agents do not use their own information at all. The point is that if the price reveals all that is worth knowing about the vector $(I_1, ..., I_n)$, agents will neglect their individual informations and look at the price only. Any aspect of the information I_i that is not contained in the price is regarded as noise. If agent i uses the information contained in the price, he can afford to disregard the information I_i.

However, if agents do not act on their own informations at all, it is unclear why the price should reflect the informations $I_1, ..., I_n$ in the first place. Grossman's proposition does not seem to capture the original idea that individual agents react to their individual informations and *therefore* the equilibrium price reflects some aggregate of the informations $I_1, ..., I_n$. Instead, it must be presumed that the "auctioneer" somehow happens to know the vector $I_1, ..., I_n$, in which case perfect aggregation of information through the price induces agents to disregard their individual informations and therefore is consistent with market clearing.

Against this objection, Grossman (1975) suggests that in fact there does not exist an equilibrium in which the information that is contained in the equilibrium price can itself be traced back to the demand decisions of those agents who had obtained the information originally.[1]

The equilibrium in which price aggregates information perfectly is unique.

Grossman's argument on this point makes heavy use of agents' awareness that their own information and the information contained in the price are statistically dependent. In particular, agents take account of the covariance between "noise" in their own information and "noise" in the price. It is this covariance which makes them neglect their own information when they pay attention to the information contained in the price.

But then, Grossman's agents are slightly schizophrenic. The covariance between "noise" in individual information and "noise" in the price is nonzero because the number of agents is finite, and each agent exerts a nonnegligible influence on the price.[2] Therefore one should expect that agents who are aware of this covariance will also notice the effect they have on the price. Yet, Grossman's agents are price takers. They do not attempt to manipulate the price and the information content of the price.

In order to avoid these difficulties, the present paper will study the aggregation of information in a large market, in which individual agents have no influence on the price. This analysis will lead to an alternative view of the aggregation of information through the price.

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[1] This is proved as a theorem under the additional restriction that the equilibrium price depend linearly on the vector of information.

[2] The alternative interpretation of a finite number of types and a unit mass continuum of agents of each type would fail to capture the notion that *each* agent has different information.

First, it will be seen that in a large market the relative importance of the information available to agent i depends on his preferences. In particular, I_i is relatively the more important, the less risk averse agent i is.

Second, in a large market, the equilibrium price will reflect only those elements of information that are common to a large number of agents. Because an individual agent does not affect the price, his information enters the price only to the extent that it is shared by other agents.

This second result suggests that the market is a good aggregator of information, if there are many agents with many independent sources of information. In this case, "noise" in the information available to any individual agent is filtered out and does not affect the price.

The plan of the paper is as follows: Sections 2 and 3 introduce a generalized version of Grossman's model with a finite number of agents. In particular, I shall allow for the presence of exogenous noise in the price system.[3] Section 4 analyses equilibrium with a finite number of agents and amplifies the foregoing critique of Grossman's approach. The aggregation of information in a large market is studied in Section 5 by means of a competitive sequence of economies. This section contains the main results of the paper.

2. THE BASIC MODEL

I shall use the following version of Grossman's model. There are n agents $i = 1 \cdots n$. Each agent i allocates his initial wealth w_{0i} between a riskless and a risky asset. For each unit purchased, the riskless asset yields 1 unit, the risky asset $\tilde{X}$ units of a single consumption good, where $\tilde{X}$ is a random variable. Using the riskless asset as numeraire, let p be the price of the risky asset. If agent i holds z_i units of the risky asset, his portfolio yields the return

$$\tilde{w}_{1i} = w_{0i} + z_i(\tilde{X} - p).$$

Agents' preferences are described by the following assumption:

A.1. For $i = 1 \cdots n$, agent i maximizes the expected utility of consumption $E_i u_i(\tilde{w}_{1i})$. The utility function u_i exhibits constant absolute risk aversion $\rho_i \in (0, \infty)$.

Under this assumption agent i's demand for the risky asset is independent of his initial wealth w_{0i} (Pratt, 1965). It depends only on the price p and the expectations operator E_i, which in turn is determined by the agent's information I_i. If the supply of the risky asset is Z, the market clearing condition takes the form

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[3] On the importance of such exogenous noise, see Grossman, 1977; Grossman and Stiglitz, 1976.

$$Z = \sum_{i=1}^{n} z_i(p, I_i). \tag{1}$$

The information I_i, on which agent i bases his expectation of $\tilde{X}$, consists of the market price p and his private information y_i. The latter is taken to be the realization of a random variable $\tilde{y}_i$, which communicates the true return $\tilde{X}$ perturbed by some noise $\tilde{\epsilon}_i$:

$$\tilde{y}_i = \tilde{X} + \tilde{\epsilon}_i. \tag{2}$$

Furthermore, the supply Z is taken to be the realization of a random variable $\tilde{Z}$. The following distributional assumption will be imposed:

A.2. The random vector $(\tilde{X}, \tilde{Z}, \tilde{\epsilon}_1 \cdots \tilde{\epsilon}_n)$ has a normal distribution with mean $(\bar{X}, \bar{Z}, 0 \cdots 0)$ and nonsingular variance–covariance matrix: $\sum = (\sigma^2, \Delta^2, s_1{}^2 \cdots s_n{}^2)\, I_{n+2}$, where I_{n+2} is the $(n+2)$-dimensional identity.

Substituting for $I_i = (y_i, p)$ in (1), one finds that the market clearing price p is given as the realization of a random variable $\tilde{p}$ which depends on $\tilde{Z}$ and the vector of signals $(\tilde{y}_1 \cdots \tilde{y}_n)$. The precise form of this relationjhip depends on the functions $z_i(\cdot)$, in particular on the way in which the information I_i affects the expectations operator E_i. Imposing the hypothesis that expectations are rational, I shall require:

A.3. For $i = 1 \cdots n$, agent i knows the actual joint distribution of the triple $(\tilde{X}, \tilde{y}_i, \tilde{p})$. For any information $I_i = (y_i, p)$, he derives the expectations operator E_i from the actual conditional distribution of $\tilde{X}$ given y_i, p.

It should be noted that under Assumption A.3 expectations formation and market clearing cannot be treated separately. The use that agent i makes of his information depends on the joint distribution of $(\tilde{X}, \tilde{y}_i, \tilde{p})$. This in turn depends on the price–supply–signals relation that is imposed by market clearing. Therefore individual expectations formation and demand cannot be analyzed by themselves; from the beginning, the system as a whole must be considered, because the market clearing condition determines the information that agents draw from the market price.

Formally, the determination of equilibrium under the rational expectations assumption A.3 can be treated as a fixed-point problem in the space of functions relating the asset price to supplies and signals. Given any function $f\colon \mathbb{R}^{n+1} \to \mathbb{R}$, suppose, initially, that agents act on the hypothesis that $\tilde{p} = f(\tilde{Z}, \tilde{y}_1 \cdots \tilde{y}_n)$. Then agent i's asset demand depends on the price p, the signal y_i *and* the function f, which determines the joint distribution of the triple $(\tilde{X}, \tilde{y}_i, \tilde{p})$ and therefore the conditional distribution of $\tilde{X}$ given any realization $I_i = (y_i, p)$. Rewriting the market clearing condition (1) in the form

$$Z = \sum_{i=1}^{n} \bar{z}_i(p, y_i ; f), \tag{1a}$$

one finds that the market clearing price depends on f as well as on the vector $(Z, y_1 \cdots y_n)$. With the function f, one can associate a new function Tf: $\mathbb{R}^{n+1} \to \mathbb{R}$, such that $Tf(Z, y_1 \cdots y_n)$ is the market clearing price for the realization $(Z, y_1 \cdots y_n)$ given that agents base their expectations on the hypothesis that $\tilde{p} = f(\tilde{Z}, \tilde{y}_1 \cdots \tilde{y}_n)$. In this formulation, expectations are rational if f is a fixed point of the mapping T, i.e., if $f(Z, y_1 \cdots y_n) = Tf(Z, y_1 \cdots y_n)$ for all $(Z, y_1 \cdots y_n) \in \mathbb{R}^{n+1}$.

Under Assumptions A.1 and A.2, the given fixed-point problem has a *linear* solution. To determine this solution, one proceeds as follows. Consider an arbitrary linear relation:

$$\tilde{p} = \pi_0 + \sum_{i=1}^{n} \pi_i \tilde{y}_i - \gamma \tilde{Z}. \tag{3}$$

Define $\pi = \sum_{i=1}^{n} \pi_i$. Given (2), (3), and Assumption A.2, the triple $(\tilde{X}, \tilde{y}_i, \tilde{p})$ has a normal distribution with mean $(\bar{X}, \bar{X}, \pi_0 + \pi\bar{X} - \gamma\bar{Z})$ and variance–covariance matrix:

$$V_i = \begin{pmatrix} \sigma^2 & \sigma^2 & \pi\sigma^2 \\ \sigma^2 & \sigma^2 + s_i{}^2 & \pi\sigma^2 + \pi_i s_i{}^2 \\ \pi\sigma^2 & \pi\sigma^2 + \pi_i s_i{}^2 & \pi^2\sigma^2 + \sum_{k=1}^{n} \pi_k{}^2 s_k{}^2 + \gamma^2\Delta^2 \end{pmatrix}.$$

From normal distribution theory, the posterior distribution of $\tilde{X}$ given a realization (y_i, p) is again normal, with mean and variance of the form

$$E(\tilde{X} \mid y_i, p) = \alpha_{0i} + \alpha_{1i} y_i + \alpha_{2i} p, \tag{4a}$$

$$\operatorname{Var}(\tilde{X} \mid y_i, p) = \beta_i, \tag{4b}$$

where the values of α_{0i}, α_{1i}, and β_i depend on the matrix V_i.

If expectations are based on relation (3), normality implies that asset demands under expected utility maximization depend only on the posterior mean and variance of returns, $E(\tilde{X} \mid y_i, p)$ and $\operatorname{Var}(\tilde{X} \mid y_i, p)$. In the particular case of constant absolute risk aversion given by Assumption A.1, one has (for a derivation, see, e.g., Grossman (1976, p. 575 f.))

$$\begin{aligned} \bar{z}_i(p, y_i ; \pi_0, \pi_1 \cdots \pi_n, \gamma) &= \frac{E(\tilde{X} \mid y_i, p) - p}{\rho_i \operatorname{Var}(\tilde{X} \mid y_i, p)} \\ &= \frac{\alpha_{0i} + \alpha_{1i} y_i + (\alpha_{2i} - 1) p}{\rho_i \beta_i}. \end{aligned}$$

Substituting for $\tilde{z}_i$ in (1a) and solving for p, one has

$$p = \left[\sum_{i=1}^{n} \frac{1-\alpha_{2i}}{\rho_i\beta_i}\right]^{-1}\left[\sum_{i=1}^{n} \frac{\alpha_{0i}+\alpha_{1i}y_i}{\rho_i\beta_i} - Z\right].\text{[4]} \qquad \text{(1b)}$$

Thus the market clearing price–supply–signals relation (1b) that is induced by the hypothesis (3) is again linear. Expectations based on (3) are rational, if and only if the coefficients π_0, π_i, γ in (3) are the same as the corresponding coefficients in (1b). This yields the following conditions:

$$\gamma = \left[\sum_{i=1}^{n} \frac{1-\alpha_{2i}}{\rho_i\beta_i}\right]^{-1}, \qquad \text{(5a)}$$

$$\pi_i = \frac{\gamma}{\rho_i}\frac{\alpha_{1i}}{\beta_i}, \qquad i = 1 \cdots n, \qquad \text{(5b)}$$

$$\pi_0 = \gamma \sum_{i=1}^{n} \frac{\alpha_{0i}}{\rho_i\beta_i}. \qquad \text{(5c)}$$

To analyze Eqs. (5), remember that the coefficients α_{1i}, α_{2i}, α_{0i}, and β_i arise from the formulas for the conditional mean and variance of $\tilde{X}$ given y_i, p. Under Assumption A.3, these coefficients will in turn depend on π_0, π_i, γ as they affect the joint distribution of $(\tilde{X}, \tilde{y}_i, \tilde{p})$. Before writing out the formulas for these coefficients, I need the following:

LEMMA 2.1. *Assume* A.1–A.3. *Then the equilibrium price–supply–signals relation* (3) *has* $\gamma \neq 0$, *and the matrices* V_i, $i = 1 \cdots n$, *are nonsingular.*

Proof. Suppose that $\gamma = 0$. Then (5b) and (5c) imply $\pi_i = 0$, $i = 1 \cdots n$, and $\pi_0 = 0$, hence $\tilde{p} = 0$. In this case, the coefficients in (4a) and (4b) are calculated as:

$$\alpha_{0i} = s_i^2/(\sigma^2 + s_i^2), \quad \alpha_{1i} = \sigma^2/(\sigma^2 + s_i^2), \quad \alpha_{2i} = 0,$$

$\beta_i = \sigma^2 s_i^2/(\sigma^2 + s_i^2)$. Then (5a) implies $\gamma \neq 0$, a contradiction. Nonsingularity of the matrices V_i follows directly from the fact that $\gamma \neq 0$.

Q.E.D.

Given that the matrices V_i are nonsingular, the coefficients α_{1i}, α_{2i}, α_{0i} and β_i are given by the formulas (Raiffa and Schlaifer, 1961, p. 250)

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[4] Given that $\Delta^2 > 0$, the implicit assumption that $\sum_{i=1}^{n} [(1 - \alpha_{2i})/\rho_i\beta_i] \neq 0$ is harmless. If $\sum_{i=1}^{n} [(1 - \alpha_{2i})/\rho_i\beta_i] = 0$, variations in p have no effect on excess demand. Then there is no way to clear the market for different realizations of $\tilde{Z}$. Since $\tilde{Z}$ has positive variance, by Assumption A.2, one cannot have an equilibrium with rational expectations. The existence of a rational expectations equilibrium with $\sum_{i=1}^{n} [(1 - \alpha_{2i})/\rho_i\beta_i] \neq 0$ is discussed separately in Proposition 3.3.

$$\alpha_{1i} = \frac{\sigma^2}{b_i}\left[\sum_{k=1}^{n} \pi_k^2 s_k^2 + \gamma^2 \Delta^2 - \pi_i \pi s_i^2\right], \tag{6a}$$

$$\alpha_{2i} = \frac{\sigma^2}{b_i}(\pi - \pi_i)\, s_i^2, \tag{6b}$$

$$\alpha_{0i} = \bar{X}\frac{s_i^2}{b_i}\left[\sum_{k=1}^{n} \pi_k^2 s_k^2 + \gamma^2 \Delta^2 - \pi_i^2 s_i^2\right] - \alpha_{2i}(\pi_0 - \gamma \bar{Z}), \tag{6c}$$

$$\beta_i = \frac{\sigma^2 s_i^2}{b_i}\left[\sum_{k=1}^{n} \pi_k^2 s_k^2 + \gamma^2 \Delta^2 - \pi_i^2 s_i^2\right], \tag{6d}$$

$$\begin{aligned} b_i = (\sigma^2 + s_i^2)&\left[\sum_{k=1}^{n} \pi_k^2 s_k^2 + \gamma^2 \Delta^2 - \pi_i^2 s_i^2\right] \\ &+ \sigma^2 s_i^2 (\pi - \pi_i)^2. \end{aligned} \tag{6e}$$

Now Eqs. (5) and (6) form a nonlinear system of equations in π_{0i}, π_{1i}, γ, α_{1i}, α_{2i}, α_{0i}, β_i. The solutions to this system correspond to the equilibria of the economy under Assumptions A.1–A.3. The problem is to analyze (5) and (6) in detail so as to obtain some insight into the way in which the price p aggregates and communicates the information contained in the signals $y_1 \cdots y_n$.

3. PRELIMINARY RESULTS

To analyze equilibrium under Assumptions A.1–A.3, I shall first use Eqs. (6) to eliminate the coefficients α_{1i}, α_{2i}, α_{0i} and β_i from (5). As a result one has

$$\pi_i = \frac{\gamma}{\rho_i s_i^2}\,\frac{\sum_{k=1}^{n} \pi_k^2 s_k^2 + \gamma^2 \Delta^2 - \pi_i \pi s_i^2}{\sum_{k=1}^{n} \pi_k^2 s_k^2 + \gamma^2 \Delta^2 - \pi_i^2 s_i^2}, \qquad i = 1 \cdots n, \tag{7a}$$

$$\frac{1}{\gamma} = \sum_{i=1}^{n} \frac{\sigma^2 + s_i^2}{\rho_i \sigma^2 s_i^2} + \sum_{i=1}^{n} \frac{(\pi - \pi_i)^2 - (\pi - \pi_i)}{\rho_i\left[\sum_{k=1}^{n} \pi_k^2 s_k^2 + \gamma^2 \Delta^2 - \pi_i^2 s_i^2\right]}, \tag{7b}$$

$$\pi_0 = \frac{\bar{X}}{\sigma^2}\sum_{i=1}^{n}\frac{1}{\rho_i} - \gamma(\pi_0 - \gamma \bar{Z})\sum_{i=1}^{n}\frac{\pi - \pi_i}{\rho_i\left[\sum_{k=1}^{n} \pi_k^2 s_k^2 + \gamma^2 \Delta^2 - \pi_i^2 s_i^2\right]}. \tag{7c}$$

Now Eqs. (7a) can be analyzed independently of (7b) and (7c). To see this, define the new variables $Q_i = \pi_i/\gamma$, $i = 1 \cdots n$, $Q = \pi/\gamma = \sum_{i=1}^{n} \pi_i/\gamma$, and reqrite (7a) as

$$Q_i = \frac{1}{\rho_i s_i^2} \, \frac{\sum_{k=1}^{n} Q_k^2 s_k^2 + \Delta^2 - Q_i s_i^2 \sum_{k=1}^{n} Q_k}{\sum_{k=1}^{n} Q_k^2 s_k^2 + \Delta^2 - Q_i^2 s_i^2}, \qquad i = 1 \cdots n. \tag{8a}$$

Equations (8a) serve to determine the variables $Q_1 \cdots Q_n$ regardless of γ and π_0. One has:

LEMMA 3.1. *Under Assumptions* A.1 *and* A.2, *Eqs.* (8a) *have a solution. Any solution* $Q_1 \cdots Q_n$ *to* (8a) *satisfies*

$$0 < Q_i < 1/\rho_i s_i^2.$$

Proof. I shall prove the last statement first. Let $Q_1 \cdots Q_n$ be a solution to (8a) and define the index set $I = \{i \mid Q_i \leqslant 0\}$. Suppose that $I \neq \emptyset$. Then there exists $i_0 \in I$, such that for all $k \in I$, $Q_{i_0} s_{i_0}^2 \geqslant Q_k s_k^2$. Therefore, for $\Delta^2 > 0$,

$$\begin{aligned} \sum_{k=1}^{n} Q_k^2 s_k^2 + \Delta^2 - Q_{i_0} Q s_{i_0}^2 &> \sum_{k \in I} Q_k^2 s_k^2 - Q_{i_0} s_{i_0}^2 \sum_{k \in I} Q_k \\ &= \sum_{k \in I} Q_k (Q_k s_k^2 - Q_{i_0} s_{i_0}^2) \geqslant 0. \end{aligned}$$

Now (8a) implies $Q_{i_0} > 0$, hence $i_0 \notin I$, a contradiction. Therefore, $Q_i > 0$ for all i, implying $Q_i \sum_{k=1}^{n} Q_k > Q_i^2$. Then (8a) implies $Q_i < 1/\rho_i s_i^2$, $i = 1 \cdots n$.

The existence of a solution to (8a) is shown by a fixed-point argument. Let $Y = \prod_{i=1}^{n} [0, 1/\rho_i s_i^2] \subset \mathbb{R}_+^n$ and define mappings $T_0 \colon Y \to \mathbb{R}^n$, $T_1 \colon Y \to Y$ by the conditions

$$(T_0 Q)_i = \frac{1}{\rho_i s_i^2} \, \frac{\sum_{k=1}^{n} Q_k^2 s_k^2 + \Delta^2 - Q_i s_i^2 \sum_{k=1}^{n} Q_k}{\sum_{k=1}^{n} Q_k^2 s_k^2 + \Delta^2 - Q_i^2 s_i^2}, \qquad i = 1 \cdots n,$$

$$\begin{aligned} (T_1 Q)_i &= 0 && \text{if } (T_0 Q)_i < 0 \\ (T_1 Q)_i &= (T_0 Q)_i && \text{if } 0 \leqslant (T_0 Q)_i \leqslant 1/\rho_i s_i^2, \qquad i = 1 \cdots n, \\ (T_1 Q)_i &= 1/\rho_i s_i^2 && \text{if } (T_0 Q)_i > 1/\rho_i s_i^2. \end{aligned}$$

Under Assumptions A.1 and A.2, Y is compact, and T_1 is continuous. By Brouwer's theorem, T_1 has a fixed point Q^*. It remains to show that $0 < Q_i^* < 1/\rho_i s_i^2$ for all i, so that $Q^* = T_0 Q_0^*$. By way of contradiction, suppose first that $Q_i^* = 0$ for some i. Then $(T_0Q^*)_i = 1/\rho_i s_i^2$, hence $(T_1Q^*)_i = 1/\rho_i s_i^2 \neq Q_i^*$, a contradiction. Hence $Q_i^* > 0$ for all i, implying $(T_0Q^*)_i < 1/\rho_i s_i^2$ and hence $(T_1Q^*)_i < 1/\rho_i s_i^2$ for all i. Thus Q^* is a fixed point of T_0 and, hence a solution to (8a). Q.E.D.

For future reference, it is useful to state explicitly the following obvious:

COROLLARY 3.2. *Any solution* $Q_1 \cdots Q_n$ *to* (8a) *satisfies*

$$\Delta^2 > Q_i s_i^2 \sum_{k=1}^{n} Q_k - \sum_{k=1}^{n} Q_k^2 s_k^2, \qquad i = 1 \cdots n.$$

Further, one now has:

PROPOSITION 3.3. *Under Assumptions* A.1 *and* A.2, *Eqs.* (5) *and* (6) *have a solution.*

Proof. It suffices to show that (7) has a solution. Substituting for $\pi_i = \gamma Q_i$, $i = 1 \cdots n$, and solving for γ and π_0, one rewrites (7b) and (7c) as

$$\gamma = \frac{1 + \sum_{i=1}^{n} \dfrac{Q - Q_i}{\rho_i \left(\sum_{k=1}^{n} Q_k^2 s_k^2 + \Delta^2 - Q_i^2 s_i^2\right)}}{\sum_{i=1}^{n} \dfrac{\sigma^2 + s_i^2}{\rho_i \sigma^2 s_i^2} + \sum_{i=1}^{n} \dfrac{(Q - Q_i)^2}{\rho_i \left(\sum_{k=1}^{n} Q_k^2 s_k^2 + \Delta^2 - Q_i^2 s_i^2\right)}}, \tag{8b}$$

$$\pi_0 = \frac{\dfrac{\bar{X}}{\sigma^2} \sum_{i=1}^{n} \dfrac{1}{\rho_i} + \bar{Z} \sum_{i=1}^{n} \dfrac{Q - Q_i}{\rho_i \left(\sum_{k=1}^{n} Q_k^2 s_k^2 + \Delta^2 - Q_i^2 s_i^2\right)}}{\sum_{i=1}^{n} \dfrac{\sigma^2 + s_i^2}{\rho_i \sigma^2 s_i^2} + \sum_{i=1}^{n} \dfrac{(Q - Q_i)^2}{\rho_i \left(\sum_{k=1}^{n} Q_k^2 s_k^2 + \Delta^2 - Q_i^2 s_i^2\right)}}. \tag{8c}$$

The proposition now follows immediately from Lemma 3.1. Q.E.D.

Most of the subsequent analysis will be based on Lemma 3.1. But for some purposes, I shall need a sharper bound on the Q_i. This is given by the following lemma, which is proved in the Appendix.

LEMMA 3.4. *Let* $Q_1 \cdots Q_n$ *be a solution to* (8a), *and let* $Q \equiv \sum_{k=1}^{n} Q_k$. *Then*

$$\frac{Q_i}{Q} > \left[\sum_{k=1}^{n} \frac{\rho_i s_i^2}{\rho_k s_k^2} + \sum_{k=1}^{n} \frac{s_i^2}{s_k^2}\right]^{-1}, \qquad i = 1 \cdots n.$$

4. The Aggregation of Information in a Small Competitive Market

I shall now turn to the question how the equilibrium price under Assumptions A.1–A.3 aggregates information. In particular, I shall be concerned with the relative strength of the weights π_i, π_j with which the signals y_i and y_j affect the equilibrium price. From (8a) and the definition $Q_i := \pi_i/\gamma$, one has

$$\frac{\pi_i}{\pi_j} = \frac{Q_i}{Q_j} = \frac{\rho_j s_j^2}{\rho_i s_i^2} \frac{\dfrac{\sum_{k=1}^{n} Q_k^2 s_k^2 + \Delta^2 - Q_i Q s_i^2}{\sum_{k=1}^{n} Q_k^2 s_k^2 + \Delta^2 - Q_i^2 s_i^2}}{\dfrac{\sum_{k=1}^{n} Q_k^2 s_k^2 + \Delta^2 - Q_j Q s_j^2}{\sum_{k=1}^{n} Q_k^2 s_k^2 + \Delta^2 - Q_j^2 s_j^2}}. \tag{9}$$

In the Appendix, I prove the following:

Proposition 4.1. *Assume* A.1–A.3 *and let* π_0, π_i, $i = 1 \cdots n$, γ *be the coefficients of an equilibrium price function.*

(a) *If* $\rho_i \geqslant \rho_j$ *and* $s_i^2 \geqslant s_j^2$, *then* $\pi_i \leqslant \pi_j$.

(b) *If* $\rho_i \geqslant \rho_j$ *and* $s_i^2 = s_j^2$, *then* $\rho_i \pi_i \geqslant \rho_j \pi_j$.

(c) *If* $\rho_i = \rho_j$ *and* $s_i^2 \geqslant s_j^2$, *then* $s_i^2 \pi_i \leqslant s_j^2 \pi_j$.

If one of the inequalities in the premises of statements (a)–(c) *is strict, the inequality in the corresponding conclusion is also strict.*

Proposition 4.1 shows the dependence of π_i/π_j on the risk aversions ρ_i, ρ_j and the variances s_i^2, s_j^2. The first part of the proposition expresses the simple fact that a precise signal available to a risk neutral agent affects the price more than an imprecise signal available to a risk averse agent. This is so because the sensitivity of agent i's demand to the signal y_i increases with the precision of the signal and decreases with the agent's risk aversion.

The other statements of Proposition 2 give some indication of the extent of the dependence of π_i/π_j on risk aversions and the variances s_i^2, s_j^2. To see the significance of these statements, consider the situation when agents condition *only* on their own signals y_i without drawing information from the price. In this situation one has $\pi_i \rho_i s_i^2 = \pi_j \rho_j s_j^2$ for all i, j, i.e., π_i is just inversely proportional to the factor $\rho_i s_i^2$. In contrast, if agents condition on both signals and price, the ratio π_i/π_j is relatively *less* sensitive to agents' risk aversions and relatively *more* sensitive to the variances s_i^2, s_j^2 (Proposition 4.1b, c)).

These effects arise mainly because agents take account of the covariance between their own signal and the price.[5] The signal y_i affects agent i's expectation about $\tilde{X}$ both directly and indirectly, through the price p. To avoid "double counting" and compensate for his indirect reaction to y_i through the price, he reduces his direct reaction to y_i. For this reason, the coefficient π_i depends less strongly on agent i's risk aversion.

In contrast, the variances s_i^2 and s_j^2 become relatively *more* important because they enter into the covariance between signals and the price. The larger the variance s_i^2, the larger the covariance between noise in y_i and noise in p, and the greater the need to scale down one's direct reaction to the signal y_i.

If the exogenous noise in supplies is large, this covariance effect is insignificant. In this case, variations in price reflect variations in supply rather than variations in signals. Therefore agents draw only little information from the price. In the limit as $\Delta^2 \rightarrow \infty$ the system goes back to the situation when agents condition only on their own signals. Formally, one has

PROPOSITION 4.2. *Assume* A.1–A.3 *and let* $\Delta^2 \rightarrow \infty$. *Then:*

(a) *the equilibrium price converges to*

$$\tilde{p}_\infty = \frac{1}{\sigma^2 B + A}\left[A\bar{X} + \sigma^2 \sum_{i=1}^{n} \frac{\tilde{y}_i}{\rho_i s_i^2} - \sigma^2 Z\right],$$

where

$$A = \sum_{i=1}^{n} \frac{1}{\rho_i}, \qquad B = \sum_{i=1}^{n} \frac{1}{\rho_i s_i^2};$$

(b) *the conditional expectation of* $\tilde{X}$ *given* y_i, p *converges to* $(s_i^2\bar{X} + \sigma^2 y_i)/(\sigma^2 + s_i^2)$;

(c) *the conditional variance of* $\tilde{X}$ *given* y_i, p *converges to* $\sigma^2 s_i^2/(\sigma^2 + s_i^2)$.

Proof. Immediate from (8a)–(8c) and (6). Q.E.D.

The other limit as $\Delta^2 \rightarrow 0$ is of greater interest. Here one has

PROPOSITION 4.3. *Assume* A.1–A.3 *and let* $\Delta^2 \rightarrow 0$. *Then*

(a) *the equilibrium price converges to*:

$$\tilde{p}_0 = \frac{1}{\sigma^2 C + 1}\left[\bar{X} + \sigma^2 \sum_{i=1}^{n} \frac{\tilde{y}_i}{s_i^2} - \frac{\sigma^2 \bar{Z}}{A}\right],$$

---

[5] Formally, the entry $\operatorname{cov}(y_i, p) = \pi\sigma^2 + \pi_i s_i^2$ in the matrix V_i is responsible for the terms $\pi_i \pi s_i^2$ and $\pi_i^2 s_i^2$ in (7), which lead to the inequality $\pi_i/\gamma < 1/\rho_i s_i^2$.

where

$$A = \sum_{i=1}^{n} \frac{1}{\rho_i}, \qquad C = \sum_{i=1}^{n} \frac{1}{s_i^2};$$

(b) *the conditional expectation of* $\tilde{X}$ *given* (y_i, p) *converges to*

$$\frac{\sigma^2 \bar{Z}}{A(\sigma^2 C + 1)} + p;$$

(c) *the conditional variance of* $\tilde{X}$ *given* (y_i, p) *converges to* $\sigma^2/(\sigma^2 C + 1)$.

Proof. Given $\Delta^2 \to 0$, let $Q_1(\Delta^2) \cdots Q_n(\Delta^2)$ be a solution to (8a) and with $Q(\Delta^2) = \sum_{i=1}^{n} Q_i(\Delta^2)$. From Corollary 3.2 one has

$$\frac{\Delta^2}{Q(\Delta^2)^2} \geqslant \sum_{k=1}^{n} \frac{Q_k(\Delta^2)}{Q(\Delta^2)} \left[\frac{Q_i(\Delta^2)}{Q(\Delta^2)} s_i^2 - \frac{Q_k(\Delta^2)}{Q(\Delta^2)} s_k^2\right], \qquad i = 1 \cdots n,$$

and therefore, by Lemma 3.4,

$$\frac{\Delta^2}{Q(\Delta^2)^2} \geqslant q \max_{i,k} \left[\frac{Q_i(\Delta^2)}{Q(\Delta^2)} s_i^2 - \frac{Q_k(\Delta^2)}{Q(\Delta^2)} s_k^2\right], \tag{10a}$$

where

$$q := \min_{j} \left[\sum_{k=1}^{n} \frac{\rho_j s_j^2}{\rho_k s_k^2} + \sum_{k=1}^{n} \frac{s_j^2}{s_k^2}\right]^{-1} > 0$$

is the smallest of the lower bounds in Lemma 3.4. Further, from (8a) and Lemma 3.4, one has

$$\lim_{\Delta^2 \to 0} Q_i(\Delta^2) = 0 \qquad \text{if and only if:} \tag{10b}$$

$$\lim_{\Delta^2 \to 0} \left[\sum_{k=1}^{n} \frac{Q_k(\Delta^2)^2}{Q(\Delta^2)^2} s_k^2 + \frac{\Delta^2}{Q(\Delta^2)^2} - \frac{Q_i(\Delta^2)}{Q(\Delta^2)} s_i^2\right] = 0.$$

The proposition now follows from two intermediate steps:

Step 1. $\lim_{\Delta^2 \to 0} Q_i(\Delta^2) = 0$, $i = 1 \cdots n$.

By way of contradiction, suppose that for some i, there exists $\epsilon > 0$ and a subsequence $\{\Delta_m^2\} \to 0$, such that $Q_i(\Delta_m^2) \geqslant \epsilon$ for all m. Since $Q(\Delta_m^2) > Q_i(\Delta_m^2)$, one has $\lim_{\Delta^2 \to 0} \Delta_m^2 / Q(\Delta_m^2) = 0$, and (10a) implies

$$\lim_{m \to \infty} \left[\frac{Q_i(\Delta_m^2)\, s_i^2 - Q_k(\Delta_m^2)\, s_k^2}{Q(\Delta_m^2)}\right] = 0$$

for all i, k. Given that

$$\sum_{i=1}^{n} \frac{Q_i(\Delta_m^2)}{Q(\Delta_m^2)} = 1,$$

one has

$$\lim_{m\to\infty} \frac{Q_i(\Delta_m^2)}{Q(\Delta_m^2)} = \frac{1}{Cs_i^2}.$$

In consequence,

$$\lim_{m\to\infty} \sum_{k=1}^{n} \left[\frac{Q_k(\Delta_m^2)^2}{Q(\Delta_m^2)^2} s_k^2 + \frac{\Delta_m^2}{Q(\Delta_m^2)^2} - \frac{Q_i(\Delta_m^2)}{Q(\Delta_m^2)} s_i^2\right] = 0,$$

and, by (10b), $\lim_{m\to\infty} Q_i(\Delta_m^2) = 0$, contrary to the assumption that $Q_i(\Delta_m^2) \geqslant \epsilon > 0$ for all m. Hence $\lim_{\Delta^2\to 0} Q_i(\Delta^2) = 0$.

Step 2. $\lim_{\Delta^2\to 0} [Q_i(\Delta^2)/Q(\Delta^2)] = 1/Cs_i^2$.

From (10b) and the fact that $Q_i(\Delta^2) \to 0$, for all i, one has

$$\lim_{\Delta^2\to 0} \left[\sum_{k=1}^{n} \frac{Q_k(\Delta^2)^2}{Q(\Delta^2)^2} s_k^2 + \frac{\Delta^2}{Q(\Delta^2)^2} - \frac{Q_i(\Delta^2)}{Q(\Delta^2)} s_i^2\right] = 0, \qquad \text{for all } i,$$

hence

$$\lim_{\Delta^2\to 0} \left[\frac{Q_i(\Delta^2)\, s_i^2}{Q(\Delta^2)} - \frac{Q_j(\Delta^2)\, s_j^2}{Q(\Delta^2)}\right] = 0, \qquad \text{for all } i, j.$$

The desired result now follows from the fact that

$$\sum_{i=1}^{n} \frac{Q_i(\Delta^2)}{Q(\Delta^2)} = 1.$$

These two steps and (10b) further imply that $\lim_{\Delta^2\to 0} \Delta^2/Q(\Delta^2)^2 = 0$. Using all these results to take limits in (8b) and (8c), one has

$$\lim_{\Delta^2\to 0} \gamma(\Delta^2) = \lim_{\Delta^2\to 0} \pi_0(\Delta^2) = \infty;$$

$$\lim_{\Delta^2\to 0} \gamma(\Delta^2)\, Q(\Delta^2) = \frac{C\sigma^2}{1 + C\sigma^2},$$

and therefore,

$$\lim_{\Delta^2\to 0} \pi_i(\Delta^2) = \lim_{\Delta^2\to 0} \gamma(\Delta^2)\, Q(\Delta^2) \frac{Q_i(\Delta^2)}{Q(\Delta^2)} = \frac{\sigma^2}{1 + C\sigma^2} \cdot \frac{1}{s_i^2};$$

$$\lim_{\Delta^2\to 0} [\pi_0(\Delta^2) - \gamma(\Delta^2)\, \bar{Z}] = \frac{\bar{X} - \sigma^2 \bar{Z}/A}{(\sigma^2 C + 1)},$$

as was to be shown. Statements (b) and (c) of the proposition are obtained in a similar way by taking limits in (6). Q.E.D.

Proposition 4.3 is basically Grossman's result. As the supply-induced noise in the market disappears, the equilibrium price of the risky asset becomes a sufficient statistic for the vector of signals. Variations in the price reflect and communicate variations in the precision weighted sum $\sum_{i=1}^{n} y_i/s_i^2$ of the signals.

The key to this result is the disappearance of the risk aversion coefficients ρ_i from the weights π_i. As Δ^2 becomes small, it hardly makes a difference whether a favorable signal is given to a risk averse or a risk neutral agent. This is an extreme consequence of the agent's awareness of the covariance between the price and their own signal. As Δ^2 becomes small, the price $\tilde{p}$ becomes a more reliable predictor of the return $\tilde{X}$. Therefore agents pay more attention to the price. Because of the price-signal covariance, this is compencompensated by paying less attention to their own signal. In consequence, the weight π_i of the signal y_i becomes less dependent on the particular properties of agent i. This effect is to some extent self-enforcing: As Δ^2 becomes small, price becomes a more reliable predictor of $\tilde{X}$, both because it reveals $\sum_{i=1}^{n} \pi_i y_i$ more precisely *and* because, with disappearance of the coefficients ρ_i from the weights π_i, $\sum_{i=1}^{n} \pi_i y_i$ becomes a more efficient predictor of $\tilde{X}$. In the limit as $\Delta^2 \to 0$, the price reveals $\sum_{i=1}^{n} \pi_i y_i$ perfectly, and, moreover, $\sum_{i=1}^{n} \pi_i y_i$ becomes a sufficient statistic for the vector $(y_1 \cdots y_n)$.

However, if $\Delta^2 = 0$, the model is no longer well specified. If the price is a sufficient statistic, agents no longer even look at their own information, because the pair (p, y_i) is no better than the price alone. But if demands are independent of the signals y_i, there is no reason why the price should vary with the sum $\sum_{i=1}^{n} y_i/s_i^2$.

One can also see this difficulty by considering price formation in a Walrasian tâtonnement. The auctioneer begins the auction by calling a price p "au hasard." Individual agents know their own signals and announce their desired asset demands knowing that trade at p will take place if and only if p clears the market. Therefore the demand announcement must take account of the information carried by p, *if* it happened to clear the market. From the formula

$$\bar{z}_i(p, y_i; \pi_0, \pi_1 \cdots \pi_n, \gamma) = \frac{E(\tilde{X} \mid y_i, p) - p}{\rho_i \operatorname{Var}(\tilde{X} \mid y_i, p)}$$

and Proposition 4.3, this revealed demand is calculated as $\bar{z}_i(p, y_i; \pi_0, \pi_1 \cdots \pi_n, \gamma) = \bar{Z}/\rho_i A$, regardless of p, y_i. But then $\sum_{i=1}^{n} \bar{z}_i(p, y_i, \pi_0, \pi_1 \cdots \pi_n, \gamma) = \bar{Z}$ regardless of $p, y_1 \cdots y_n$. No matter where the auctioneer begins the auction, and what signals agents have received, the market will clear. There is no mechanism by which the auctioneer can, for given $y_1 \cdots y_n$, find the price prescribed by Grossman's formula.

In summary, Grossman's result gives an *approximate* description of communication through the market when the supply-induced noise is small. It should *not* be regarded as an exact result for the case $\Delta^2 = 0$. In this case, the communication process simply is not well defined.

5. THE AGGREGATION OF INFORMATION IN A LARGE MARKET

The preceding model of communication in a finite market is a bit schizophrenic. On the one hand agents are aware of the covariance between the price and their own signals and actions. On the other hand they behave as price takers. To remove this difficulty I shall now look at the aggregation of information in a competitive sequence of economies.

Let $(\Omega, \mathcal{F}, \nu)$ be a probability space and $\mathcal{N}$ the set of normal random variables on Ω. An *economy* $\mathcal{E}$ is defined by a finite set $\mathcal{A}$ of economic agents, a mapping $e\colon \mathcal{A} \to \mathbb{R}_+^2$, and two random variables $\tilde{X} \in \mathcal{N}$, $\tilde{Z} \in \mathcal{N}$ with the following interpretation:

(i) $\tilde{X}$ is the per-unit return, $\tilde{Z}$ the supply of the risky asset.

(ii) For $i \in \mathcal{A}$, $\mathrm{proj}_1\, e(i) = \rho_i$, agent i's coefficient of risk aversion under A.1.

(iii) For $i \in \mathcal{A}$, $\mathrm{proj}_2\, e(i) = s_i^2$, the variance of the error $\tilde{\epsilon}_i = \tilde{y}_i - \tilde{X}$ in agent i's signal, where the vector $(\tilde{X}, \tilde{Z}, \tilde{\epsilon}_i \cdots \tilde{\epsilon}_n)$ satisfies Assumption A.2.

The *characteristic distribution* of the economy $\mathcal{E} = (\mathcal{A}, e, \tilde{X}, \tilde{Z})$ is a measure $\mu_{\mathcal{E}} \in \mathcal{M}(\mathbb{R}_+^2)$, defined by the usual formula, $\mu_{\mathcal{E}}(B) = \#e^{-1}(B)/\#\mathcal{A}$, for every Borel subset B of $\mathbb{R}_+^2$.

A sequence of economies $\mathcal{E}_n = \{\mathcal{A}^n, e^n, \tilde{X}^n, \tilde{Z}^n\}$ will be called *competitive*, if it satisfies the following conditions:

(B.1) $\#\mathcal{A}^n = n \to \infty$.

(B.2) There exist two random variables $\tilde{X} \in \mathcal{N}$, $\tilde{Z} \in \mathcal{N}$, such that for all n, $\tilde{X}^n = \tilde{X}$ and $\tilde{Z}^n/n = \tilde{Z}$.

(B.3) There exists a closed rectangle $[\underline{\rho}, \bar{\rho}] \times [\underline{s}^2, \bar{s}^2]$ in the interior of $\mathbb{R}_+^2$, such that for all n, $e^n(\mathcal{A}^n) \subset [\underline{\rho}, \bar{\rho}] \times (\underline{s}^2, \bar{s}^2]$.

(B.4) The sequence of characteristic distributions $\{\mu_{\mathcal{E}_n}\}$ converges weakly to a measure $\mu \in \mathcal{M}(\mathbb{R}_+^2)$.

If one writes $\rho_i(n)$, $s_i^2(n)$ for the characteristics $e^n(i)$ of agent $i \in \mathcal{A}^n$, one obtains:

PROPOSITION 5.1. *Let $\{\mathscr{E}_n\}$ be a competitive sequence of economies. Any solution $Q_i(n)$, $i \in \mathscr{A}^n$ of* Eqs. (8a) *for economy $\mathscr{E}_n$ satisfies*

$$Q_i(n) = \frac{1}{p_i(n)\, s_i^2(n)} + o(1/n), \qquad i \in \mathscr{A}^n.$$

Proof. See the Appendix.

Further let $\bar{Z}(n)$, $\Delta^2(n)$ denote the mean and variance of total supply $\tilde{Z}^n$, and let Z^*, Δ^{*2} denote the mean and variance of per capita supply $\tilde{Z}$. Clearly, (B.2) implies $\bar{Z}(n) = nZ^*$ and $\Delta^2 = n^2 \Delta^{*2}$.

With obvious notation, write the market-clearing price for economy $\mathscr{E}_n$ under the rational expectations hypothesis A.3 as

$$\tilde{p}^n = \pi_0(n) + \sum_{i \in \mathscr{A}^n} \pi_i(n)\, \tilde{y}_i^n - \gamma(n)\, \tilde{Z}^n.$$

In the Appendix, I prove:

PROPOSITION 5.2. *Let $\{\mathscr{E}_n\}$ be a competitive sequence of economies and let $\{\tilde{p}^n\}$ be a corresponding sequence of market-clearing prices under* A.3. *Then the sequence $\{\tilde{p}\}$ converges in probability to $\tilde{p}^* = \pi_0^* + \pi^*\tilde{X} - \gamma^*\tilde{Z}$, where*

$$\pi_0^* = \frac{\bar{X}\Delta^{*2}A^* + \sigma^2 Z^* A^* B^*}{A^*\Delta^{*2} + \sigma^2 B^* \Delta^{*2} + \sigma^2 A^* B^{*2}},$$

$$\pi^* = \frac{\sigma^2 B^* \Delta^{*2} + \sigma^2 A^* B^{*2}}{A^*\Delta^{*2} + \sigma^2 B^* \Delta^{*2} + \sigma^2 A^* B^{*2}},$$

$$\gamma^* = \frac{\sigma^2 \Delta^{*2} + \sigma^2 A^* B^*}{A^*\Delta^{*2} + \sigma^2 B^* \Delta^{*2} + \sigma^2 A^* B^{*2}},$$

$$A^* = \int \frac{1}{\rho}\, d\mu, \qquad B^* \equiv \int \frac{1}{\rho s^2}\, d\mu.$$

As the number of agents becomes large, the weights $\pi_i(n)$ converge to zero. Individual agents can no longer affect the price. This has two important consequences: First, the relative weights

$$\frac{\pi_i(n)}{\pi_j(n)} = \frac{Q_i(n)}{Q_j(n)}$$

of signals given to two agents i, j become approximately equal to $\rho_j(n)\, s_j^2(n)/\rho_i(n)\, s_i^2(n)$. As $\pi_i(n)$ converges to zero, the covariance between the noise $\tilde{\epsilon}_i^n$ and the equilibrium price $\tilde{p}^n$ disappears. The relative importance of an agent's information becomes inversely proportional to his degree of risk aversion. The aggregation of information in a large market depends on agents' preferences as well as on the precision of their signals.

Second, as the number of agents increases, the equilibrium price comes to depend only on the actual return $\tilde{X}$, which is the common element in all the signals $\tilde{y}_i$. The noises $\tilde{\epsilon}_i$" disappear from the equilibrium price, by the weak law of large numbers. This is an instance of the well known result that individual uncertainty does not affect the equilibrium price in a large economy (see, e.g., Malinvaud (1972)). In the more general case where the common element in the individual signals $\tilde{y}_i$ involves some noise $\tilde{\delta}$, so that $\tilde{y}_i = \tilde{X} + \tilde{\delta} + \tilde{\epsilon}_i$, the equilibrium price will depend on $\tilde{\delta}$ as well as $\tilde{X}$, but the $\tilde{\epsilon}_i$ will still cancel out.

In summary, Propositions 5.1 and 5.2 suggest that communication through the market relies on the presence of a large number of independent sources of information. Because of differences in preferences, the equilibrium price is not, in general, an "efficient" aggregator of information. However, this inefficiency is irrelevant, if the market draws on many independent sources of information, so that individual errors cancel out.

Even so, individuals cannot actually read $\tilde{X}$ off the equilibrium price. Because of the noise in supplies, they cannot distinguish whether a high price is due to a high realization of $\tilde{X}$ or a low realization of $\tilde{Z}$. Therefore, they find it worthwhile to draw information from their own signal as well as the price. Formally one has:

PROPOSITION 5.3. *Let $\tilde{p}^*$ be the equilibrium price as in Proposition* 5.2, *and let $\tilde{y} = \tilde{X} + \tilde{\epsilon} \in \mathcal{N}$ be a signal with $E\tilde{\epsilon} = 0$, $E\tilde{\epsilon}^2 = s^2$, $E\tilde{\epsilon}\tilde{Z} = 0$. Given a realization (y, p) of the random pair $(\tilde{y}, \tilde{p}^*)$ the posterior mean and variance of returns are*

$$E(\tilde{X} \mid y, p) = \frac{1}{D}\left[\frac{\Delta^{*2}}{B^{*2}}\, s^2 \bar{X} + \frac{\Delta^{*2}}{B^{*2}}\, \sigma^2 y + \sigma^2 s^2\, \frac{p - \pi_0^* + \gamma^* Z^*}{\pi^*}\right],$$

$$\operatorname{Var}(\tilde{X} \mid y, p) = \frac{1}{D}\, \frac{\Delta^{*2}}{B^{*2}}\, s^2 \sigma^2,$$

where

$$D = \frac{\Delta^{*2}}{B^{*2}}\, \sigma^2 + \frac{\Delta^{*2}}{B^{*2}}\, s^2 + \sigma^2 s^2,$$

and π_0^*, π^*, γ^*, B^* are the coefficients defined in Proposition 5.2.

Proposition 5.3 shows the importance of preferences for the communication of information through the market. The distribution of degrees of risk aversion and of signal variances determines the coefficient $B^* = \int (1/\rho s^2)\, d\mu$. This coefficient is equal to the ratio π^*/γ^* of the weights of $\tilde{X}$ and $\tilde{Z}$ in the equilibrium price. For given variances σ^2, Δ^{*2}, this ratio determines the relative contributions of $\tilde{X}$ and $\tilde{Z}$ to variations in the equilibrium price. The point is that the strength of agents' reactions to their signals—and implicitly to $\tilde{X}$—is inversely related to both their degree of risk aversion and the level of noise in their signal. If agents are less risk averse, variations in $\tilde{X}$ will lead to larger variations in asset demands, which in turn induce larger variations in the equilibrium price.

The larger the coefficient B^*, the more reliable the market as a means of communication. If variations in the equilibrium price arise mainly from variations in the return, there is little chance that a given price change will reflect a change in supply rather than a change in return. Therefore, agents can rely heavily on the price as a source of information.

From this point of view, the ratio Δ^{*2}/B^{*2} in Proposition 5.3 is a natural measure of the level of noise in the market communication process. The smaller the ratio Δ^{*2}/B^{*2}, the more attention agents pay to the price and the less to their own signals. Then also the posterior variance of returns decreases. There is no noise in the market, if either $\Delta^{*2} = 0$ *or* $B^{*2} = \infty$. The latter situation arises when a positive set of agents is risk neutral and reacts so strongly to its information that variations in supplies have no effect on the price ($\gamma^* = 0$). In this case *all* agents can actually infer the realization of returns from the price.[6]

APPENDIX

Proof of Lemma 3.4

By Lemma 3.1,

$$Q < \sum_{k=1}^{n} \frac{1}{\rho_k s_k^2}.$$

Hence, from (8a),

$$\frac{Q_i}{Q} > \left[\sum_{k=1}^{n} \frac{\rho_i s_i^2}{\rho_k s_k^2}\right]^{-1} \cdot \frac{\sum_{k=1}^{n} Q_k^2 s_k^2 + \Delta^2 - Q_i Q s_i^2}{\sum_{k=1}^{n} Q_k^2 s_k^2 + \Delta^2 - Q_i^2 s_i^2}$$

$$= \left[\sum_{k=1}^{n} \frac{\rho_i s_i^2}{\rho_k s_k^2}\right]^{-1} \left[1 - \frac{Q_i(Q - Q_i)\, s_i^2}{\sum_{k=1}^{n} Q_k^2 s_k^2 + \Delta^2 - Q_i^2 s_i^2}\right]$$

---

[6] However, one again has the paradox that agents inferring the realization of $\tilde{X}$ from the price pay no attention to their own signals, so that the question is, Why should the price reveal the realization of $\tilde{X}$ in the first place.

$$\geq \left[\sum_{k=1}^{n} \frac{\rho_i s_i^2}{\rho_k s_k^2}\right]^{-1} \left[1 - \frac{Q_i(Q - Q_i)\, s_i^2}{\sum_{k\neq i} Q_k^2 s_k^2}\right]$$

$$= \left[\sum_{k=1}^{n} \frac{\rho_i s_i^2}{\rho_k s_k^2}\right]^{-1} \left[1 - \frac{Q_i}{Q - Q_i} \sum_{k\neq i} \frac{s_i^2}{s_k^2}\right] = \frac{1 - \dfrac{Q_i}{Q} \displaystyle\sum_{k=1}^{n} \frac{s_i^2}{s_k^2}}{\displaystyle\sum_{k=1}^{n} \frac{\rho_i s_i^2}{\rho_k s_k^2}\left(1 - \dfrac{Q_i}{Q}\right)},$$

where the last inequality is based on the fact that $\sum_{k\neq i} Q_k^2 s_k^2 \geqslant (Q - Q_i)^2/\sum_{k\neq i} 1/s_k^2$.[7]

Multiplying by

$$\sum_{k=1}^{n} \frac{\rho_i s_i^2}{\rho_k s_k^2}\left(1 - \frac{Q_i}{Q}\right)$$

and rearranging terms, one has

$$\frac{Q_i}{Q}\left[\sum_{k=1}^{n} \frac{\rho_i s_i^2}{\rho_k s_k^2} + \sum_{k=1}^{n} \frac{s_i^2}{s_k^2}\right] > 1 + \frac{Q_i^2}{Q^2} \sum_{k=1}^{n} \frac{\rho_i s_i^2}{\rho_k s_k^2} > 1,$$

and the lemma follows immediately. Q.E.D.

Proof of Proposition 4.1

Define $L = \sum_{k=1}^{n} Q_k^2 s_k^2 + \Delta^2$ and rewrite (9) as

$$\frac{Q_i \rho_i s_i^2}{Q_j \rho_j s_j^2} = \frac{(L - Q_i Q s_i^2)(L - Q_j^2 s_j^2)}{(L - Q_i^2 s_i^2)(L - Q_j Q s_j^2)}. \tag{9'}$$

Further, define

$$M := (L - Q_i Q s_i^2)(L - Q_j^2 s_j^2) - (L - Q_i^2 s_i^2)(L - Q_j Q s_j^2)$$

and note that, by Lemma 3.1 and Corollary 3.2,

$$Q_i \rho_i s_i^2 \gtrless Q_j \rho_j s_j^2 \qquad \text{as} \quad M \gtrless 0. \tag{9''}$$

(a) Let $\rho_i \geqslant \rho_j$, $s_i^2 \geqslant s_j^2$ and suppose that $Q_i > Q_j$. Then one calculates

$$M = Q_j(Q - Q_j)(s_j^2 - s_i^2)L + (Q_j - Q_i)[(Q - Q_i - Q_j)\, s_i^2 L + Q_i Q_j Q s_i^2 s_j^2] < 0.$$

Now (9″) implies $Q_i \rho_i s_i^2 < Q_j \rho_j s_j^2$, hence $Q_i < Q_j$, a contradiction. Therefore, $Q_i \ngtr Q_j$, hence $\pi_i/\gamma \leq \pi_j/\gamma$. From (8b), $\gamma > 0$, hence $\pi_i \leqslant \pi_j$.

---

[7] To see this, minimize $\sum_{k\neq i}^{n} Q_k^2 s_k^2$ subject to $\sum_{k\neq i}^{n} Q_k = Q - Q_i$.

(b) Let $\rho_i \geqslant \rho_j$, $s_i^2 = s_j^2$ and suppose that $\rho_i Q_i < \rho_j Q_j$. Then $Q_j > Q_i$, hence $M > 0$ and by (9″), $\rho_i Q_i > \rho_j Q_j$, a contradiction. Therefore $\rho_i Q_i \nless \rho_j Q_j$ and $\rho_i \pi_i \geqslant \rho_j \pi_j$ because $\gamma > 0$, again by (8b).

(c) Let $\rho_i = \rho_j$, $s_i^2 \geqslant s_j^2$ and suppose that $s_i^2 Q_i > s_j^2 Q_j$. Rearranging terms in the expression for M, one has

$$M = (Q_j s_j^2 - Q_i s_i^2)(Q - Q_j)\, L - (Q_j - Q_i)(L - Q_j Q s_j^2)\, Q_i s_i^2 \leq 0,$$

because $Q_j \geqslant Q_i$, by (a), and $L > Q_j Q s_j^2$, by Corollary 3.2. Now (9″) implies $Q_i s_i^2 < Q_j s_j^2$, a contradiction. Therefore $s_i^2 Q_i \ngtr s_j^2 Q_j$ and $s_i^2 \pi_i \leqslant s_j^2 \pi_j$, because $\gamma > 0$ by (8b).

The sharpening of the conclusions when the inequalities in the premises are strict is proved by the same argument. Q.E.D.

Proof of Proposition 5.1

From (8a) and Lemma 3.1 one has

(i) $Q_i(n)\, \rho_i(n)\, s_i^2(n) < 1$,

(ii)
$$\begin{aligned} Q_i(n)\, \rho_i(n)\, s_i^2(n) &= 1 - Q_i(n)\, s_i^2(n) \sum_{k \in \mathscr{A}^n} Q_k(n)/\Delta^2(n) \\ &= 1 - \sum_{k \in \mathscr{A}^n} \frac{1}{\rho_k(n)\, s_k^2(n)} \Big/ \rho_i(n)\, \Delta^2(n) \qquad \text{by (i)} \\ &\geq 1 - \frac{1}{n} \frac{1}{\underline{\rho}^2 \underline{s}^2 \Delta^{*2}}, \qquad \text{by (B.2) and (B.3).} \end{aligned}$$

Q.E.D.

Proof of Proposition 5.2

For any n, define

$$A(n) = \frac{1}{n} \sum_{i \in \mathscr{A}^n} \frac{1}{\rho_i(n)}, \qquad B(n) = \frac{1}{n} \sum_{i \in \mathscr{A}^n} \frac{1}{\rho_i(n)\, s_i^2(n)}.$$

Using (B.2) and Proposition 5.1 to substitute for $\bar{Z}(n) = nZ^*$, $\Delta^2(n) = n^2 \Delta^{*2}$ and $Q_i(n) = 1/\rho_i(n)\, s_i^2(n) + o(1/n)$ in (8b) and (8c), one has, by elementary algebra

$$n\gamma(n) = \frac{\sigma^2 \Delta^{*2} + \sigma^2 A(n)\, B(n)}{\Delta^{*2} A(n) + \sigma^2 B(n)\, \Delta^{*2} + \sigma^2 A(n)\, B(n)^2} + o(1/n),$$

$$\pi_0(n) = \frac{\bar{X} A(n)\, \Delta^{*2} + \sigma^2 Z^* A(n)\, B(n)}{\Delta^{*2} A(n) + \sigma^2 B(n)\, \Delta^{*2} + \sigma^2 A(n)\, B(n)^2} + o(1/n).$$

Note that

$$A(n) = \int \frac{1}{\rho}\, d\mu_{\mathscr{E}_n} \qquad \text{and} \qquad B(n) = \int \frac{1}{\rho s^2}\, d\mu_{\mathscr{E}_n},$$

so that (B3) and (B4) imply $\lim_{n\to\infty} A(n) = A^*$ and $\lim_{n\to\infty} B(n) = B^*$. Then one has $\lim_{n\to\infty} \pi_0(n) = \pi_0^*$, $\lim_{n\to\infty} \gamma(n) = \gamma^*$, and $\lim_{n\to\infty} \sum_{i\in\mathscr{A}^n} \pi_i(n) = \lim_{n\to\infty} \sum_{i\in\mathscr{A}^n} Q_i(n)\gamma(n) = \lim_{n\to\infty} \sum_{i\in\mathscr{A}^n} Q_i(n) \lim_{n\to\infty} n\gamma(n) = B^*\gamma^*$, by Lemma 5.1.

Rewriting the equation for $\tilde{p}^n$ as

$$\tilde{p}^n = \pi_0(n) + \sum_{i\in\mathscr{A}^n} \pi_i(n)\, \tilde{X} + \sum_{i\in\mathscr{A}^n} \pi_i(n)\, \tilde{\epsilon}_i{}^n - \gamma(n)\, n\, \frac{\tilde{Z}^n}{n},$$

one sees that the proposition follows, if $\sum_{i\in\mathscr{A}^n} \pi_i(n)\, \tilde{\epsilon}_i{}^n$ converges to zero in probability. To prove the latter statement, it suffices to note that $\sum_{i\in\mathscr{A}^n} \pi_i(n)\tilde{\epsilon}_i{}^n$ has mean zero and variance

$$\begin{aligned}\sum_{i\in\mathscr{A}^n} \pi_i(n)^2\, s_i{}^2(n) &= (\gamma(n)\, n)^2 \frac{1}{n^2} \sum_{i\in\mathscr{A}^n} Q_i(n)^2\, s_i{}^2(n)\\ &= (\gamma(n)\, n)^2 \frac{1}{n^2} \sum_{i\in\mathscr{A}^n} \left[\frac{1}{\rho_i(n)^2\, s_i{}^2(n)} + o\left(\frac{1}{n}\right)\right]\\ &\leqslant (\gamma(n)\, n)^2 \frac{1}{n} \left[\frac{1}{\underline{\rho}^2 \underline{s}^2} + o\left(\frac{1}{n}\right)\right],\end{aligned}$$

which converges to zero as $n \to \infty$. Q.E.D.

Proof of Proposition 5.3

The proposition follows from normal distribution theory (Raiffa and Schlaifer, 1961, p. 250) after noting that the triple $(\tilde{X}, \tilde{y}, \tilde{p}^*)$ is normally distributed with mean $(\bar{X}, \bar{X}, \pi_0^* + \pi^*\bar{X} - \gamma^* Z^*)$ and variance-covariance matrix:

$$\begin{pmatrix} \sigma^2 & \sigma^2 & \pi^*\sigma^2 \\ \sigma^2 & \sigma^2 + s^2 & \pi^*\sigma^2 \\ \pi^*\sigma^2 & \pi^*\sigma^2 & \pi^{*2}\sigma^2 + \gamma^{*2}\Delta^{*2} \end{pmatrix}.$$

Q.E.D.

ACKNOWLEDGMENT

I would like to thank Sanford Grossman and Michael Rothschild for several stimulating conversations. Research support from the National Science Foundation under Grant SOC 75-13437 at Princeton University and from the Deutsche Forschungsgemeinschaft is gratefully acknowledged.

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JOURNAL OF ECONOMIC THEORY 22, 499–504 (1980)

Notes, Comments, and Letters to the Editor

How the Law Circumvents Starrett's Nonconvexity

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Received November 15, 1977; revised July 19, 1979

Starrett has made a significant contribution to our understanding of the mathematics of externalities,[1] but the nonconvexity which he identifies has a remedy in law. The mechanism by which the law circumvents this nonconvexity has gone unnoticed in the literature, as evidenced by a recent exchange in this Journal.[2] This note relates nonconvexity to the law of liability.

THE PROBLEM

We shall identify the nonconvexity by recalling Arrow's formulation of a market for externalities and then describing the remedy.[3] Let y_{ik} be the ith firm's net supply of the kth good, where $i = 1, 2, \ldots, m$ and $k = 1, 2, \ldots, n$. Let $y_{)i(}$ indicate the net supply of all firms except the ith: $y_{)i(} = (y_{11}, \ldots, y_{(i-1)n}, y_{(i+1)1}, \ldots, y_{mn})$. Externality is indicated by a production function whose arguments include the net supply of other firms: $0 \geqslant f^i(y_{i1}, \ldots, y_{in}; y_{)i(})$. Arrow defines a variable y_{jik} which may be described as j's

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[1] David Starrett, Fundamental non-convexities in the theory of externalities, *J. Econ. Theory* 4 (1972), 180–199.

[2] In an exchange on the Coase theorem, the Editors of this Journal remark that Starrett has shown that there are "non-convexities inherent in the problem of external diseconomies, non-convexities which destroy the validity of the Coase Theorem." These comments are a response to Shapiro's example where setup costs invalidate the Coase theorem if exchange opportunities are restricted to marginal bargains. See David Shapiro, A note on rent and the Coase theorem, *J. Econ. Theory* 7 (1974), 125–128; and the Editors, Addendum to "A note on rent and the Coase theorem" by David Shapiro, *J. Econ. Theory* 14 (1977), 221–222.

[3] K. J. Arrow, The organization of economic activity: Issues pertinent to the choice of market versus non-market allocation, *in* U. S. Congress, Joint Economic Committee, "The Analysis and Evaluation of Public Expenditures: The PPB System," Vol. 1, pp. 47–64, Government Printing Office, Washington, D.C.

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observation of i's net supply of good k; by an obvious extension of notation we let $y_{i)i(}$ indicate i's observation of $y_{)i(}$. We imagine that institutions can be established so that there is an equivalence relation between net supplies and their observation: $y_{jik} = y_{ik}$ for all j, i, k. Consequently we may eliminate the externality by substituting observations for commodities in the production function: $0 \geqslant f^i(y_{ii1}, \dots, y_{iin}; y_{i)i(})$.

The equivalence relation is interpreted by Arrow as the market clearing condition for observations. Obviously there is bilateral monopoly in these markets, so competitive behavior is not to be anticipated; Starrett observed that there is also a nonconvexity. The difficulty is that externalities can be observed beyond the point where own production is nil, as indicated by the example of the laundry and smoke in Fig. 1. Equilibrium may fail to exist in the market for observations of smoke because of this nonconvexity.

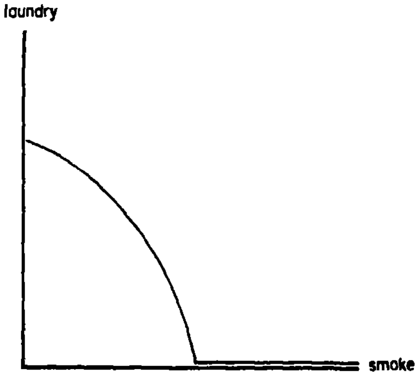

FIG. 1. Starrett's nonconvexity.

THE LIABILITY SOLUTION

The institutional problem is to eliminate the externality by a mechanism which can distinguish between global and local optima. The courts can solve this problem by assigning liability rights according to a calculation of benefits and costs, as we shall show for the laundry and smokey electric company. Benefits and costs are measured by the profit functions for the two firms. Let y_{js} indicate j's smoke with price $p_s = 0$, where $(p_1, p_2, \dots, p_n)$.

$$\text{Laundry:} \quad \pi^i = \max_{y_{ik}} \sum_k p_k y_{ik} \quad \text{s.t.} \quad f^i(y_{i1}, \dots, y_{in}; y_{js}) \leqslant 0,$$

$$\text{Electric Company:} \quad \pi^j = \max_{y_{jk}} \sum_k p_k y_{jk} \quad \text{s.t.} \quad f^j(y_{j1}, \dots, y_{jn}) \leqslant 0.$$

The cost of pollution is the minimum loss in profits to the laundry caused by the smoke, or the loss when the laundry maximizes profits, written

$\pi^i(p, 0) - \pi^i(p, y_{js})$.[4] If the electric company is liable for the full cost of smoke, then its profit net of compensation is[5]

$$\bar{\pi}^j = \max_{y_{jk}} \sum_k p_k y_{jk} - (\pi^i(p, 0) - \pi^i(p, y_{js})) \qquad \text{s.t.} \quad f^j(y_{j1}, \dots, y_{jn}) \leq 0.$$

The term $\pi^i(p, 0)$ is constant, so the electric company is maximizing joint profits as it would if the firms merged. Thus the externality is internalized. The joint maximization problem exhibits Starrett's nonconvexity, as illustrated in Fig. 2, but it is likely that the electric company will be able to identify the global maximum if the laundry divulges information about profits.[6]

Starrett was concerned that the pollutee (laundry) might supply an infinite number of pollution rights to a competitive market, but this problem does not arise under liability. The laundry may sell its legal claim for compensation by signing a contract which waives its right to recover damages. We may describe this transaction as selling coupons granting their owner the right to pollute. The laundry moves down the production frontier as it sells coupons to the electric company, until further damages are impossible. The right to recover further damages is worthless in the nonconvex region, because further damages are nil. In effect, the electric company must pay a positive price for coupons up to the point where the landry can sustain no further damages, and then the price becomes nil. Exchange of pollution rights is not competitive under the liability solution, because the price varies with the quantity.

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[4] It is essential for economic efficiency to compensate for the minimum profit loss, written $\pi^i(p,0) - \pi^i(p,y_{js})$, rather than the *actual* profit loss, written $\pi^i(p,0) - \Sigma_k p_k y_{ik}$. If actual losses are the basis of compensation, then the Laundry's profit net of compensation is constant, $\Sigma_k p_k y_{ik} + (\pi^i(p, 0) - \Sigma_k p_k y_{ik}) = \pi^i(p, 0)$, and it has no incentive for efficient production. It is difficult in practice for courts to distinguish between actual and minimum losses.

[5] This presentation is simplified by not discussing court costs or settlement costs; we avoid the theory of litigation.

[6] Here is a *simple proof on nonconvexity. Let the objective function in the optimization* problem defining $\bar{\pi}^j$ be denoted $\Psi_p(y_j) \equiv \Sigma_k p_k y_{jk} - (\pi^i(p, 0) - \pi^i(p, y_{js}))$. We shall prove that $\Psi_p(y_j) \geq \alpha$ is a nonconvex set.

(i) Starrett's nonconvexity, as drawn in Fig. 1, implies that laundry output approaches zero continuously as smoke approaches some finite level. (The proof is slightly different when this level is infinite.) Hence, there exist $(\hat{p}, \hat{y})$ such that $\pi^i(\hat{p}, y_{js}) > 0$ for $y_{js} < \hat{y}_{js}$ ($= 0$ otherwise). Choose $\bar{y}_j$ and $\bar{\bar{y}}_j$ so that $\bar{y}_{js} < \hat{y}_{js} < \bar{\bar{y}}_{js}$, and $\Sigma_k \hat{p}_{jk}(\bar{y}_{jk} - \bar{\bar{y}}_{jk}) + \pi^i(\hat{p}, \bar{y}_{js}) = 0$. Hence for $p = \hat{p}$ we have $\Psi_{\hat{p}}(\bar{y}_j) = \Psi_{\hat{p}}(\bar{\bar{y}}_j) \equiv \alpha$.

(ii) Choose t so that $0 < t < 1$ and $t\bar{y}_{js} + (1 - t)\bar{\bar{y}}_{js} = y_{js}$.

(iii) We wish to show that $\Psi_{\hat{p}}(t\bar{y}_j + (1 - t)\bar{\bar{y}}_j) < \alpha$. Assume the contrary; then $\Psi_{\hat{p}}(t\bar{y}_j + (1 - t)\bar{\bar{y}}_j) \geqslant \alpha = t\Psi_{\hat{p}}(\bar{y}_j) + (1 - t)\Psi_{\hat{p}}(\bar{\bar{y}}_j) \Rightarrow \pi^i(\hat{p}, t\bar{y}_{js} + (1 - t)\bar{\bar{y}}_{js}) \geqslant t\pi^i(\hat{p}, \bar{y}_{js}) + (1 - t)\pi^i(\hat{p}, \bar{\bar{y}}_{js}) \Rightarrow 0 \geqslant t\pi^i(\hat{p}, \bar{y}_{js}) > 0$, which is a contradiction.

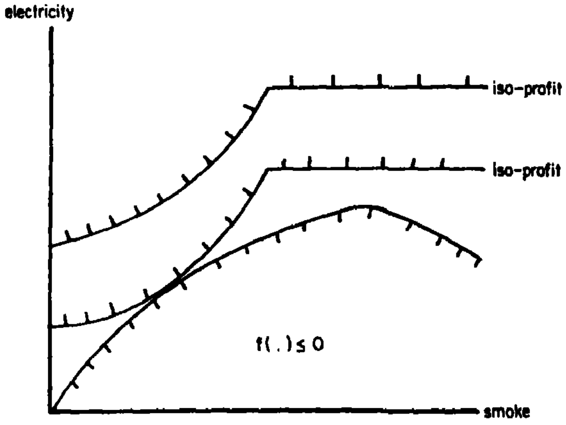

FIG. 2. Joint profits.

One source of confusion concerns whether the electric company will have enough profits to compensate the laundry. This is a question of whether quasi-rents are adequate in the short run, or rents are adequate in the long run. The solution is essentially the same in both cases; we shall explain the long-run case. In the long run the electric company's profits net of compensation are nonnegative because it can produce zero output and pay no compensation: $\bar{\pi}^j \geq 0$. In other words the electric company's rent is always sufficient to compensate for the laundry's lost rent at the optimum. The rule of pollutor liability creates sufficient rent for efficient compensation.

LOCATION

Another source of confusion concerns the optimal location of plants. Locational choice does not change the essential nature of the problem, but it increases the computational difficulties. Let z_r be a variable indicating the site chosen for plant r and let $w(z_r)$ be the corresponding rent. Location affects production by changing the transportation inputs needed for a particular output. Furthermore, the amount of smoke observed by the laundry y_{ijs} depends upon the electric company's emissions y_{js}, and the location of the plants: $y_{ijs} = y_{ijs}(z_i, z_j, y_{js})$. Thus the laundry solves

$$\pi^i = \max_{y_i z_i} \sum_k p_k y_{ik} - w(z_i) \qquad \text{s.t.} \quad f^i(y_i, z_i; y_{ijs}(z_i, z_j, y_{js})) \leq 0.$$

Under full liability, the electric company solves

$$\bar{\pi}^j = \max_{y_j z_j} \sum_k p_k y_{jk} - w(z_j) - (\pi^i(p, 0, z_j) - \pi^i(p, y_{js}, z_j))$$
$$\text{s.t.} \quad f^j(y_j, z_j) \leq 0.$$

We may assume that zero smoke emissions results in zero smoke observations regardless of location, so $\pi^i(p, o, z_j)$ is constant with respect to z_j.

The electric company is choosing net supplies and location to maximize joint profits, as required for social efficiency, but the computations are difficult when location is a choice variable. Efficiency requires that compensation be based upon the losses which the laundry would suffer when its location maximizes its profits. The electric company and the courts may have difficulty distinguishing between losses at the laundry's actual location and hypothetical losses at its ideal location.

There is an added difficulty in identifying the global maximum, due to a locational nonconvexity which is different from Starrett's nonconvexity[7]. Suppose that the laundry and electric company are located far apart, and the production set is convex over the region where laundry is produced in positive quantities, as shown in Fig. 3. As the laundry changes its location and moves closer to the electric company, the production frontier dips toward the origin for all interior points, as shown, which indicates that smoke is interfering more with laundry production. However, the corners of the frontier are unaltered. If this process continues as the laundry is moved ever closer, the production frontier's interior eventually becomes nonconvex, as shown.

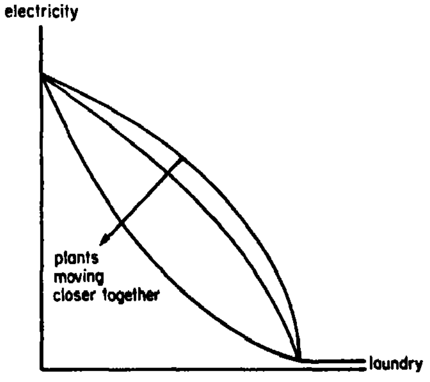

FIG. 3. Baumol's nonconvexity from locating plants closer together.

LIMITED INFORMATION AND BARGAINING

The liability solution requires courts to base compensation upon the pollutee's minimum profit loss, as given by the profit function, rather than the actual profit loss. (See footnote 4.) The courts may lack the information needed for such a computation. It is possible to construct models in which the courts overcome limited information by a sequence of court decisions which converge to the optimum, but convergence assumes full convexity, which is absent in our case.[8] Consequently, it would be desirable if efficiency could be achieved by private bargaining, irrespective of the law's structure, as postulated by the Coase theorem.[9]

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[7] See W. J. Baumol, On taxation and control of externalities, *Amer. Econ. Rev.* **62** (1972), 307–322.

Even in the two-person case, the nature of bargaining is such that efficient or group rational outcomes cannot be anticipated with confidence. For example, the parties may fail to agree because each holds out for more than half of the surplus from cooperation. In such a strategic setting, the electric company may be unable to extract the information needed to distinguish between global and local maxima. The likelihood of efficiency is still lower when there are several pollutors and pollutees, all of whom must become parties to the bargain.

How does the existence of several pollutors or pllutees influence the liability solution? The essential nature of the problem is unchanged when there are several laundries and a single electric company, which corresponds to class action against a single defendent. Adding additional terms to the joint profit function requires the electric company to have more information, but the nature of its maximization problem is unchanged. However, adding several electric companies creates serious difficulties for the full liability solution. The courts would need to compute the global optimum in order to divide damages among the pollutors in a way which would achieve efficiency. It would be difficult for the courts to make such a computation.[10]

ACKNOWLEDGMENT

I would like to thank an anonymous reviewer for correcting errors in an earlier draft.

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[8] See R. Cooter, L. Kornhauser, and D. Lane, Liability rules, limited information, and the role of precedent, *Bell J. Econ.* (Spring 1979).

[9] The symmetry of compensation rules is asserted by Coase in his classic paper, The problem of social cost, *J. Law and Econ.* 3 (1960). His failure to consider the implications of the strategic problems in bargaining is discussed in R. Cooter, The cost of believing Coase's theorem, *UCB* Working Paper No. 76, revised 1978.

[10] Let $y^*_{1\cdot},\ldots, y^*_{n\cdot}$ be the efficient pollution. Firm j will choose the efficient value for $y_{j\cdot}$ if compensation is based upon the damages j's smoke imposes when other pollutors are at the efficient point. Specifically, $y^*_{j\cdot}$ is the value of $y_{j\cdot}$ which solves $\bar{\pi}^j \equiv \max_{y_{jk}} \sum_k p_k y_{jk} - \sum_i \pi^i(p, 0) - \pi^i(p, y^*_{\backslash j(\cdot)}, y_{j\cdot}))$ s.t. $f^j(y_{j1},\ldots, y_{jn}) < 0$. With nonconvexities, it is difficult to think of an interactive process by which the courts could discover the globally efficient pollution levels; see footnote 8.

JOURNAL OF ECONOMIC THEORY 22, 505 (1980)

Author Index for Volume 22

B

Bewley, Truman F., 377
Blackorby, Charles, 87
Border, Kim C., 395

C

Cohen, Susan I., 37
Cooter, Robert D., 499

D

Donaldson, David, 67
Dubey, Pradeep, 339, 364

G

Gabszewicz, J. Jaskold, 327
Green, Edward J., 155
Greenberg, Joseph, 12

H

Hart, Oliver D., 279
Hellwig, Martin F., 477

J

Jordan, J. S., 395

K

Kalai, Ehud, 23
Kannai, Yakar, 115
Knoer, Elsie M., 439
Kurz, Mordecai, 99

L

Laitner, John P., 56

M

MacDougall, William, 113
Makowski, Louis, 208, 222
Mas-Colell, Andreu, 121, 339
McLennan, Andrew, 1

N

Novshek, William, 243, 313

O

Ostroy, Joseph M., 183

P

Primont, Daniel, 87

R

Radner, Roy, 136
Ritz, Zvi, 23
Roberts, Kevin, 256
Roemer, John E., 451

S

Sertel, Murat R., 423
Shah, Anup, 107
Shubik, Martin, 339
Sonnenschein, Hugo, 243
Spivak, Avia, 465
Suzumura, Kotaro, 407

T

Thisse, J.-F., 327

V

Van der Bellen, Alexander, 423

W

Weymark, John A., 67

Printed in Belgium

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